

MEFE 2

Experimentos de conteo: Poisson

Intervalos de Poisson

Usando

$$Pois(N | \mu) = \frac{e^{-\mu} \mu^N}{N!} \quad E(N) = \mu \quad \sigma(N) = \sqrt{\mu}$$

uno comunmente aproxima:

$$N_{\text{obs}} \simeq E(N) = \mu \quad \Rightarrow \quad \sigma(N) = \sqrt{\mu} \simeq \sqrt{N_{\text{obs}}} \quad \Rightarrow \quad N_{\text{obs}} \pm \sqrt{N_{\text{obs}}}$$

Probablemente la ecuación más usada a diario en física experimental.

- La medición N_{obs} tiene un error 1-sigma de $\pm\sqrt{N_{\text{obs}}}$ con 68% CL.
- Es p.ej lo que programas gráficos dibujan en histogramas como error del bin.
- Pero el intervalo $N_{\text{obs}} \pm \sqrt{N_{\text{obs}}}$ en un intervalo 68% CL ?
- Se necesita determinar su cobertura.

Cobertura de $N_{\text{obs}} \pm \sqrt{N_{\text{obs}}}$

N_{obs}	$\mu_1 - \mu_2$
0	0.00 - 0.00
1	0.00 - 2.00
2	0.58 - 3.41
3	1.27 - 4.73
4	2.00 - 6.00
5	2.76 - 7.23

Cobertura de $N_{\text{obs}} \pm \sqrt{N_{\text{obs}}}$

$$\mu = 1.9$$

N_{obs}	$\mu_1 - \mu_2$
0	0.00 - 0.00
1	0.00 - 2.00
2	0.58 - 3.41
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Cobertura de $N_{\text{obs}} \pm \sqrt{N_{\text{obs}}}$

N_{obs}	$\mu_1 - \mu_2$	$P(N_{\text{obs}})$
0	0.00 - 0.00	0.15
1	0.00 - 2.00	0.28
2	0.58 - 3.41	0.27
3	1.27 - 4.73	0.17
4	2.00 - 6.00	0.08
5	2.76 - 7.23	0.03

$$\mu = 1.9$$

Cobertura de $N_{\text{obs}} \pm \sqrt{N_{\text{obs}}}$

N_{obs}	$\mu_1 - \mu_2$	$P(N_{\text{obs}})$	in?
0	0.00 - 0.00	0.15	—
1	0.00 - 2.00	0.28	✓
2	0.58 - 3.41	0.27	✓
3	1.27 - 4.73	0.17	✓
4	2.00 - 6.00	0.08	—
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Cobertura de $N_{\text{obs}} \pm \sqrt{N_{\text{obs}}}$

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2	0.58 - 3.41	0.27	✓
3	1.27 - 4.73	0.17	✓
4	2.00 - 6.00	0.08	—
5	2.76 - 7.23	0.03	—
		0.72	

$$\mu = 1.9$$

Cobertura de $N_{\text{obs}} \pm \sqrt{N_{\text{obs}}}$

N_{obs}	$\mu_1 - \mu_2$	$\mu = 1.9$	in?	$\mu = 2.1$
		$P(N_{\text{obs}})$		$P(N_{\text{obs}})$
0	0.00 - 0.00	0.15	—	0.12
1	0.00 - 2.00	0.28	✓	0.26
2	0.58 - 3.41	0.27	✓	0.27
3	1.27 - 4.73	0.17	✓	0.19
4	2.00 - 6.00	0.08	—	0.10
5	2.76 - 7.23	0.03	—	0.04
		0.72		0.56

Cobertura de $N_{\text{obs}} \pm \sqrt{N_{\text{obs}}}$

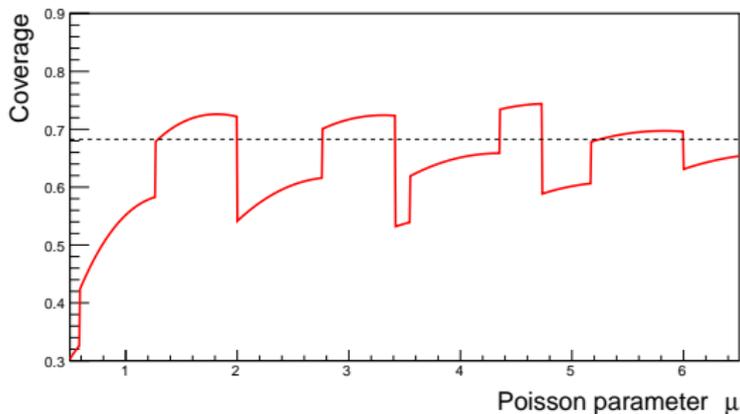
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		$P(N_{\text{obs}})$	in?	$P(N_{\text{obs}})$	in?
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3	1.27 - 4.73	0.17	✓	0.19	✓
4	2.00 - 6.00	0.08	—	0.10	✓
5	2.76 - 7.23	0.03	—	0.04	—
		0.72		0.56	

Cobertura de $N_{\text{obs}} \pm \sqrt{N_{\text{obs}}}$

N_{obs}	$\mu_1 - \mu_2$	$\mu = 1.9$		$\mu = 2.1$		$\mu = 1.3$	
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1	0.00 - 2.00	0.28	✓	0.26	—	0.35	✓
2	0.58 - 3.41	0.27	✓	0.27	✓	0.23	✓
3	1.27 - 4.73	0.17	✓	0.19	✓	0.10	✓
4	2.00 - 6.00	0.08	—	0.10	✓	0.03	—
5	2.76 - 7.23	0.03	—	0.04	—	0.01	—
		0.72		0.56		0.68	

Cobertura de $N_{\text{obs}} \pm \sqrt{N_{\text{obs}}}$

N_{obs}	$\mu_1 - \mu_2$	$\mu = 1.9$		$\mu = 2.1$		$\mu = 1.3$	
		$P(N_{\text{obs}})$	in?	$P(N_{\text{obs}})$	in?	$P(N_{\text{obs}})$	in?
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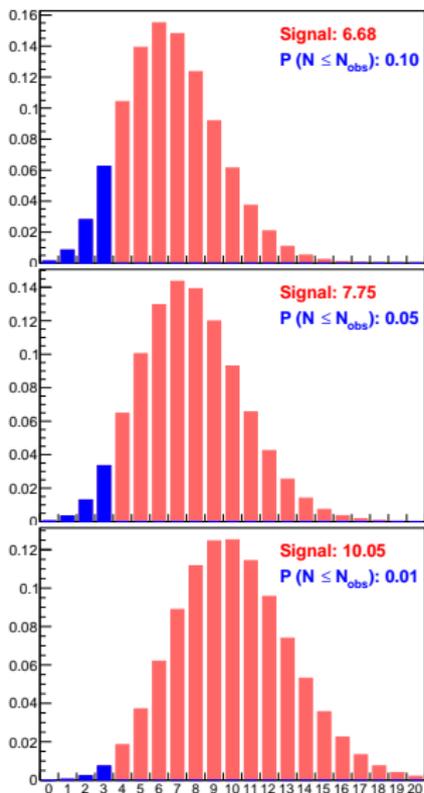
Límites superiores de Poisson

- Los intervalos de confianza dan información sobre parámetros.
- Los intervalos de dos lados corresponden a mediciones (usualmente CL 68%).
- Los de un solo lado son la cota superior o inferior (usualmente CL 90% o 95%).

Los límites superiores e inferiores son importantes en resultados negativos:

- El flujo de neutrinos extragalácticos de UHE sobre la Tierra es menor que ...
- La sección eficaz \times la densidad de materia oscura es menor que ...
- La fracción de una cierta sustancia en sangre es menor que ...
- La eficiencia de un cierto proceso no es menor que ...
- La masa de los electrones supersimétricos es mayor que ...

Poisson Upper Limits. Frequentist



- Measure N_{obs} from Poisson with μ unknown

In this example $N_{\text{obs}} = 3$

- What values of μ are compatible with N_{obs} ?

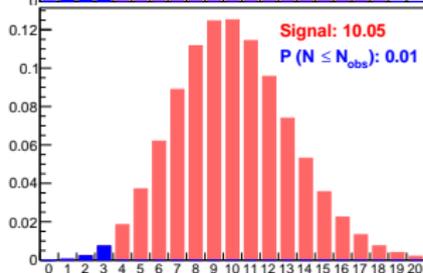
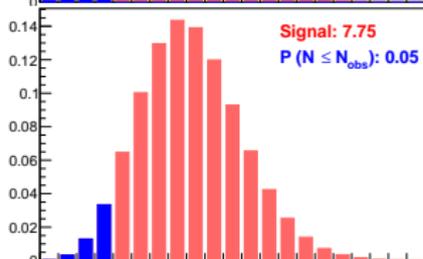
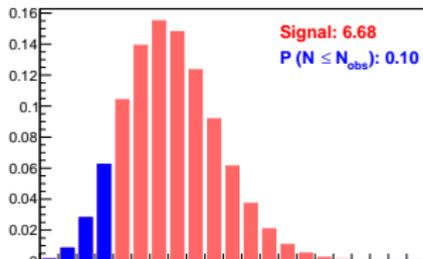
We quantify “compatibility” via $P(N \leq N_{\text{obs}} | \mu)$

- $P(N \leq N_{\text{obs}} | \mu)$ decreases as μ grows

- Upper Limit:

Highest value of μ we consider compatible with N_{obs}

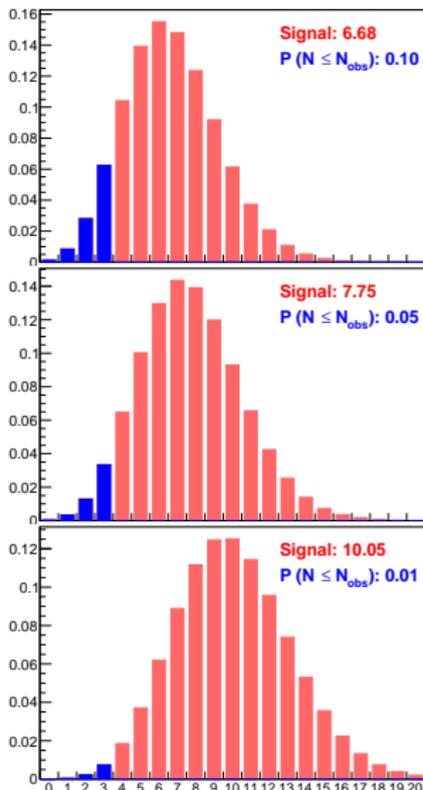
Poisson Upper Limits. Frequentist



⇒ Decide when a small $P(N \leq N_{\text{obs}} | \mu)$ is considered

- a random fluctuation
- a sign of incompatibility between μ and N_{obs}

Poisson Upper Limits. Frequentist



- Decide when a small $P(N \leq N_{\text{obs}} | \mu)$ is considered
 - a random fluctuation
 - a sign of incompatibility between μ and N_{obs}

- Choose a significance α or confidence level CL such that

$$P(N \leq N_{\text{obs}} | \mu_{\text{up}}) \leq \alpha = 1 - CL$$

Significance	CL	μ_{up}
0.10	90%	6.68
0.05	95%	7.75
0.01	99%	10.05

Poisson Upper Limits. Coverage.

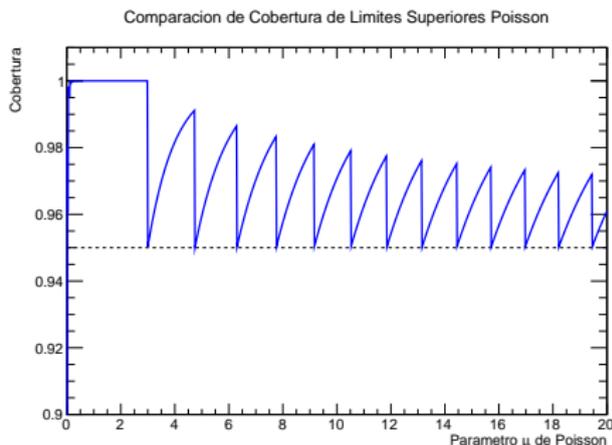
This method yields μ_{up} at 95% CL for each N_{obs} .

For these values the coverage is 95% ,by construction.

For other values of μ there will be overcoverage.

N_{obs}	μ_{up}
0	3.00
1	4.74
2	6.30
3	7.75
4	9.15
5	10.51
6	11.84
...	...

$$\text{Coverage}(\mu) = \sum_{N:\mu < \mu_{up}(N)} P(N|\mu)$$



Poisson Limits. Bayesian

Remember Bayes:

$f(x | \mu)$: probability of measuring x if the true value of the parameter is μ .

$p(\mu | x)$: the posterior, pdf of the parameter μ given that we measured x .

They are related by Bayes' Theorem:

$$p(\mu | x) \propto f(x | \mu) \pi(\mu)$$

$$p(\mu | x) = C f(x | \mu) \pi(\mu)$$

The constant C does not depend on μ and is calculated by normalization:

$$\int p(\mu | x) d\mu = 1 \quad \Rightarrow \quad C \int f(x | \mu) \pi(\mu) d\mu = 1 \quad \Rightarrow \quad C = \frac{1}{\int f(x | \mu) \pi(\mu) d\mu}$$

Poisson Limits. Bayesian

Bayes for the Poisson case:

$P(N | \mu)$: probability of measuring N if the true value of the parameter is μ .

$p(\mu | N)$: posterior distribution of the parameter μ given that we measured N .

$$p(\mu | N) \propto P(N | \mu) \pi(\mu)$$

$$p(\mu | N) \propto \frac{e^{-\mu} \mu^N}{N!} \propto e^{-\mu} \mu^N$$

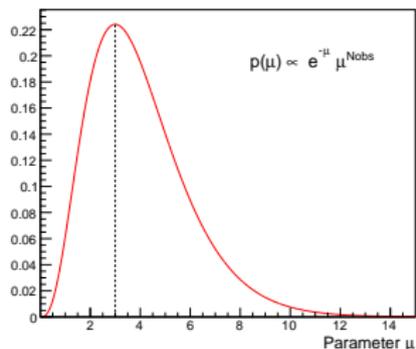
If we measured N_{obs} :

$$p(\mu | N_{\text{obs}}) \propto e^{-\mu} \mu^{N_{\text{obs}}}$$

Poisson Limits. Bayesian

Using Bayes for the case $N_{\text{obs}} = 3$:

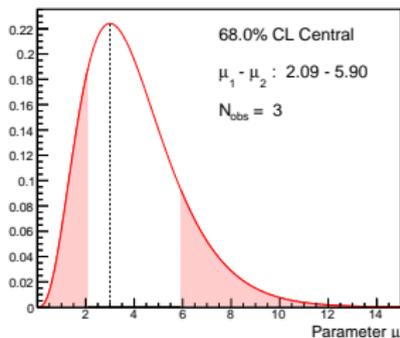
$$p(\mu | N_{\text{obs}}) \propto e^{-\mu} \mu^{N_{\text{obs}}} \quad (\text{for } \pi(\mu) \text{ flat})$$



Poisson Limits. Bayesian

Using Bayes for the case $N_{\text{obs}} = 3$:

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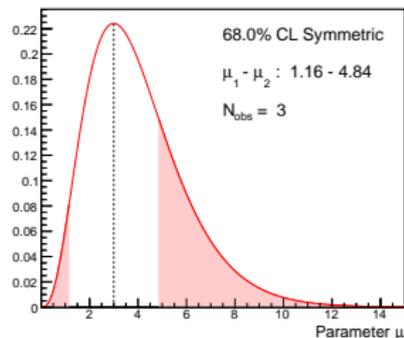
$$\mu = 3^{+2.90}_{-0.91}$$

$$\int_0^{\mu^1} p(\mu) d\mu = \int_{\mu^2}^{\infty} p(\mu) d\mu$$

Poisson Limits. Bayesian

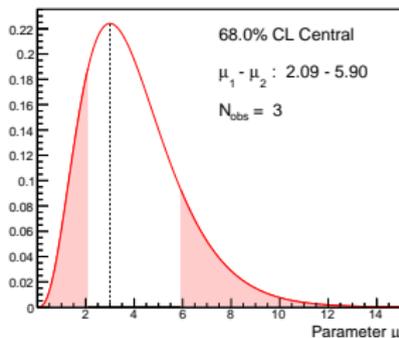
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$$\mu = 3 \pm 1.84$$

$$\mu_2 - \mu = \mu - \mu_1$$



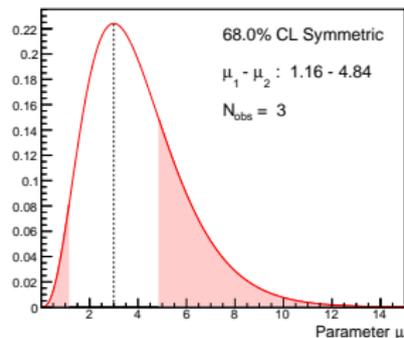
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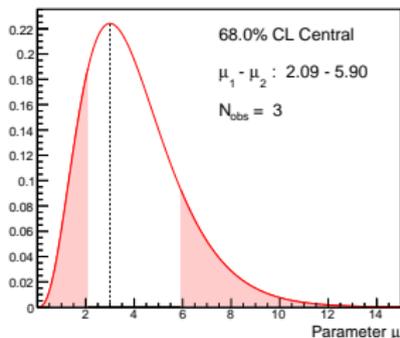
Using Bayes for the case $N_{\text{obs}} = 3$:

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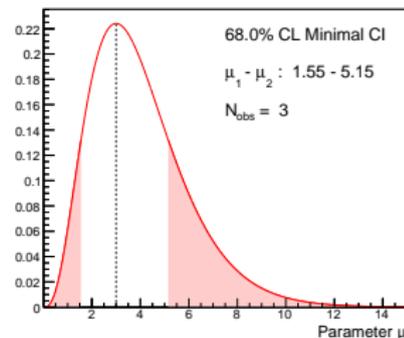
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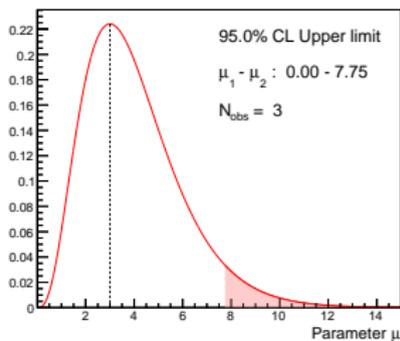
$$\mu = 3^{+2.15}_{-1.45}$$

$$p(\mu_1) = p(\mu_2)$$

Poisson Limits. Bayesian

Using Bayes for the case $N_{\text{obs}} = 3$:

$$p(\mu | N_{\text{obs}}) \propto e^{-\mu} \mu^{N_{\text{obs}}} \quad (\text{for } \pi(\mu) \text{ flat})$$



$$\mu_{\text{up}} = 7.75$$

$\pi(\mu)$ flat

Poisson Limits. Bayesian

A special case, $N_{\text{obs}} = 0$.

$$p(\mu | N_{\text{obs}}) \propto e^{-\mu} \mu^{N_{\text{obs}}} = e^{-\mu} \quad (\text{for } \pi(\mu) \text{ flat})$$

The equation for the upper limit is:

$$1 - CL = \int_{\mu_{\text{up}}}^{\infty} p(\mu | N_{\text{obs}}=0) d\mu = \int_{\mu_{\text{up}}}^{\infty} e^{-\mu} d\mu = e^{-\mu_{\text{up}}}$$

And thus

$$\mu_{\text{up}} = -\log(1 - CL) = -\log(1 - 0.95) = 3.00 \quad \text{for } CL = 0.95$$

Summary Frequentist vs. Bayesian Upper Limits

Given N_{obs} , the upper limit at CL confidence level is obtained as

Frequentist

$$P(N \leq N_{\text{obs}} | \mu_{\text{up}}) = 1 - CL$$

where

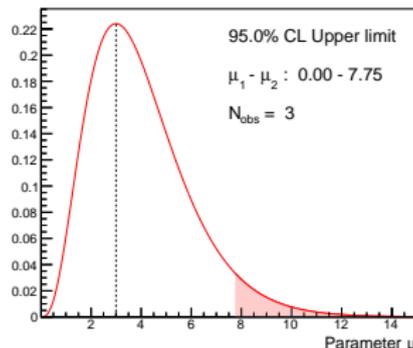
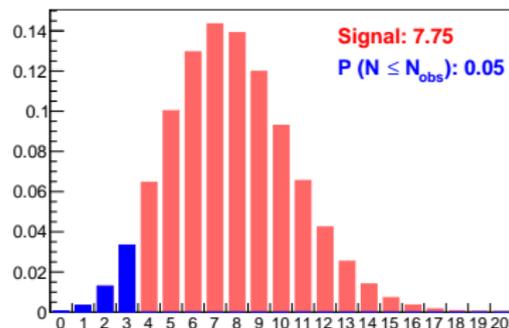
$$P(N \leq N_{\text{obs}} | \mu_{\text{up}}) = \sum_{N=0}^{N_{\text{obs}}} P(N | \mu_{\text{up}})$$

Bayesian

$$P(\mu \geq \mu_{\text{up}} | N_{\text{obs}}) = 1 - CL$$

where

$$P(\mu \geq \mu_{\text{up}} | N_{\text{obs}}) = \int_{\mu_{\text{up}}}^{\infty} p(\mu | N_{\text{obs}}) d\mu$$



Summary Frequentist vs. Bayesian Upper Limits

Frequentist and Bayesian upper limits are identical only for flat prior.

This is due to a mathematical property of the Poisson distribution.

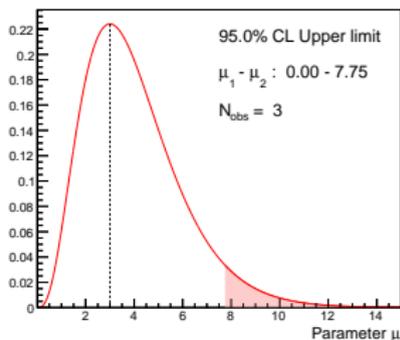
The equivalence Frequentist/Bayesian limits breaks for other priors.

N_{obs}	Frequentist	Bayes flat	Bayes $\pi(\mu) = \mu$	Bayes $\pi(\mu) = 1/\sqrt{\mu}$
0	3.00	3.00		
1	4.74	4.74		
2	6.30	6.30		
3	7.75	7.75		
5	10.51	10.51		
10	16.96	16.96		

Bayesian prior dependence

The relation between $f(\mu | N_{\text{obs}})$ and $P(N_{\text{obs}} | \mu)$ depends on the prior.

$$p(\mu | N_{\text{obs}}) \propto e^{-\mu} \mu^{N_{\text{obs}}} \pi(\mu)$$



$$\mu_{\text{up}} = 7.75$$

$$\pi(\mu) = 1/\sqrt{\mu}$$

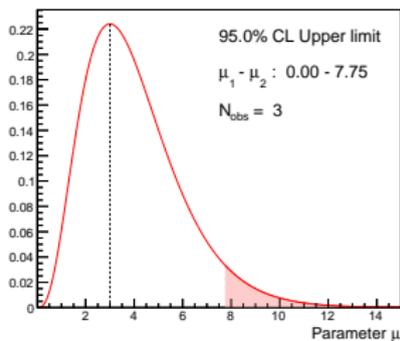
$$\pi(\mu) \text{ flat}$$

$$\pi(\mu) = \mu$$

Bayesian prior dependence

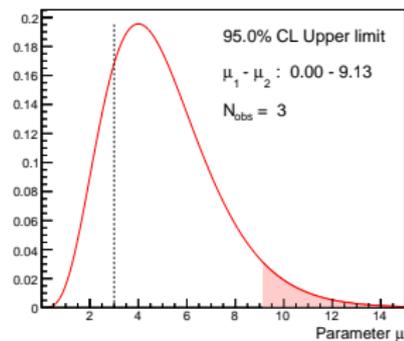
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$$\mu_{\text{up}} = 7.75$$

$\pi(\mu)$ flat



$$\mu_{\text{up}} = 9.13$$

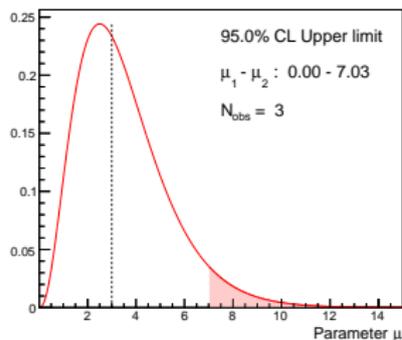
$\pi(\mu) = \mu$

$$\pi(\mu) = 1/\sqrt{\mu}$$

Bayesian prior dependence

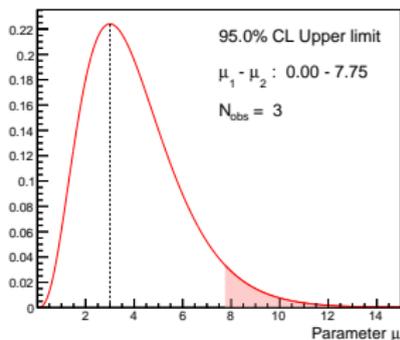
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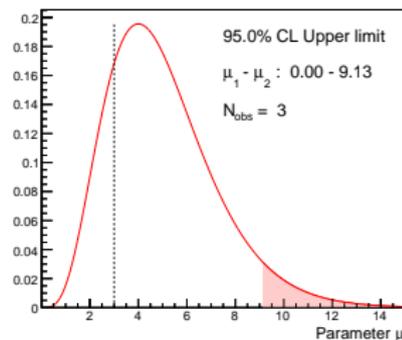
$$\mu_{\text{up}} = 7.03$$

$$\pi(\mu) = 1/\sqrt{\mu}$$



$$\mu_{\text{up}} = 7.75$$

$$\pi(\mu) \text{ flat}$$



$$\mu_{\text{up}} = 9.13$$

$$\pi(\mu) = \mu$$

Bayesian prior dependence

Frequentist and Bayesian upper limits are identical only for flat prior.

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The equivalence Frequentist/Bayesian limits breaks for other priors.

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N_{obs}	Frequentist	Bayes	Bayes	Bayes
		flat	$\pi(\mu) = \mu$	$\pi(\mu) = 1/\sqrt{\mu}$
0	3.00	3.00	4.73	
1	4.74	4.74	6.23	
2	6.30	6.30	7.48	
3	7.75	7.75	9.13	
5	10.51	10.51	11.84	
10	16.96	16.96	18.21	

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1	4.74	4.74	6.23	3.90
2	6.30	6.30	7.48	5.50
3	7.75	7.75	9.13	7.03
5	10.51	10.51	11.84	9.84
10	16.96	16.96	18.21	16.34

Poisson Upper Limits in the presence of background

Poisson Upper Limits with background

Expected number of events μ : sum of a signal s and background b

$$\mu = s + b$$

- Signal/Noise
- HEP: New Physics / SM Physics
- Cosmology: Source / Average Sky

Assume background known. What is the upper limit on the signal?

Poisson Upper Limits with background: Bayesian

The number of events observed depends on two parameters: s (unknown), b (known).

The inversion of the conditional probability concerns only s :

$$P(N | s, b) \rightarrow p(s | N, b) \quad \text{where} \quad P(N | s, b) = \frac{e^{-(s+b)} (s+b)^N}{N!}$$

The inversion using Bayes:

$$p(s | N, b) \propto P(N | s, b) \pi(s)$$

$$p(s | N, b) \propto \frac{e^{-(s+b)} (s+b)^N}{N!} \propto e^{-(s+b)} (s+b)^N$$

Poisson Upper Limits with background: Frequentist

$$P(N_{\text{obs}} | \mu = s + b) = \frac{e^{-\mu} \mu^{N_{\text{obs}}}}{N_{\text{obs}}!}$$

- Finding μ given N_{obs} , is exactly like the problem without background.
- Knowing b and μ , get the signal parameter as $s = \mu - b$

$CL = 95\%$	$b = 0$		$b = 0.5$		$b = 1.5$		$b = 2.5$		$b = 3.5$	
N_{obs}	0	2	0	2	0	2	0	2	0	2
Bayesian	3.0	6.3	3.0	5.8	3.0	5.1	3.0	4.6	3.0	4.3
Frequentist	3.0	6.3	2.5	5.8						

Poisson Upper Limits with background: Frequentist

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N_{obs}	0	2	0	2	0	2	0	2	0	2
Bayesian	3.0	6.3	3.0	5.8	3.0	5.1	3.0	4.6	3.0	4.3
Frequentist	3.0	6.3	2.5	5.8	1.5	4.8				

Poisson Upper Limits with background: Frequentist

$$P(N_{\text{obs}} | \mu = s + b) = \frac{e^{-\mu} \mu^{N_{\text{obs}}}}{N_{\text{obs}}!}$$

- Finding μ given N_{obs} , is exactly like the problem without background.
- Knowing b and μ , get the signal parameter as $s = \mu - b$

$CL = 95\%$	$b = 0$		$b = 0.5$		$b = 1.5$		$b = 2.5$		$b = 3.5$	
N_{obs}	0	2	0	2	0	2	0	2	0	2
Bayesian	3.0	6.3	3.0	5.8	3.0	5.1	3.0	4.6	3.0	4.3
Frequentist	3.0	6.3	2.5	5.8	1.5	4.8	0.5	3.8		

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Frequentist	3.0	6.3	2.5	5.8	1.5	4.8	0.5	3.8	-0.5	2.8

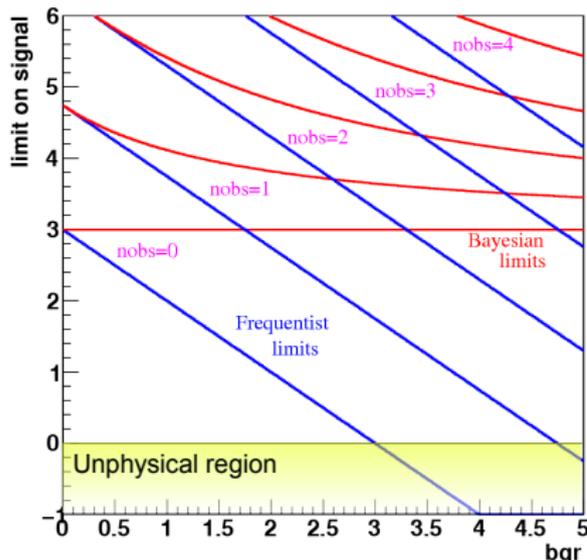
Frequentist vs. Bayesian

Frequentist:

- Empty confidence intervals are possible.
- $N_{\text{obs}} = 0$: Limit on signals depends on expected bkg.
- Experiments with more background can get stricter (more impressive) limits.

Bayesian:

- No empty signals.
- $N_{\text{obs}} = 0$: no background dependence.
- $N_{\text{obs}} > 0$: with increasing bkg, limit tends to $N_{\text{obs}}=0$ case.



Poisson Limits with Systematic Errors

Auger: an experiment measuring a Poisson $P(\mu)$ random variable

The number of neutrino-like showers recorded in a year is a Poisson variable with

$$\mu = \varepsilon (s + b)$$

where:

s : signal, integrated cosmic UHE neutrino flux over a time T

$\varepsilon \pm \sigma_\varepsilon$: detection efficiency

$b \pm \sigma_b$: integrated background

Let's assume for the moment that the errors are Gaussian

Poisson experiment with $\mu = \varepsilon (s + b)$

$$\mathcal{L}(N_{\text{Obs}}, b_{\text{Obs}}, \varepsilon_{\text{Obs}} | s, b, \varepsilon) = \underbrace{\frac{e^{-\varepsilon(s+b)} [\varepsilon(s+b)]^{N_{\text{Obs}}}}{N_{\text{Obs}}!}}_{\text{event counting}} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sigma_b} e^{-\frac{1}{2} \left(\frac{b_{\text{Obs}}-b}{\sigma_b}\right)^2}}_{\text{measurement of } b} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sigma_\varepsilon} e^{-\frac{1}{2} \left(\frac{\varepsilon_{\text{Obs}}-\varepsilon}{\sigma_\varepsilon}\right)^2}}_{\text{measurement of } \varepsilon}$$

⇒ \mathcal{L} is the likelihood to observe $N_{\text{Obs}}, b_{\text{Obs}}, \varepsilon_{\text{Obs}}$ given the (unknown) values of s, b, ε

⇒ Probability to measure N_{Obs} depends on the three parameters s, b, ε :

$$P(N_{\text{Obs}} | \mu) = \frac{e^{-\mu} \mu^{N_{\text{Obs}}}}{N_{\text{Obs}}!} \quad \text{with} \quad \mu = \varepsilon (s + b)$$

⇒ b and ε are ‘nuisance’ parameters, that correspond to systematic uncertainties :

Their value is unimportant, but are needed to determine s , the parameter of interest.

Poisson experiment with $\mu = \varepsilon (s + b)$

$$\mathcal{L}(N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}} | s, b, \varepsilon) = \underbrace{\frac{e^{-\varepsilon(s+b)} [\varepsilon(s+b)]^{N_{\text{obs}}}}{N_{\text{obs}}!}}_{\text{event counting}} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sigma_b} e^{-\frac{1}{2} \left(\frac{b_{\text{obs}}-b}{\sigma_b}\right)^2}}_{\text{measurement of } b} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sigma_\varepsilon} e^{-\frac{1}{2} \left(\frac{\varepsilon_{\text{obs}}-\varepsilon}{\sigma_\varepsilon}\right)^2}}_{\text{measurement of } \varepsilon}$$

Gaussian systematic errors mean:

✦ **Frequentist:** If b is the true background, the probability to measure b_{obs} is

$$f(b_{\text{obs}} | b) = \frac{1}{\sqrt{2\pi}\sigma_b} e^{-\frac{1}{2} \left(\frac{b_{\text{obs}}-b}{\sigma_b}\right)^2}$$

✦ **Bayesian:** Having measured b_{obs} , the probability density of the true background b is

$$f(b | b_{\text{obs}}) = \frac{1}{\sqrt{2\pi}\sigma_b} e^{-\frac{1}{2} \left(\frac{b-b_{\text{obs}}}{\sigma_b}\right)^2}$$

The Bayesian approach

$$\mathcal{L}(N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}} | s, b, \varepsilon) = \underbrace{\frac{e^{-\varepsilon(s+b)} [\varepsilon(s+b)]^{N_{\text{obs}}}}{N_{\text{obs}}!}}_{\text{event counting}} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sigma_b} e^{-\frac{1}{2} \left(\frac{b_{\text{obs}}-b}{\sigma_b}\right)^2}}_{\text{measurement of } b} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sigma_\varepsilon} e^{-\frac{1}{2} \left(\frac{\varepsilon_{\text{obs}}-\varepsilon}{\sigma_\varepsilon}\right)^2}}_{\text{measurement of } \varepsilon}$$

- Use Bayes to find the probability density of the parameters :

$$p(s, b, \varepsilon | N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}}) \propto \mathcal{L}(N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}} | s, b, \varepsilon) \pi(s) \pi(b) \pi(\varepsilon)$$

- Integrate (marginalise) over nuisance parameters :

$$p(s | N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}}) \propto \iint \mathcal{L}(N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}} | s, b, \varepsilon) \pi(s) \pi(b) \pi(\varepsilon) db d\varepsilon$$

- Use $p(s | N_{\text{obs}})$ as a regular probability density on s :

$$(90\% \text{ CL}) \quad s_{\text{sup}} : \int_0^{s_{\text{sup}}} p(s | N_{\text{obs}}) ds = 0.90$$

The hybrid Frequentist-Bayesian approach (Cousins-Highland)

$$\mathcal{L}(N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}} | s, b, \varepsilon) = \underbrace{\frac{e^{-\varepsilon(s+b)} [\varepsilon(s+b)]^{N_{\text{obs}}}}{N_{\text{obs}}!}}_{\text{event counting}} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sigma_b} e^{-\frac{1}{2} \left(\frac{b_{\text{obs}}-b}{\sigma_b}\right)^2}}_{\text{measurement of } b} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sigma_\varepsilon} e^{-\frac{1}{2} \left(\frac{\varepsilon_{\text{obs}}-\varepsilon}{\sigma_\varepsilon}\right)^2}}_{\text{measurement of } \varepsilon}$$

- Use Bayes only for nuisance parameters (bayesian treatment of systematics)

$$\mathcal{L}(N_{\text{obs}}, b, \varepsilon | s, b_{\text{obs}}, \varepsilon_{\text{obs}}) \propto \mathcal{L}(N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}} | s, b, \varepsilon) \pi(b) \pi(\varepsilon)$$

- Integrate (marginalise) over nuisance parameters

$$\mathcal{L}(N_{\text{obs}} | s, b_{\text{obs}}, \varepsilon_{\text{obs}}) \propto \iint \mathcal{L}(N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}} | s, b, \varepsilon) \pi(b) \pi(\varepsilon) db d\varepsilon$$

- Use $\mathcal{L}(N_{\text{obs}} | s)$ as a regular likelihood:

$$(90\% \text{ CL}) \quad s_{\text{up}} : \sum_{N=0}^{N_{\text{obs}}} \mathcal{L}(N | s_{\text{up}}) = 0.10$$

The Frequentist approach

$$\mathcal{L}(N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}} | s, b, \varepsilon) = \underbrace{\frac{e^{-\varepsilon(s+b)} [\varepsilon(s+b)]^{N_{\text{obs}}}}{N_{\text{obs}}!}}_{\text{event counting}} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sigma_b} e^{-\frac{1}{2} \left(\frac{b_{\text{obs}}-b}{\sigma_b}\right)^2}}_{\text{measurement of } b} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\sigma_\varepsilon} e^{-\frac{1}{2} \left(\frac{\varepsilon_{\text{obs}}-\varepsilon}{\sigma_\varepsilon}\right)^2}}_{\text{measurement of } \varepsilon}$$

- For each s , estimate b and ε via maximum likelihood: $\hat{b}(s)$ and $\hat{\varepsilon}(s)$

$$\forall s : \quad \mathcal{L}(N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}} | s, \hat{b}(s), \hat{\varepsilon}(s)) \geq \mathcal{L}(N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}} | s, b, \varepsilon)$$

- Define \mathcal{L}_p as the likelihood profiled over the nuisance parameters

$$\mathcal{L}_p(N_{\text{obs}} | s) \equiv \mathcal{L}(N_{\text{obs}}, b_{\text{obs}}, \varepsilon_{\text{obs}} | s, \hat{b}(s), \hat{\varepsilon}(s))$$

- Use $\mathcal{L}_p(N_{\text{obs}} | s)$ as a regular likelihood:

$$(90\% \text{ CL}) \quad s_{\text{up}} : \quad \sum_{N=0}^{N_{\text{obs}}} \mathcal{L}_p(N | s_{\text{up}}) = 0.10$$

Systematic errors need not be Gaussian

$$\mathcal{L}(N_{\text{obs}}, b_{\text{obs}}, n_{\text{obs}} | s, b, \varepsilon) = \underbrace{\frac{e^{-\varepsilon(s+b)} [\varepsilon(s+b)]^{N_{\text{obs}}}}{N_{\text{obs}}!}}_{\text{event counting}} \cdot \underbrace{\frac{e^{-\tau b} (\tau b)^{b_{\text{obs}}}}{b_{\text{obs}}!}}_{\text{Poisson error}} \cdot \underbrace{\binom{n}{n_{\text{obs}}} \varepsilon^{n_{\text{obs}}} (1-\varepsilon)^{n-n_{\text{obs}}}}_{\text{Binomial error}}$$

- One parameter of interest,
 - ⇒ s : signal, integrated cosmic UHE neutrino flux over a time T
- Two nuisance parameters: background b , and efficiency ε
- b measured in a Poisson experiment:
 - ⇒ b_{obs} counts observed from background sky during a period $\tau \times T$
- ε measured in a binomial experiment:
 - ⇒ n_{obs} events observed out of n total events.

Example of limits from hybrid method including systematics

- Change in limit for different combinations of relative systematic errors on background b and efficiency ε :

$$\mu = \varepsilon(s + b)$$

- Systematic errors make limits worse

CL _s limits	$b = 0.5$		$b = 3.5$	
	$N_{\text{obs}} = 0$	$N_{\text{obs}} = 2$	$N_{\text{obs}} = 0$	$N_{\text{obs}} = 2$
No syst.	2.5	5.8	-0.5	2.8
$\sigma_b/b = 50\%$	2.5	5.8	0.5	3.5
$\sigma_\varepsilon/\varepsilon = 10\%$	2.6	5.9	-0.4	2.9
Both syst.	2.6	5.9	0.6	3.6