

# Connecting thermodynamics to students' calculus

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I describe subtle calculus ideas that are essential for thermodynamics, but are typically not encountered by students in calculus or prior physics classes. I argue that these previously unencountered subtleties are a substantial cause of the difficulty that many students encounter in learning thermodynamics and that thermodynamics can be taught more effectively by introducing the subtleties within an environment with which students are familiar rather than insisting that students learn them at the same time that they encounter new physics concepts such as entropy and thermodynamic potentials. I show how Legendre transforms can be used to illustrate the important calculus concepts and the nature of thermodynamics calculations. An added advantage of this approach is that it provides a coherent picture of the thermodynamic potentials. © 2004 American

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## I. INTRODUCTION

Thermodynamics uses simple mathematics, consisting of nothing more advanced than partial derivatives and differentials. Given this simple foundation, it is remarkable how many accomplished students find its mathematics bewildering.

I discuss in Sec. II several calculus subtleties that contribute to this bewilderment. In Sec. III, I show how instructors can address the problem by briefly demonstrating the subtleties that are important for thermodynamics in the symbolic world in which students are used to working rather than introducing the subtleties in an environment where students must simultaneously learn new physics concepts (such as entropy and thermodynamic potentials). In addition to illustrating the essential calculus concepts, this approach can assist students to construct a coherent mathematical picture of thermodynamics in general and of thermodynamic potentials in particular.

My approach functions as a short mathematical introduction akin to the way vectors are introduced and reviewed in as many as four courses during an undergraduate student's career. However, in contrast to the introduction and review of vectors, little, if any, class time is lost to physics with this investment. It is brief, but more importantly, a simple change of symbols yields the various thermodynamic potentials and Maxwell relations.

## II. SOME CHALLENGES POSED BY THERMODYNAMICS

### A. Calculus subtleties

Thermodynamics involves the use of partial derivatives and differentials which is complicated by the following considerations.

(1) Independent variables (in the formal mathematical sense) are physically (and therefore mathematically) related through an equation of state. In mechanics and electricity and magnetism, the Cartesian variables  $x$ ,  $y$ , and  $z$ , for example, are physically as well as mathematically independent; we can choose any  $x$ ,  $y$ , and  $z$  triplet. In contrast, for an ideal gas, a change in the volume  $V$  will, in general, produce a change in the pressure  $P$ .

(2) The proper independent variables are not always obvious (particularly to students). Furthermore, the independent variables change from application to application. In contrast, in mechanics and electricity and magnetism, the spatial variables and time are almost always the independent variables. We would not in normal circumstances consider, for example, writing  $x$  as a function of the electric field  $\mathbf{E}$ .

To deal with these considerations, physicists specify the independent variables by subscripts when they write partial derivatives in thermodynamics. For example, the expression,<sup>1</sup>

$$\left(\frac{\partial E}{\partial S}\right)_V, \quad (1)$$

tells us that the independent variables are  $V$  and  $S$ , and that  $V$  is held constant. This information could be conveyed equally well by writing  $\partial E(S, V)/\partial S$ .<sup>2</sup>

Note how strange the standard thermodynamics notation appears to a student who has previously had calculus, mechanics, and electricity and magnetism. To use the notation  $(\partial f/\partial x)_y$  in calculus or  $E_x = (\partial V/\partial x)_{y,z}$  in electricity and magnetism would be redundant and nonsensical. The student would justifiably be puzzled if a text introduced this notation. Yet many thermal physics texts introduce this subscript notation with nothing but a short sentence which repeats the typical definition of a partial derivative found in calculus textbooks where subscripts were unnecessary (see, for example, Ref. 3). Students are left to discover on their own why physicists, who value notational parsimony, have added apparently unneeded symbols.

While I am concerned primarily with equipping students to use and understand standard thermodynamics notation, it is useful for physics professors to be aware that the meaning of partial derivatives' subscripts in thermodynamics may differ from the meaning of such subscripts (on occasions when they do appear) in calculus classes. It is increasingly common in mathematics and in calculus textbooks to use the subscript to specify the value(s) of independent variables at which the derivative is evaluated.<sup>4</sup> For example,  $(\partial f_1/\partial x)_{x_0, y_0}$  implies that the derivative is taken at  $x = x_0$  and  $y = y_0$ .

## B. Revealing the coherence and beauty of thermodynamics

In addition to cultivating an appreciation for and an ability to apply the necessary mathematical subtleties, the thermodynamics professor faces the challenge of unveiling the beauty and coherence of thermodynamics.

One element of this challenge involves the thermodynamic potentials. They sometimes seem to appear in a textbook as if they have fallen from the sky as random and unrelated drops of rain.<sup>5</sup> For example, Kittel and Kroemer's textbook introduces the Helmholtz free energy  $F$  abruptly, saying that it "plays the part in thermal physics at constant temperature that the energy  $E$  plays in ordinary mechanical processes ..."<sup>6</sup> How many students can articulate the role energy plays? A similar sentence introduces enthalpy,<sup>7</sup> and a briefer comment introduces the Gibbs free energy.<sup>8</sup>

Schroeder's textbook improves on Ref. 3 by introducing  $F$ ,  $G$ , and  $H$  at the same time. (Kittel introduces  $G$  194 pages after discussing  $F$ , and introduces  $H$  in the chapter succeeding the introduction of  $G$ .) His textbook also gives a physical interpretation for each variable. For example, for  $F$  he writes that "This is the total energy needed to create the system, minus the heat you can get for free from an environment at temperature  $T$  ... . Thus  $F$  is the energy that comes out as work if you're creating the system out of nothing. Or if you annihilate the system, the energy that comes out as work is  $F$ , because you have to dump some heat, equal to  $TS$ , into the environment in order to get rid of the system's entropy. The *available*, or "free," energy is  $F$ " (italics in original).<sup>9</sup> By introducing  $F$ ,  $G$ , and  $H$  together, Schroeder shows that the thermodynamic potentials have similar characteristics. Nevertheless, I think that this introduction can be further improved.

Although a physical explanation of  $F$  may seem useful to students, it is arguably more important to help students place  $F$  within a cognitive framework and cultivate the style of thought that will allow students to more effectively use  $F$  and develop a coherent picture of thermodynamics. Furthermore, the physical explanation Schroeder provides seems to invite the questions, "Why should I be interested in the total energy needed to create the system, minus the heat you can get for free from an environment at temperature  $T$ ?" and "How does the total energy needed to create the system, minus the heat you can get for free from an environment at temperature  $T$  relate to the total energy (the energy plus the work needed) to create a system out of nothing in an environment with constant pressure  $P$ ?" (Schroeder's definition of enthalpy).<sup>10</sup> Do the definitions help students understand why physicists invented the Helmholtz free energy and enthalpy? Are they related to how the variables are used? In my opinion, the answer to both questions is no. The quest to understand the thermodynamic potentials inevitably leads us to mathematics.

The thermodynamic potentials are related mathematically by Legendre transforms. They are useful because our knowledge of mathematics allows us to calculate useful physical quantities such as the work done on the system at constant temperature (equal to  $\Delta F$ ), the energy added by heating in isobaric processes ( $\Delta H$ ), and the wealth of information obtained from the four Maxwell relations. Because mathematics unifies the thermodynamic potentials and a knowledge of the appropriate mathematics makes them useful, explicitly

and clearly introducing this simple but elegant mathematics should help students appreciate the beauty and coherence of thermodynamics.

## III. ILLUSTRATING THERMODYNAMICS' METHODS WITHIN STUDENTS' SYMBOLIC ENVIRONMENT

Students typically have worked many problems with functions of two independent variables [for example,  $f(x,y)$ ], including problems involving partial derivatives. They also are likely to be familiar with differentials involving two variables such as

$$df_1 = g dx + h dy, \quad (2)$$

which is formally identical to the fundamental thermodynamic identity  $dE = TdS - PdV$ .<sup>11</sup> Most students would recognize that the independent variables of  $f_1$  are  $x$  and  $y$ , that

$$g = \frac{\partial f_1}{\partial x}, \quad (3a)$$

$$h = \frac{\partial f_1}{\partial y}, \quad (3b)$$

and would easily grasp the Maxwell relation  $\partial g / \partial y = \partial h / \partial x$ .

In the cases that concern us,  $g$  and  $h$  are not constants, but are functions of  $x$  and  $y$ . Unfortunately, students will likely not have encountered this case in the usual problems in calculus textbooks or elsewhere (one welcome exception is Ref. 12). Consequently, they are unlikely to recognize that

- (1) holding "all other independent variables" constant when taking a partial derivative is not equivalent to holding "all other variables" constant. Students are unlikely to have calculated a partial derivative in problems where "all other independent" and "all other" variables are not synonymous;<sup>13</sup>
- (2) the value of  $f_1$  can be determined by specifying the values of  $g$  and  $h$  rather than  $x$  and  $y$  (for example, as when the values of  $P$  and  $T$  can determine  $E$  rather than  $S$  and  $V$ );
- (3) three related functions can be created that have different independent variables (as when  $F$ ,  $H$ , and  $G$  are created from  $E$  and the fundamental thermodynamic identity).

### A. Legendre transforms

Students must grasp these concepts to successfully perform thermodynamic calculations. A simple way to highlight the concepts and to give students a framework in which to think through thermodynamics calculations in subsequent work is to introduce Legendre transforms of  $f(x,y)$ .<sup>18</sup> I have found that it is useful to show the abstract  $f(x,y)$  formalism followed by a simple thermodynamics calculation. The first illustrates the mathematical subtleties in a context where they can be recognized. The second shows why this procedure is useful.

To be concrete, consider the function

$$f_2 = f_1 - gx, \quad (4)$$

where Eq. (2) defines  $df_1$  gives

$$df_2 = df_1 - g dx - x dg \quad (5a)$$

$$= -x dg + h dy. \quad (5b)$$

Students would naturally understand from Eq. (4) that  $f_2$  is a function of  $x$  and  $y$  because it is composed of the product  $gx$  added to a function of  $x$  and  $y$ . In fact, any two variables can be used as independent variables for the quantity represented by  $f_2$ , but based on Eq. (5b), we conveniently think of  $f_2$  as a function of the independent variables  $g$  and  $y$ . Furthermore, in standard thermodynamic fashion, Eq. (5b) implies that

$$x = -\frac{\partial f_2}{\partial g}, \quad (6a)$$

$$h = \frac{\partial f_2}{\partial y}, \quad (6b)$$

$$\frac{\partial x}{\partial y} = -\frac{\partial h}{\partial g}. \quad (6c)$$

These simple steps illustrate the existence of a new related function with different independent variables (and provide the opportunity to review differentials and independent variables). More importantly, we have the opportunity to illustrate why we choose to subscript our partial derivatives in thermodynamics, and the importance of being aware of independent variables in differentiation. In particular, by using Eqs. (2) and (5b) students can now express  $h$  in terms of a seemingly identical partial derivative of two different functions:

$$h = \frac{\partial f_2}{\partial y} = \frac{\partial f_1}{\partial y}. \quad (7)$$

Equation (7) is precisely the situation that forces the subscripts (or some other notation) on us. A naive response [one that many students would probably make, despite Eq. (4)] would be to think incorrectly that the functions  $f_1$  and  $f_2$  are identical. At this point, the instructor can lead the student to recognize from Eqs. (2) and (5b) that what is different are the two functions' independent variables (the situation is nonsensical if they do not recognize it), or equivalently, what is held constant during differentiation. Thus, this situation illustrates why thermodynamics-style notation, which specifies what is held constant during differentiation, is useful:

$$h = \left(\frac{\partial f_2}{\partial y}\right)_g = \left(\frac{\partial f_1}{\partial y}\right)_x. \quad (8)$$

At this point, students have observed in a simple context how to obtain the equivalent of the Helmholtz free energy, that a partial derivative no longer means holding "all" other variables constant, that at least one related function can be created from a differential of two independent variables, and why thermodynamics practitioners choose to subscript their partial derivatives. What remains is to show the equivalents of the other thermodynamic potentials, how they are related, and why this mathematical formalism is useful.

It is simple to introduce (or have students determine for those who prefer active approaches) the other Legendre transforms and their differentials:

$$f_3 = f_1 - hy, \quad (9)$$

$$f_4 = f_2 - hy = f_1 - gx - hy, \quad (10)$$

$$df_3 = gdx - ydh, \quad (11)$$

$$df_4 = -xdg - ydh. \quad (12)$$

The functions  $f_3$  and  $f_4$  are conveniently thought of as functions of  $x$  and  $h$  and  $g$  and  $h$ , respectively. With some guided reflection (or experimentation), students will find that it is not possible to obtain any other similar functions.

We now have four related functions that have four different sets of independent variables. Each of the variables  $g$ ,  $x$ ,  $h$ , and  $y$  can be determined by differentiation from two of the four functions, and there are four Maxwell relations relating these four variables,

$$\left(\frac{\partial g}{\partial y}\right)_x = \left(\frac{\partial h}{\partial x}\right)_y, \quad (13a)$$

$$\left(\frac{\partial x}{\partial y}\right)_g = -\left(\frac{\partial h}{\partial g}\right)_y, \quad (13b)$$

$$\left(\frac{\partial g}{\partial h}\right)_x = -\left(\frac{\partial y}{\partial x}\right)_h, \quad (13c)$$

$$\left(\frac{\partial x}{\partial h}\right)_g = \left(\frac{\partial y}{\partial g}\right)_h. \quad (13d)$$

An ancillary benefit of teaching physics is the opportunity to appreciate and reveal elegant mathematics. The development we have just completed is an example. Having done nothing difficult, most students find it quite surprising that the mere structure of the differential with two independent variables [Eq. (2)] implies a host of useful mathematical relations. It includes the four constraints given by the Maxwell relations between the variables  $g$ ,  $h$ ,  $x$ , and  $y$ , and 8 ways (2 each) to determine these variables [e.g.,  $h = (\partial f_2 / \partial y)_g$ ]. These simple mathematical implications are the basis for our ability to make thermodynamic calculations. Understanding and appreciating these relations (as students should now be able to do) should assist them as they shift from  $x$ ,  $y$ , and  $f$  to quantities such as the entropy and the Helmholtz and Gibbs free energies.

## B. Reconnecting to thermodynamics

Because Eq. (2) is equivalent to the fundamental thermodynamic identity, a change of variables gives the basic thermodynamic equations and Maxwell relations:  $x \rightarrow S$ ,  $g \rightarrow T$ ,  $y \rightarrow V$ ,  $h \rightarrow -P$ ,  $f_1 \rightarrow E$ ,  $f_2 \rightarrow F$ ,  $f_3 \rightarrow H$ , and  $f_4 \rightarrow G$ . Importantly,  $E$ ,  $F$ ,  $G$ , and  $H$  and the associated differential relations now have a simple transparent mathematical framework that unifies them, providing a context in which students can fit the thermodynamic potentials as they are reintroduced and used during the semester.

In a typical thermal physics course students encounter many examples of why this exercise is useful, particularly if the instructor reconnects to and reinforces the mathematical foundation just described. However the introduction to thermodynamic thinking and the mathematics of thermodynamics I have outlined will likely be most effective if several brief examples are provided. Some concepts worth emphasizing include the freedom to choose and to use convenient independent variables, the Maxwell relations, and the usefulness of the thermodynamic potentials in different situations.

Calculating the relation between  $T$  and  $V$  during an adiabatic expansion of an ideal gas can illustrate the first two concepts. In brief, we take advantage of constant entropy and choose  $V$  and  $T$  as independent variables,

$$dS=0=\left(\frac{\partial S}{\partial T}\right)_V dT+\left(\frac{\partial S}{\partial V}\right)_T dV, \quad (14)$$

use the Maxwell relation from the Helmholtz free energy differential

$$\left(\frac{\partial S}{\partial V}\right)_T=\left(\frac{\partial P}{\partial T}\right)_V=\frac{Nk}{V}, \quad (15)$$

and the definition of  $C_V$  to obtain

$$\frac{1}{T}C_VdT+\frac{Nk}{V}dV=0. \quad (16)$$

Equation (16) yields the familiar constant  $TV^{\gamma-1}$  (where  $\gamma=C_P/C_V$ ) for adiabatic expansion.

Students can obtain an initial glimpse into the thermodynamic potentials' usefulness by leading them to recognize that each has two independent variables. Thus for any process in which one variable is held constant, there will be two thermodynamic potentials that depend on only one variable, a useful property that can be exploited. For example, in a quasistatic isothermal process, the change in the Helmholtz free energy equals the work done on the system because  $dF=-SdT-Pdv=-PdV$ . In a quasistatic isobaric process the change in enthalpy equals the energy added by heating ( $dH=TdS+VdP=TdS$ ). A knowledge of  $\Delta F$  and  $\Delta H$  thus gives access to important physical quantities.

#### IV. A RELATED CONCEPTUAL PROBLEM

Because of the freedom to choose independent variables, the distinction between a physical variable and the function that represents it also can be a conceptual problem for students.<sup>19</sup> For example, it is sometimes emphasized that the natural independent variables of the energy  $E$  are  $S$  and  $V$ , meaning that the function that relates changes in  $E$  (the fundamental thermodynamic identity) has independent variables  $S$  and  $V$ . This language can be a problem if students are not helped to separate the concept of a physical quantity (in this case energy), which can be determined in multiple ways, from the mathematical function that describes it. Soon after learning that  $E$ 's independent variables are  $S$  and  $V$ , a student will see an equation showing that for an ideal monatomic gas,  $E$  is a function of  $T$  alone:  $E(T)=3/2NkT$ . Obviously, in this equation  $E$  is not formally a function of  $S$  and  $V$ . Although simple to well-trained and experienced professors, this distinction can be troubling to students. Perhaps a more precise and therefore preferable choice of words is to say that the independent variables of the fundamental thermodynamic identity that describes changes in  $E$  are  $S$  and  $V$ .

#### V. SUMMARY

I have illustrated how the subtleties of calculus important to thermodynamics can be introduced by performing Legendre transforms. I argue that doing so builds a more solid and more coherent understanding of the mathematics that is essential to thermodynamics calculations.<sup>20</sup>

#### ACKNOWLEDGMENTS

I gladly acknowledge the influence of Professor A. C. Anderson of the University of Illinois in Urbana-Champaign who had the wisdom to introduce Legendre transforms and the four thermodynamic potentials on the first day of his thermal physics class. The modified form I suggest here comes from reflection over why some students (including me) could still take considerable time to appreciate what "those little subscripts" were all about. I also thank Professor David Van Baak of Calvin College and Michael E. McCracken of Washington and Jefferson College for their helpful comments on this manuscript. This work was supported in part from Washington and Jefferson College's Undergraduate Science Education Grant No. 52002683 from the Howard Hughes Medical Institute.

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<sup>1</sup>Here  $E$  is the internal energy,  $F$  the Helmholtz free energy,  $H$  the enthalpy, and  $G$  the Gibbs free energy.

<sup>2</sup>Craig F. Bohren argues that this notation should be adopted in thermodynamics. See Craig F. Bohren and Bruce A. Albrecht, *Atmospheric Thermodynamics* (Oxford U. P., New York, 1998).

<sup>3</sup>Charles Kittel and Herbert Kroemer, *Thermal Physics* (Freeman, San Francisco, 1980), 2nd ed., p. 40. The authors write that "The notation  $(\partial g_1/\partial U_1)_{N_1}$  means that  $N_1$  is held constant in the differentiation of  $g_1(N_1, U_1)$  with respect to  $U_1$ . That is, the partial derivative with respect to  $U_1$  is defined as  $\partial g_1/\partial U_1 = \lim_{\Delta U_1 \rightarrow 0} [g_1(N_1, U_1 + \Delta U_1) - g_1(N_1, U_1)]/\Delta U_1$ . For example, if  $g(x, y) = 3x^4y$ , then  $(\partial g/\partial x)_y = 12x^3y$  and  $(\partial g/\partial y)_x = 3x^4$ ." Note that the description the authors give is identical to the partial derivative definition found in calculus books where this subscript is unnecessary. See, for example, James Stewart, *Calculus* (Brooks/Cole, Pacific Grove, CA, 1999), 4th ed., p. 931.

<sup>4</sup>George B. Thomas and Ross L. Finney, *Calculus and Analytical Geometry* (Addison-Wesley, Reading, MA, 1992), 8th ed., p. 834; Dale Varberg and Edwin Purcell, *Calculus with Analytical Geometry* (Prentice-Hall, Englewood Cliffs, NJ, 1992), 6th ed., p. 684. I thank Professor Dean Morrow of the Washington and Jefferson College Mathematics Department for alerting me to how mathematics texts use subscripts.

<sup>5</sup>Because they were introduced by different people at different times, they did in some sense fall "from the sky as random and unrelated drops of rain." However, there is no need for the students to relive this history when the concepts can be introduced in a unified manner.

<sup>6</sup>Reference 3, p. 68.

<sup>7</sup>Reference 3, p. 246.

<sup>8</sup>Reference 3, p. 262.

<sup>9</sup>Daniel V. Schroeder, *An Introduction to Thermal Physics* (Addison-Wesley-Longman, Reading, MA, 2000), p. 150. Note that this definition assumes a quasistatic process.

<sup>10</sup>Reference 9, p. 149.

<sup>11</sup>Reference 3, p. 68. I have replaced Kittel's  $\tau$ ,  $\sigma$ , and  $U$  with  $T$ ,  $S$ , and  $E$ . The title for the equation varies. Reif calls it the "fundamental thermodynamic relation." See F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965), p. 161. Schroeder calls it the "thermodynamic identity," see Ref. 9, p. 111.

<sup>12</sup>Reference 4, Thomas and Finney, p. 862, features a section on "Partial Derivatives with Constrained Variables."

<sup>13</sup>In a survey of commonly used calculus textbooks (Refs. 14–17). I found no problems involving partial derivatives where "all other independent" and "all other" variables were not equivalent.

<sup>14</sup>James Stewart, *Calculus* (Brooks/Cole, Pacific Grove, CA, 1999), 4th ed., p. 840.

<sup>15</sup>Dale Varberg and Edwin Purcell, *Calculus with Analytical Geometry* (Prentice-Hall, Englewood Cliffs, NJ, 1992), 6th ed., p. 687.

<sup>16</sup>Earl W. Swokowski, *Calculus with Analytical Geometry* (PWS Kent, Boston, 1983), 2nd alternate ed., pp. 679–680.

<sup>17</sup>Reference 4, Thomas and Finney, p. 840.

<sup>18</sup>Ashley Carter and Ralph Baierlein explicitly introduce and discuss the properties of Legendre transforms in their texts (all texts implicitly introduce Legendre transforms, when they define  $F$ ,  $G$ , and  $H$ ). Similar to what I propose here Carter illustrates the Legendre transform's use with abstract variables (rather than  $E$ ,  $S$ , and  $F$ , etc.). What I suggest in this paper is that these introductions can be better targeted to deal with misunderstandings that have not been previously dealt with, and to confront the

student with the reason for and the meaning of thermodynamic notation. Ashley H. Carter, *Classical and Statistical Thermodynamics* (Prentice-Hall, Upper Saddle River, NJ, 2001), p. 130. Ralph Baierlein, *Thermal Physics* (Cambridge U. P., Cambridge, 1999), p. 225.

<sup>19</sup>I have heard Professor Robert J. Hardy of the University of Nebraska express similar concerns.

<sup>20</sup>A copy of a class handout that implements this procedure can be obtained by writing to the author.

#### A FLASH AT FERMILAB

Alongside the great machine of physics, the physicists move in daily routines like priests repeating rituals of transubstantiation. If physics may be pictured as a single corpus of belief—as one work with no author and thousands of successive editors—then the bright flash at the accelerator target is perhaps the best symbol, the summary in a single action, of the whole achievement of physics.

Philip J. Hiltz, *Scientific Temperaments* (Simon and Schuster, Inc., 1982). Reprinted in *The World Treasury of Physics, Astronomy, and Mathematics* (Little, Brown and Company, Boston, MA, 1991), p. 700.

## LETTERS TO THE EDITOR

Letters are selected for their expected interest for our readers. Some letters are sent to reviewers for advice; some are accepted or declined by the editor without review. Letters must be brief and may be edited, subject to the author's approval of significant changes. Although some comments on published articles and notes may be appropriate as letters, most such comments are reviewed according to a special procedure and appear, if accepted, in the Notes and Discussions section. (See the "Statement of Editorial Policy" in the January issue.) Running controversies among letter writers will not be published.

### COMMENT ON "CONNECTING THERMODYNAMICS TO STUDENTS' CALCULUS," BY JOEL W. CANNON [AM. J. PHYS. 72 (6), 753–757 (2004)]

In a recent paper ["Connecting thermodynamics to students' calculus," Am. J. Phys. 72 (6), 753–757 (2004)], Joel W. Cannon makes some very good points about the usefulness of introducing Legendre transforms to students in order to clarify the difference between the functions and the independent variables in a system when calculating partial derivatives. I offer here an example with which students are usually very familiar, that is, the electrical power,  $P$ , of a resistor with resistance  $R$  when the potential difference across the terminals is  $V$  and the current is  $I$ . Recall that  $P = IV = I^2R = V^2/R$ . Depending on the variables held constant, that is, depending on which variables are independent, partial derivatives yield different results:  $(\partial P/\partial I)_V = V$ , whereas  $(\partial P/\partial I)_R = 2IR = 2V$ . Similarly,  $(\partial P/\partial R)_I = I^2$ , while  $(\partial P/\partial R)_V = -V^2$  has the opposite sign. Derivatives provide the answer to questions such as "Does the power go up or down, if the resistance is increased?" These results show that the answer is not unique if all conditions are not explicitly stated (for example, constant current or constant voltage?), or, in other words, if it is not clarified which are the independent variables.

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### PARALLEL UNIVERSES IN THE STATISTICS LITERATURE

York *et al.* recently considered straight-line regression when random errors are

present in both the dependent and independent variables;<sup>1</sup> moreover, these errors may be correlated with each other. They compared the least-squares and maximum likelihood approaches for estimating the regression parameters and their standard errors, and found that the two approaches are equivalent when the least-squares approach uses properly adjusted data.

In the mathematical statistics community there is a great deal of literature on regression with errors in both the dependent and independent variables, including generalizations such as nonlinear curve fits, multiple predictor variables, and non-normal errors. In statistics, models of this type are called *measurement error models* or *errors-in-variables models*. For linear models of this type, the definitive monograph is by Fuller;<sup>2</sup> a more recent (and perhaps more accessible) monograph is by Cheng and Van Ness.<sup>3</sup> Nonlinear models are discussed by Carroll *et al.*<sup>4</sup> This literature seems to inhabit a "parallel universe" in that Ref. 1 and the papers they cite make no reference to these books, and *vice versa*. However, the two parallel sets of literature cite common origins, including the work of Adcock in the 1870s and that of Deming, a physicist turned statistician who wrote about the problem in the 1930s and 40s.

It might appear that the two parallel universes have diverged in terms of the problems they address, but there is a strong possibility that at least subtle connections exist between them. Such connections are probably obscured by what looks (at first glance) to be an excessively baroque theoretical apparatus that has emerged in the mathematical statistics literature. This baroque-ness is due to the statisticians' goal of systematically addressing a vast array of general data analysis problems far beyond those that typically arise in the physical sciences. It does not help that Refs. 2–4 are addressed to professional statisticians rather than experimental scientists.

One can find comparisons of least-squares and maximum likelihood techniques in the mathematical statistics literature. For instance, in discussing a methodology called *modified least squares*, Ref. 3 states that "The beauty of modified least squares is that it is a unified approach; however, it does not introduce any new estimators that were not already available from maximum likelihood or the method of moments" (p. 89). The modified least squares approach is unified in that it can be applied to several different statistical models using various assumptions for the nature of the errors. Unfortunately, as a nonexpert I have been unable to determine the relation between this approach and that of Ref. 1. I surmise that, with effort, the specific problem of Ref. 1 could be formulated in the framework of mathematical statistics.

The authors of Ref. 1 deserve much praise for providing a direct, succinct solution to their problem which has not appeared elsewhere in its current form. Perhaps physicists with similar data analysis problems also will find the literature in mathematical statistics worthy of further exploration.

<sup>1</sup>D. York, N. M. Evensen, M. López Martínez, and J. De Basabe Delgado, "Unified equations for the slope, intercept, and standard errors of the best straight line," Am. J. Phys. 72, 367–375 (2004).

<sup>2</sup>W. A. Fuller, *Measurement Error Models* (Wiley, New York, 1987).

<sup>3</sup>C.-L. Cheng and J. W. Van Ness, *Statistical Regression with Measurement Error* (Arnold, London, 1999).

<sup>4</sup>R. J. Carroll, D. Ruppert, and L. A. Stefanski, *Measurement Error in Nonlinear Models* (Chapman & Hall, London, 1995).

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