CONTENTS

10

The Speed of Light

c AS A FUNDAMENTAL CONSTANT OF NATURE

The speed of light[1] in vacuum c is one of the fundamental constants of physics. Some of its characteristics are

1 It is the speed at which all electromagnetic radiation travels in free space, independent of the frequency of the radiation.

2 No signal can be transmitted by any means whatsoever, in free space or in a material medium, at a speed faster than the speed of light c.

3 The speed of light in free space is independent of the reference frame from which it is observed. If the speed of a light signal is observed to be $c = 2.99793 \times 10^{10}$ cm/s in one galilean frame, it will be observed to be c and not $c + V$ (or $c - V$) in a second galilean frame moving parallel to the signal with a speed V with respect to the first frame.

4 Maxwell's equations in electromagnetic theory and the Lorentz force equation involve the speed of light. This is particularly apparent when they are written in gaussian units.

5 The dimensionless constant (which is called the reciprocal of the fine-structure constant)

$$\frac{\hbar c}{e^2} \approx 137.04$$

involves the speed of light. Here $2\pi\hbar$ is Planck's constant and e is the charge on the proton. This constant plays an important role in atomic physics and will be discussed in Volume 4. We do not have a theory that predicts the value of this constant.

This chapter is concerned chiefly with experiments and experimental results. We discuss the measurement of the speed of light and experimental evidence for the invariance of the speed of light with respect to the velocity of any inertial frame. We leave for Volume 3 questions about the electromagnetic nature of light and the propagation of light in refractive and dispersive media such as solids and liquids. (A refractive medium is one in which the refractive index, the ratio of the speed

[1] Note that the phrase *speed of light* should always be understood to mean the speed of light in free space (c), unless it is explicitly stated otherwise. Thus the speed of light in a material medium is less than c and may even be less than the speed of a charged particle in the same medium (Cerenkov effect).

of light in vacuum to that in the medium, is not exactly unity. A dispersive medium is one in which the refractive index is a function of the frequency.)

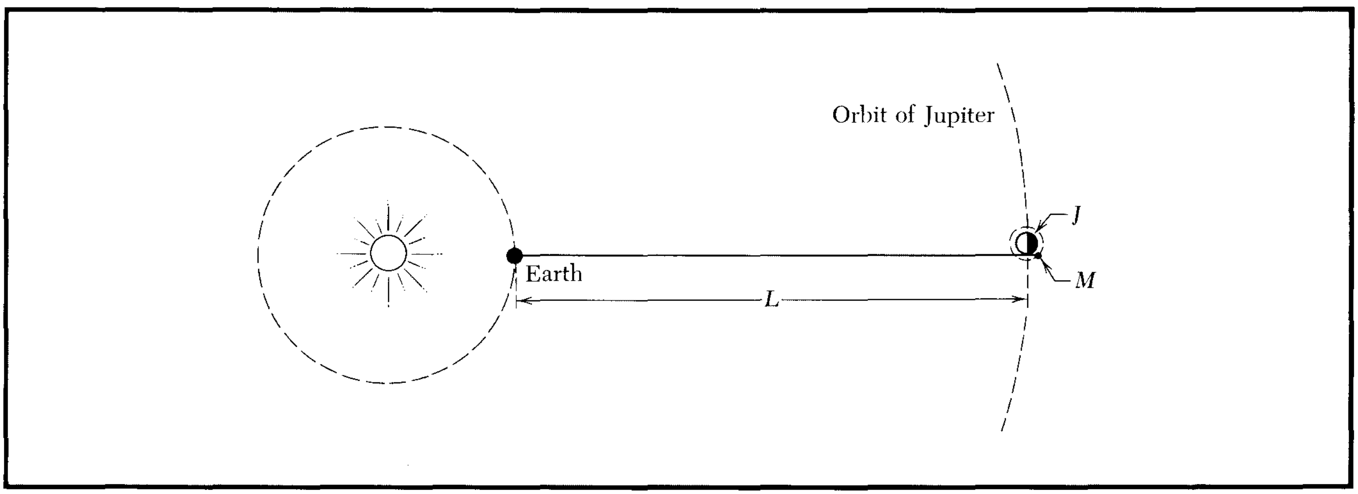
MEASUREMENTS OF c

Many methods have been employed to determine the speed of light.[1] We list and sketch several of the methods here.

Transit Time of Light across the Orbit of the Earth For some centuries before there was experimental proof, it was believed that the speed of light must be finite. The first experimental evidence of the finite speed of light was due to Roemer in 1676. He observed that the motion of Io, the innermost moon of Jupiter, did not follow an entirely regular timetable. There was a slight variation in the periods of the eclipses of Io by Jupiter. When at one time of year (see Fig. 10.1) he predicted the time of eclipse 6 months later (see Fig. 10.2), he was about 22 min in error. He postulated that this was the transit time of light across the orbit of the earth. His best estimate of the average diameter D of the earth's orbit about the sun was 2.83×10^{13} cm, and so he calculated for c

[1] An excellent review in English of measurements of the speed of light is given by E. Bergstrand in "Handbuch der Physik," S. Flügge (ed.), vol. 24, pp. 1–43 (Springer-Verlag OHG, Berlin, 1956). The values of c we quote are those listed by Bergstrand. See also J. F. Mulligan and D. F. McDonald, *Am. J. Phys.*, 25:180 (1957).

FIG. 10.1 Eclipse of Jupiter's moon M occurs when M disappears behind J as viewed from the earth. The actual time of observation on the earth is L/c later because of the finite speed of light. The period of M is about 42 h.



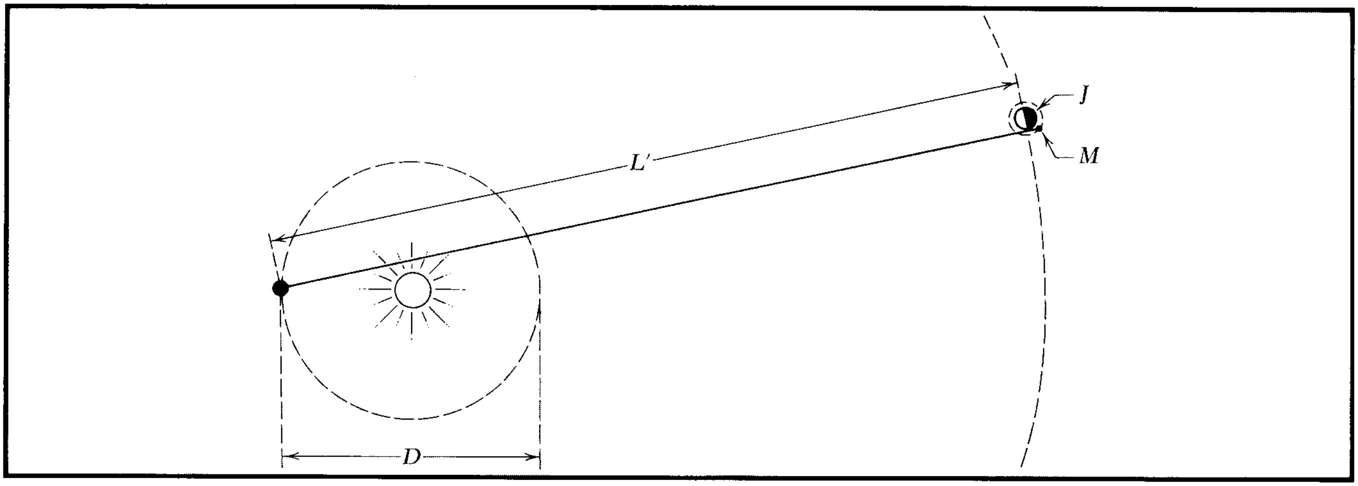


FIG. 10.2 Six months later the earth has completed a semicircle but Jupiter has moved only about 15°. The eclipse is now observed L'/c later where $L' \approx L + D$.

$$c = \frac{2.83 \times 10^{13}}{22 \times 60} = 2.14 \times 10^{10} \text{ cm/s}$$

For the time at which he made the estimate, this value is in good agreement with 3.0×10^{10} cm/s. The angular motion of Jupiter about the sun is slower (12 yr vs 1 yr) than that of the earth; thus it is the diameter of the earth's orbit and not Jupiter's orbit which is chiefly involved in the calculation. The Roemer method is not very accurate, but it did show astronomers that in analyzing planetary observations to find the true motion of a planet or moon, it is necessary to make allowance for the propagation time of the light signal.

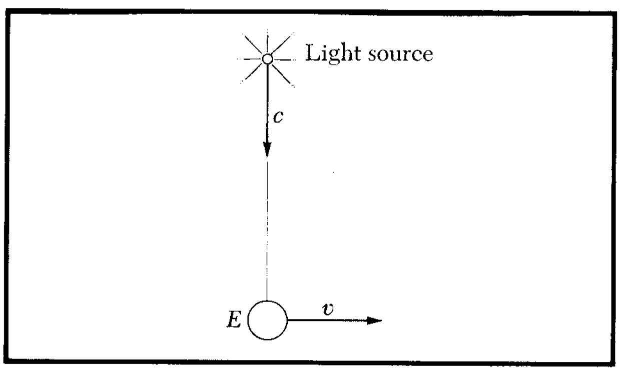


FIG. 10.3 Bradley in 1725 utilized the phenomenon of aberration to determine c. Suppose light from a distant source illuminates object E, which has velocity v normal to the incoming light.

Aberration of Starlight In 1725 James Bradley started an interesting series of precise observations of an apparent seasonal change in the position of stars, in particular of a star called γ Draconis. He observed that (after all other corrections had been applied) a star at the zenith (directly over the plane of the earth's orbit) appeared to move in a nearly circular orbit with a period of a year, with an angular diameter of about 40.5″. He also observed that stars in other positions had a somewhat similar motion—in general, elliptical.

The phenomenon Bradley observed is called *aberration*, and it is illustrated in Figs. 10.3 to 10.5. It has nothing to do with the true motion, if any, of the star; it arises from the finite speed of light and from the speed of the earth in its orbit about the sun. This was really the first direct experiment to suggest that the sun was a better inertial frame than the earth—i.e., that it is better to think of the earth as moving around the

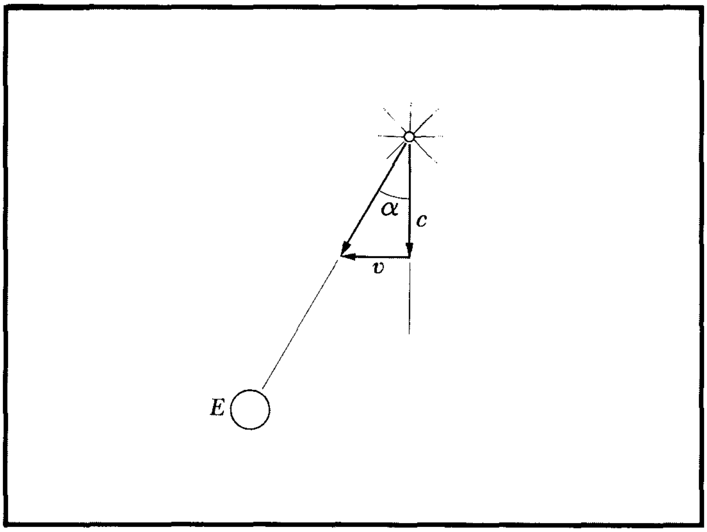


FIG. 10.4 According to an observer on E, the light has horizontal velocity component v as well as vertical component c. Thus the light ray from the source is inclined at angle α, where $\tan \alpha = v/c$.

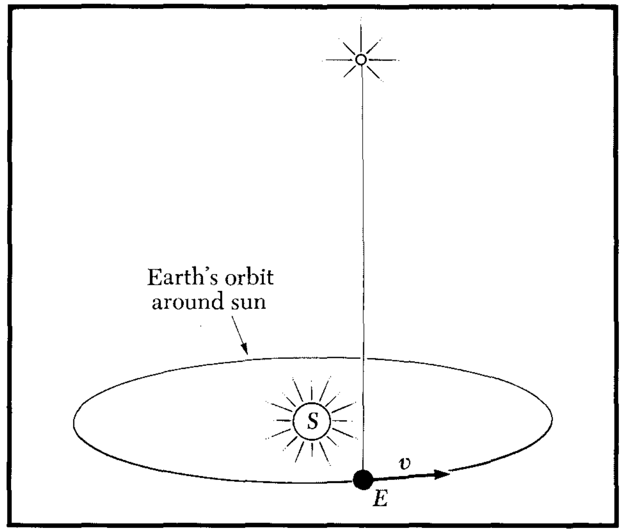


FIG. 10.5 Bradley used light from a distant star at zenith and the known velocity of the earth ($v_e = 30$ km/s) to determine c from measurements of α; $\tan \alpha = v_e/c$.

sun than of the sun as moving around the earth, for this experiment detects directly the annual change in the direction of the velocity of the earth relative to the stars.

The simplest explanation of aberration is the analogy of light propagation to the fall of raindrops (see Fig. 10.6). If no wind is blowing, raindrops fall vertically and a man at rest with an umbrella directly over his head does not get wet. If the man runs, holding the umbrella in the same position, the front of his coat will get wet. Relative to the moving person, the raindrops do not fall exactly vertically.

We quote from an account[1] of how the explanation of his observations came to Bradley:

> At last, when he despaired of being able to account for the phenomena which he had observed, a satisfactory explanation of it occurred to him all at once, when he was not in search of it.[2] He accompanied a pleasure party in a sail upon the river

[1] T. Thomson, "History of the Royal Society," p. 346, London, 1812.

[2] Many inventions and discoveries are made when, after an initial failure, the scientist has taken his thoughts away from the problem. A distinguished mathematician discusses this effect in a fascinating and important little book: J. Hadamard, "An Essay on the Psychology of Invention in the Mathematical Field," Princeton University Press, Princeton, N.J., 1945, reprint Dover Publications, Inc., New York, 1954.

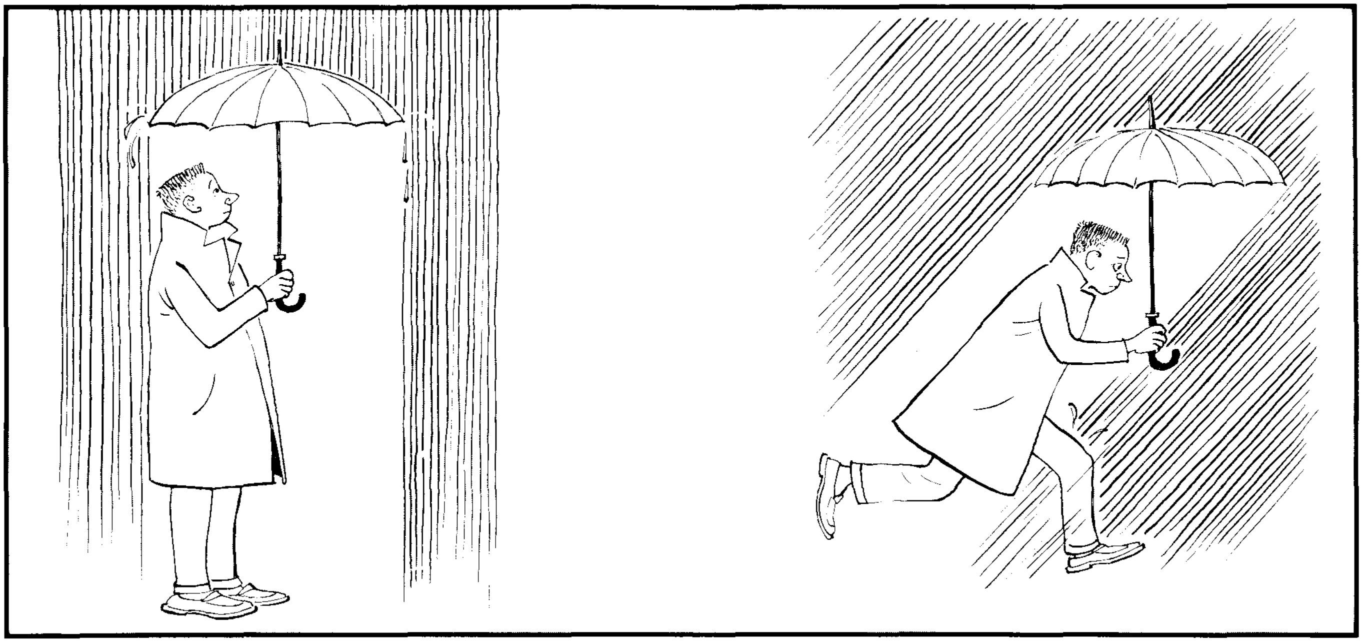


FIG. 10.6 A homely example of aberration: This student is caught in rain coming straight down. If he stands still under his umbrella, he keeps dry. But if he runs for it he gets wet. In his new reference frame the rain has horizontal velocity $-v$, where v is his velocity with respect to the ground.

Thames. The boat in which they were was provided with a mast, which had a [weather] vane at the top of it. It blew a moderate wind, and the party sailed up and down the river for a considerable time. Dr. Bradley remarked that, every time the boat put about, the vane at the top of the boat's mast shifted a little, as if there had been a slight change in the direction of the wind. He observed this three or four times without speaking; at last he mentioned it to the sailors, and expressed his surprise that the wind should shift so regularly every time they put about. The sailors told him that the wind had not shifted, but that the apparent change was owing to the change in the direction of the boat, and assured him that the same thing invariably happened in all cases. This accidental observation led him to conclude, that the phenomenon which had puzzled him so much was owing to the combined motion of light and of the earth.

This is the explanation of aberration in Bradley's words:[1]

I considered this matter in the following manner. I imagined CA [see Fig. 10.7] to be a ray of light, falling perpendicularly upon the line BD; then if the eye is at rest at A, the object must appear in the direction AC, whether light be propagated in time or in an instant. But if the eye is moving from B towards A, and light is propagated in time, with a velocity that is to

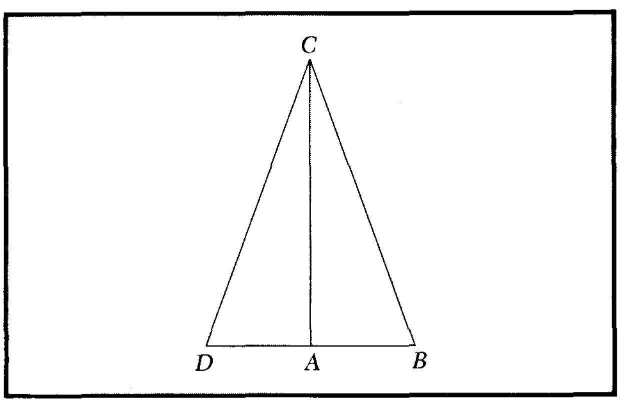


FIG. 10.7 Velocity diagram used by Bradley.

[1] J. Bradley, *Phil. Trans. Roy. Soc.*, London, **35**:637 (1728).

the velocity of the eye, as *CA* to *BA*; then light moving from *C* to *A*, whilst the eye moves from *B* to *A*, that particle of it by which the object will be discerned when the eye is in motion comes to *A*, is at *C* when the eye is at *B*. Joining the points *B*, *C*, I supposed the line *CB* to be a tube (inclined to the line *BD* in the angle *DBC*) of such a diameter as to admit of but one particle of light; then it was easy to conceive that the particle of light at *C* (by which the object must be seen when the eye, as it moves along, arrives at *A*) would pass through the tube *BC*, if it is inclined to *BD* in the angle *DBC*, and accompanies the eye in its motion from *B* to *A*; and that it could not come to the eye, placed behind such a tube, if it had any other inclination to the line *BD*.

For a star directly overhead the maximum aberration occurs when the earth's velocity is perpendicular to the line of observation. Then the tilt angle, or aberration, of the telescope is seen from Figs. 10.4 and 10.5 to be given by

$$\tan\alpha = \frac{v_e}{c} \tag{10.1}$$

where v_e is the speed of the earth. The orbital speed of the earth about the sun is 3.0×10^6 cm/s; the speed due to rotation about the earth's own axis, which is about 100 times slower, may be neglected here. The angle α in Eq. (10.1) will be half of Bradley's observed angular diameter of 40.5″. So by using $\alpha = 20''$ and solving for c in Eq. (10.1), we obtain (with $\tan\alpha \approx \alpha$)

$$c = \frac{v_e}{\alpha} = \frac{3 \times 10^6}{\frac{20}{3600} \times 1/57.3} = 3.1 \times 10^{10} \text{ cm/s}$$

This compares well with present values.

Toothed Wheels and Rotating Mirrors The first terrestrial determination of the speed of light was carried out by Fizeau in 1849. He found (see Fig. 10.8*a* to *c*)

$$c = (315{,}300 \pm 500) \text{ km/s}$$

for the speed of light in air.[1] He used a rotating toothed wheel as a light switch to determine the transit time of a light flash over a path length of 2×8633 m.

The toothed-wheel apparatus was soon replaced by a rotating-mirror device, which gives more light and better focusing. The arrangement used by Foucault in 1850 is shown

[1]The speed of light in vacuum is calculated to be about 91 km/s faster than in air.

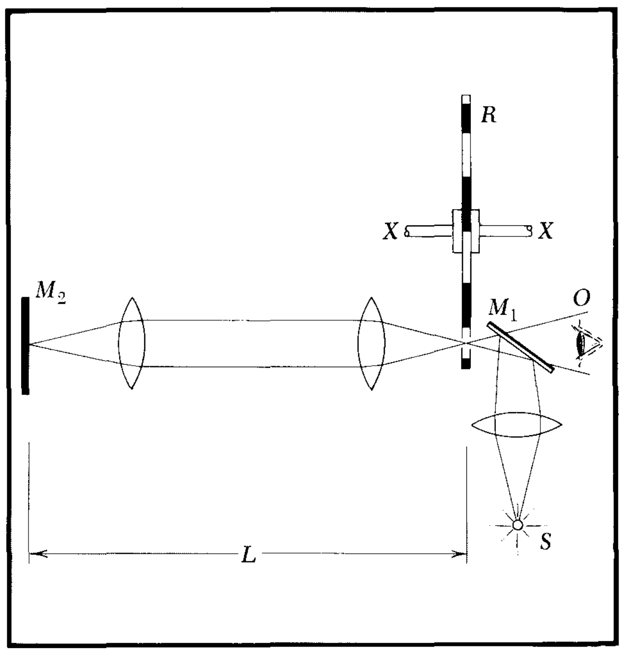


FIG. 10.8 (*a*) Fizeau's cogwheel apparatus, 1849. Light from a point source *S* is reflected from a half-silvered mirror M_1 past the cogwheel *R* rotating on axis *X-X*. Light then goes to mirror M_2 and returns to observer *O* through *R* and M_1. A half-silvered mirror reflects half the light incident and transmits half.

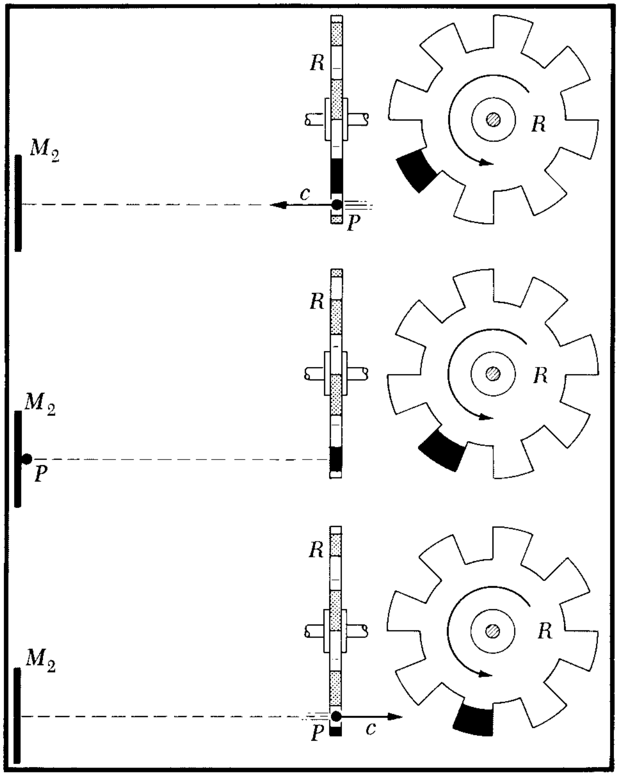


FIG. 10.8 *(cont'd)* (*b*) The pulse P with velocity c must travel to M_2 and return to R (total distance $2L$) in a time during which the cogs move over one space, if the pulse is to be transmitted to O. Fizeau determined c from L and the angular velocity of R.

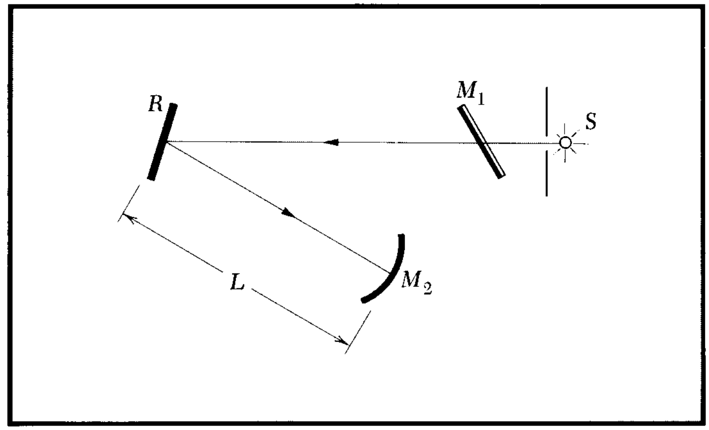


FIG. 10.9 (*a*) Foucault's rotating mirror apparatus, 1850, consisting of a source S, half-silvered mirror M_1, rotating mirror R (axis of rotation normal to the page), and spherical mirror M_2. Beam path from S to M_2 shown.

in Fig. 10.9a to c. His best value (1862) for the speed of light in air is

$$c = (298{,}000 \pm 500)\ \text{km/s}$$

A development of the rotating-mirror arrangement was used by Michelson (1927) over a path of 22 mi between Mt. Wilson and Mt. San Antonio in California. His arrangement has the light source at the focal point of a lens, giving parallel light over a long path. He found

$$c = (299{,}796 \pm 4)\ \text{km/s}$$

This work greatly exceeded in accuracy all previous work. (Further details are given in Prob. 3.)

Cavity Resonator It is possible to determine very accurately the frequency at which a resonant cavity of known dimensions (a metal box) contains a known number of half wavelengths of electromagnetic radiation. The speed of light is then calculated from the theoretical relation

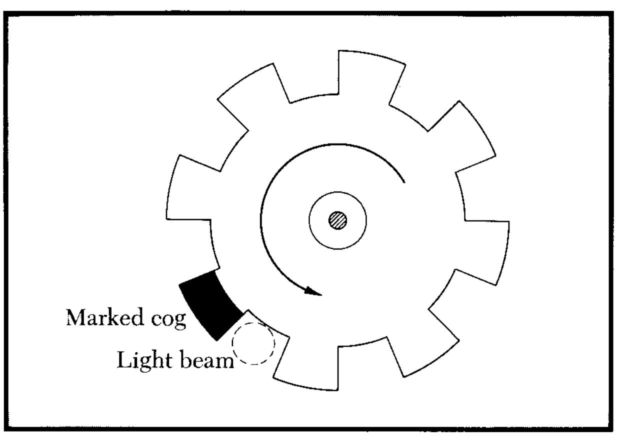


FIG. 10.8 *(cont'd)* (*c*) View of the light beam and cogwheel R seen by observer O. Rotation of R chops the light beam from S, M_1 into short pulses. (Light can only pass from M_1 to M_2 if no cog is in the way.)

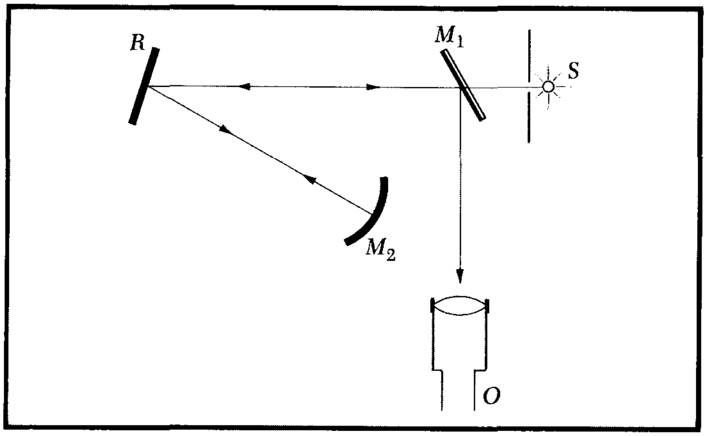


FIG. 10.9 (*cont'd*) (*b*) With R stationary, a light beam from M_1 to R to M_2 is reflected back along same path to M_1, and detected by O.

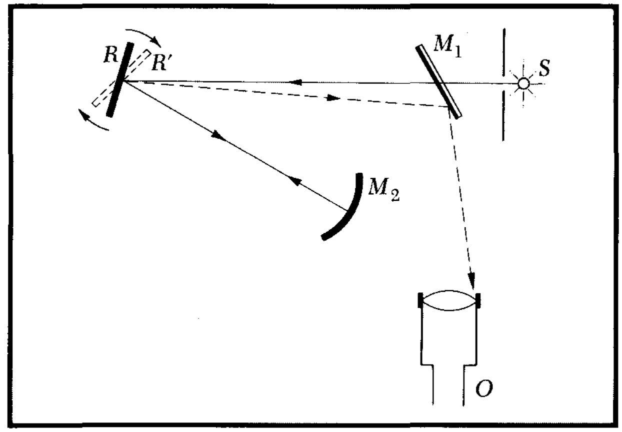


FIG. 10.9 (*cont'd*) (*c*) If mirror R rotates, light from S to R to M_2 returns when rotating mirror is in a new position R'. Thus O observes a displaced image on M_1. Foucault determined c from L, the image displacement, and the mirror angular velocity.

$$c = \lambda\nu \tag{10.2}$$

connecting the wavelength λ and the frequency ν. The cavity is usually evacuated. It is necessary to correct the inside dimensions of the cavity for the small penetration[1] of the electromagnetic field into the surface of the metal. Essen (1950) used frequencies of 5960, 9000, and 9500 Mc/s to find

$$c = (299{,}792.5 \pm 1)\ \text{km/s}$$

Kerr Cell When polarized light passes through a Kerr cell (a liquid in which an electric field can affect the transmission of polarized light) the intensity of light emerging and polarized in the initial direction can be modulated by varying the voltage between the plates producing the electric field. If the same frequency voltage is used to modulate the sensitivity of a photocell that detects the light, then a measurement of the speed of light can be made with the apparatus diagramed in Fig. 10.10. The response of the detector D will be a maximum if light of maximum intensity reaches D at a time of maximum sensitivity. If we assume that maximum intensity and maximum sensitivity occur at the same time, this maximum

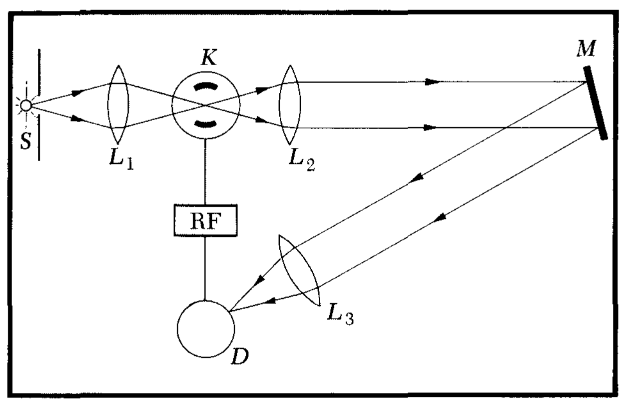


FIG. 10.10 A modern method for determination of c. Light from source S is amplitude-modulated in the Kerr cell K, then passed to mirror M and the photoelectric detector D through lenses $L_{2,3}$. The photodetector sensitivity and Kerr cell are synchronized by a modulated radio-frequency voltage generator RF.

[1]The penetration region is known as the *skin depth*. It is of the order of 1 micron (abbreviated μ; $1\ \mu \equiv 10^{-4}$ cm) in thickness in copper at room temperature at 10^{10} cps. There are also other corrections to be applied.

The intensity of light entering the Kerr-cell system from the source is steady, . . .

but the light coming out of the Kerr-cell system is modulated. The time of transit of the light from K to D can be varied by moving M: M can be adjusted so light arrives at D as shown.

If we move M out a little, the light arrives later . . .

for M *further* out, light arrival is *still* later . . .

for M *further* out, light arrival is *still* later . . .

for M further out, light arrival is still later.

Now suppose the sensitivity of the detector is modulated as shown here . . .

The detector responds only when it is sensitive and when light is coming in.

Thus we have this detector response for condition a.

For condition b we have this: The incoming light and the detector sensitivity are in phase.

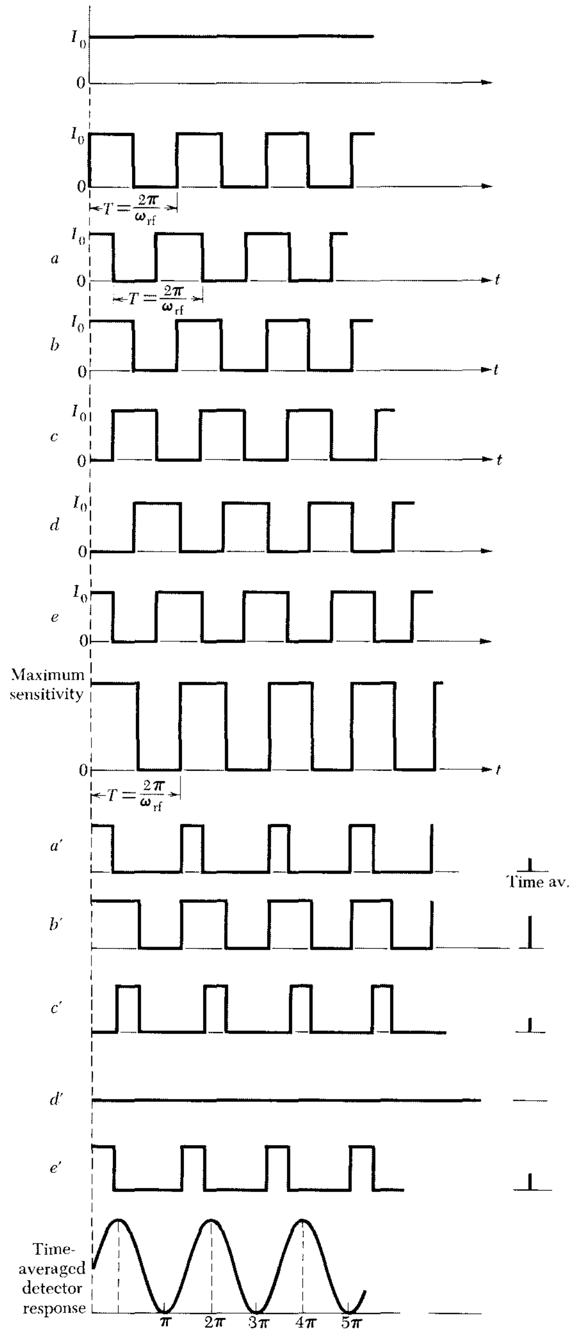
For c we have this.

For d the arriving light and the detector sensitivity are 180° out of phase, so there is no response.

For e we have this.

As we continuously vary the position of M, we obtain this average detector response.

The distance between two successive maxima of this curve corresponds to a change in the light path $2\Delta L$ caused by displacement of M.



response will occur if the time taken by the light to go from the Kerr cell K to mirror M and back to D is an integral number N of periods of the radio frequency ν. This time is N/ν and so

$$c = \frac{L\nu}{N}$$

where L is the distance from K to D. In the actual experiment it is of the order of 10 km. Some details of the method are given in Fig. 10.11.

Using this method, Bergstrand measured

$$c = (299{,}793.1 \pm 0.3)\ \text{km/s}$$

Note that the estimated error is very low. The same device is used (together with a standard value for c) to determine geodetic lengths over distances up to 40 km; in this application it is known as a *geodimeter*.

Hundreds of measurements of c have been made in the past hundred years by these and a dozen or so other methods. The present accepted value is

$$\boxed{c = (2.997\,925 \pm 0.000\,001) \times 10^{10}\ \text{cm/s}} \qquad (10.3)$$

This represents a consensus of the most reliable recent measurements by different methods in which electromagnetic waves from 10^8 cps (radio frequency) to 10^{22} cps (γ-rays) have been investigated. The precision at the highest frequency is not as great as at radio or optical frequencies, but there is at present no reason to believe that c varies with the frequency of the radiation.

SPEED OF LIGHT IN INERTIAL FRAMES IN RELATIVE MOTION

An elementary application of the galilean transformation to the problem of a moving receiver requires that the speed of light in the frame of the receiver be different from c. According to common sense we expect the speed of light c_R relative to the moving receiver to be given by

$$c_R = c \pm V \qquad (10.4)$$

FIG. 10.11 Bergstrand's measurement of c is based on the method of "phase-sensitive detection" and is similar to the experiment described here.

where V is the speed of the receiver that is supposed to be moving toward (+) or away (−) from the source. This seems a perfectly reasonable way to add velocities and is illustrated in Fig. 10.12*a* and *b*. The same relation should hold when the source and receiver are at rest and the medium moves with velocity V. The relation [Eq. (10.4)] is apparently obeyed in countless everyday experiences, at least where light is not involved. It holds for sound waves, if the velocity of sound is written for c. But it is *not true*, even approximately, for light waves in free space. It is found experimentally that (as shown in Fig. 10.12*c* and *d*)

$$c_R = c \tag{10.5}$$

for any frame *regardless of its velocity*, and regardless of the velocity relative to an imagined propagation medium. This demonstrated fact lies at the root of the relativistic formulation of physical laws.

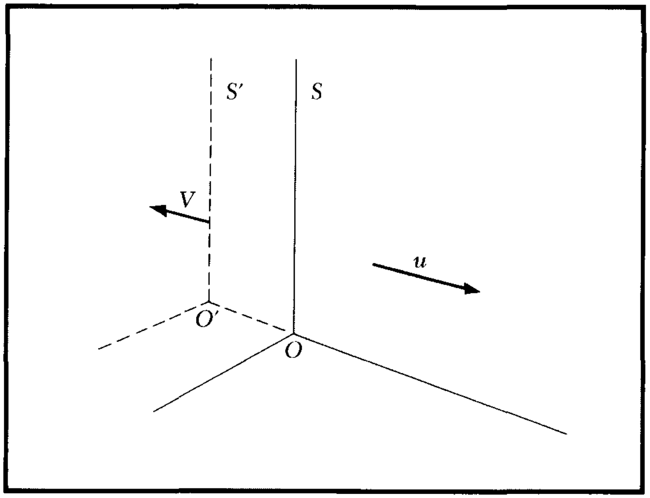
We now examine the experimental basis of Eq. (10.5). There are many different types of experiments which support the special theory of relativity; those leading to Eq. (10.5) make a convenient takeoff point. We consider the experiments which show that the velocity of light is independent of the velocity (3×10^6 cm/s) of the earth in its orbit.

First suppose, as did the physicists of the nineteenth century, that light propagates as an oscillation in a medium, just as sound propagates as an oscillation of atoms in a liquid, solid, or gas. The luminiferous medium through which light waves propagate in free space was called the *ether*.

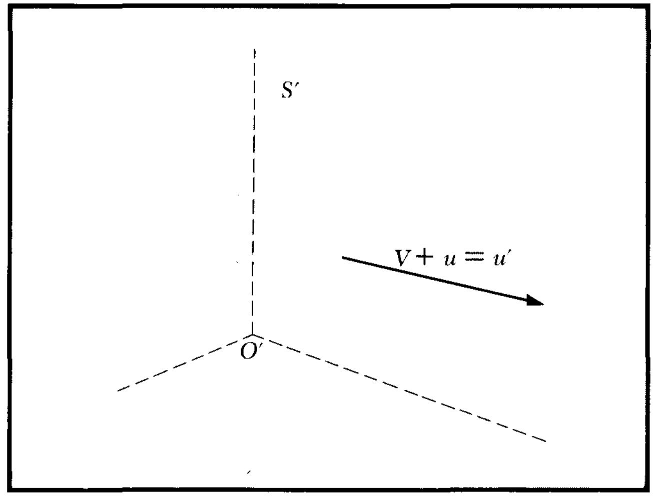
What is the ether? Today we consider ether as only another word for vacuum. But Maxwell and many others could not imagine a field as a self-supporting entity propagating in free space. Maxwell argued:

> But in all these theories the question naturally occurs:—If something is transmitted from one particle to another at a distance, what is its condition after it has left the one particle and before it has reached the other? If this something is the potential energy of the two particles, as in Neumann's theory, how are we to conceive this energy as existing in a point of space, coinciding neither with the one particle nor with the other? In fact, whenever energy is transmitted from one body to another in time, there must be a medium or substance in which the energy exists after it leaves one body and before it reaches the other, for energy, as Torricelli remarked, "is a quintessence of so subtle a nature that it cannot be contained in any vessel except the inmost substance of material things." Hence all these theories lead to the conception of a medium

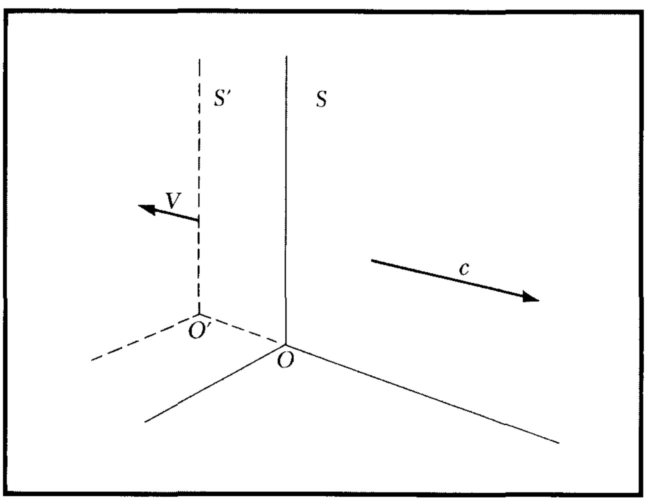
FIG. 10.12 Velocity addition predicted by the galilean transformation (*a,b*) and as actually observed for light (*c,d*).



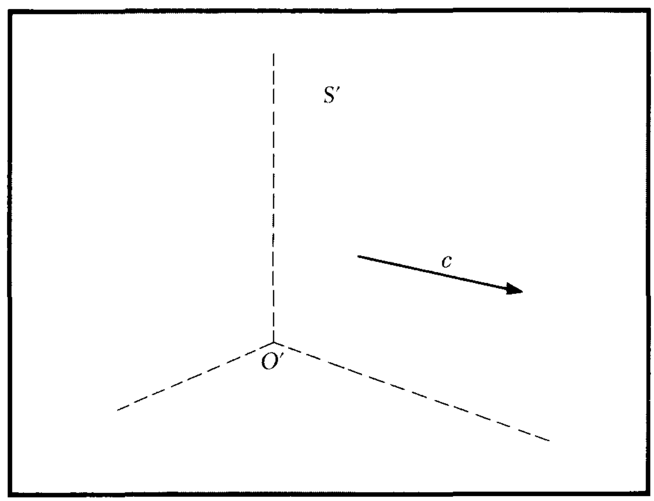
(*a*) If u is an ordinary terrestrial speed as observed in inertial frame S,



(*b*) the galilean transformation tells us that in inertial frame S' we will observe $u' = V + u$.



(*c*) However, experiments show that if an object has speed c in S,



(*d*) it also has speed c in S'.

> in which the propagation takes place, and if we admit this medium as an hypothesis, I think it ought to occupy a prominent place in our investigations, and that we ought to endeavour to construct a mental representation of all the details of its action, and this has been my constant aim in this treatise.

The obvious direct experiment to test the possible dependence of the velocity of light on the motion of the earth is to time accurately the one-way passage of a pulse of light over a measured path. This would be done separately in both directions on a north-south line, and then on an east-west line, and finally over again after 6 months, when the velocity of the earth about the sun has a reversed direction. With the development of lasers, sufficiently accurate clocks exist to permit such a direct experiment; the limiting technological factor at present appears to be the rise time of a pulse. At 10^{-9} s this introduces an effective error of $10^{-9}\,c = 30$ cm in the length of the path. The clocks in such an experiment would have to be synchronized at one spot and then separated slowly to their final positions.

A number of experiments have been performed to test Eq. (10.4), that is, to detect *ether drift* (see Fig. 10.13). All have failed to show a movement of the earth through the ether; very important and conceptually straightforward were those carried out by Michelson and Morley.[1]

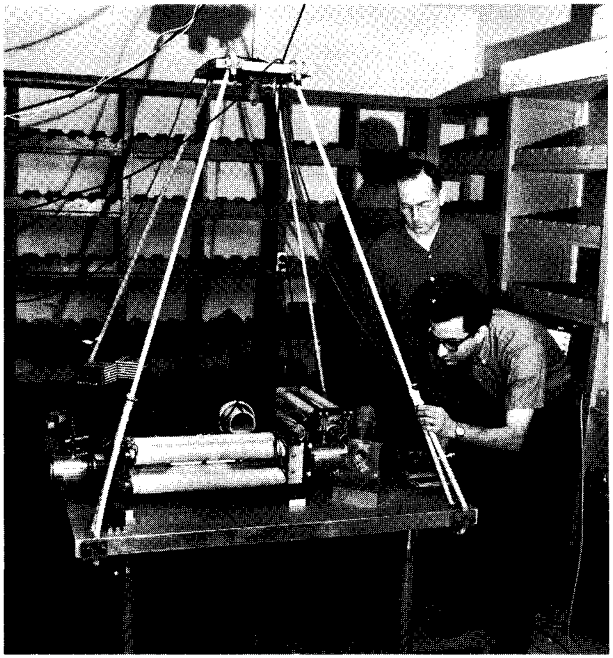


FIG. 10.13 A precision apparatus for a relativistic optical experiment using two gas lasers. The site is a former wine cellar in Round Hill, Mass. The workers are Charles H. Townes and Ali Javan.

Michelson–Morley Experiments Two sets of light waves derived from a common monochromatic source may interfere constructively or destructively at a point, according to the relative phase of the waves at that point. The relative phase may be changed by requiring one wave train to travel farther than the other. Michelson and Morley constructed an elaborate interferometer, the essential parts of which are shown in Figs. 10.14 and 10.15*a*. A beam of light from a single source *s* was split by a half-silvered mirror at *a*. We continue the description of the experiment in essentially the words and notation of the original workers:[2]

> Let *sa* [see Fig. 10.15*a* to *h*] be a ray of light which is partly reflected in *ab*, and partly transmitted in *ac*, being returned

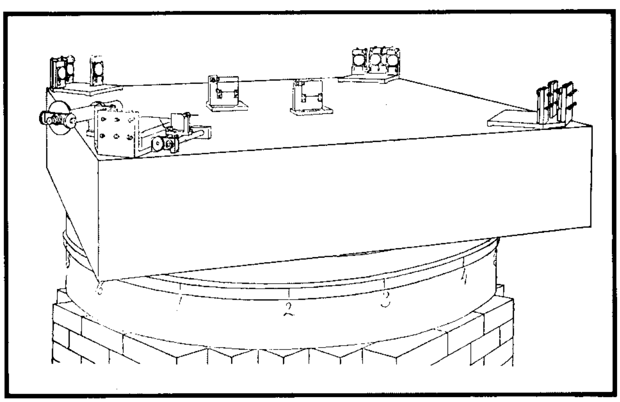


FIG. 10.14 Perspective of the apparatus described by Michelson and Morley in their 1887 paper.

[1]The influence of this experiment on Einstein in his work is discussed in an interesting article by Holton, *Am. J. Phys.*, 37:968 (1969).

[2]A. A. Michelson and E. W. Morley, *Am. J. Sci.*, 34:333 (1887). This was one of the most remarkable experiments of the nineteenth century. Simple in principle, the experiment led to a scientific revolution with far-reaching consequences. Note that the ratio of the speed of the earth in its orbit to the speed of light is about 10^{-4}. In reproducing the excerpt, we have written c for their V, and V for their v; interpolated remarks are enclosed in brackets.

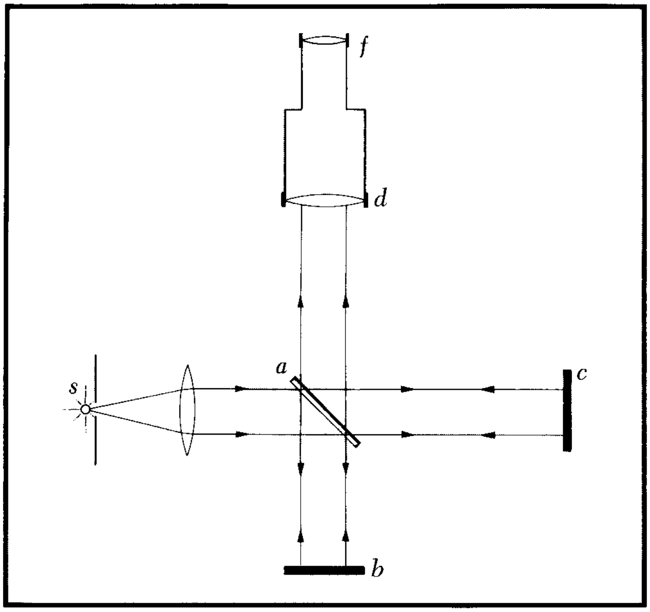
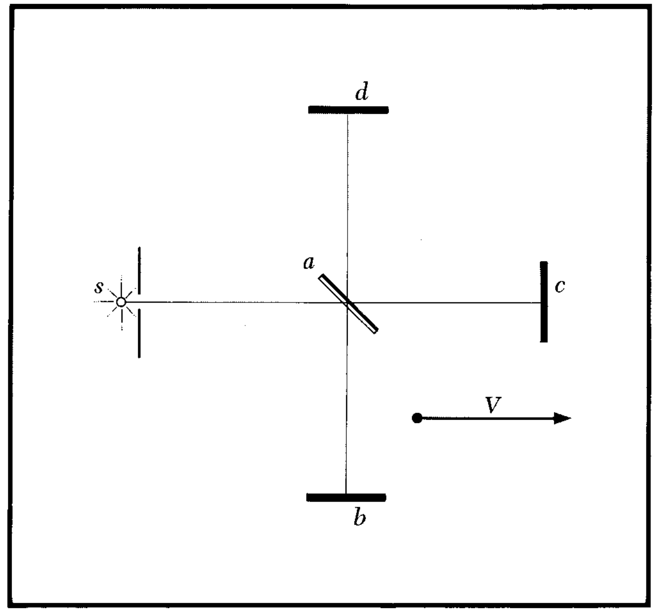
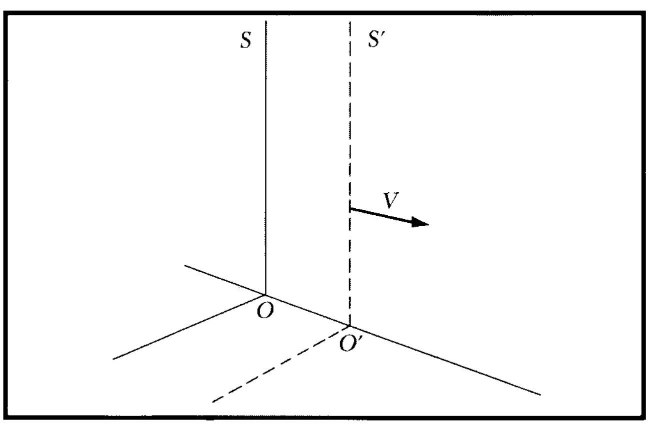


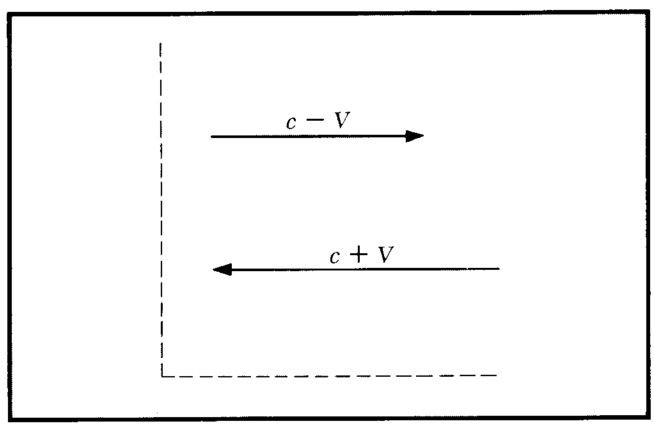
FIG. 10.15 (*a*) The Michelson–Morley experimental interferometer consists of a light source s, half-silvered mirror a, mirrors b and c, and a telescope detector d; f represents the focus of the telescope.



(*b*) If the interferometer is at rest in the ether, an interference pattern between the beams ***aba*** and ***aca*** is observed at d. If the apparatus (and earth) have velocity V with respect to the hypothetical ether, we would expect the interference pattern to change at d, since the times to traverse ***aba***, ***aca*** would now change by different amounts.



(*c*) To see this consider a galilean frame S' moving with earth and interferometer. S is a galilean frame at rest in the ether.



(*d*) According to the galilean transformation, light moving to right has speed $c - V$ in S'; light moving to left has speed $c + V$ in S'.

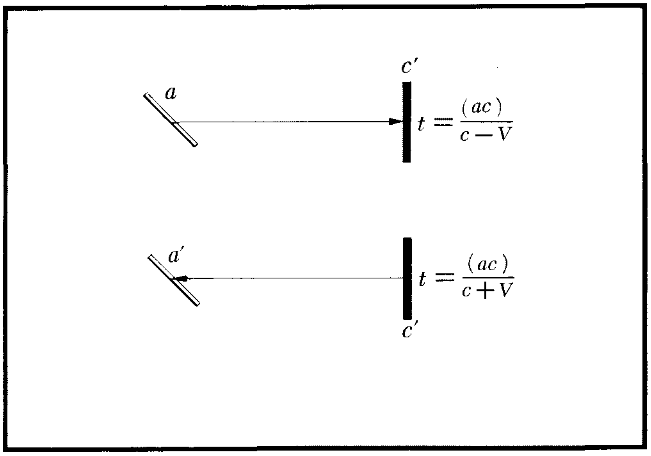
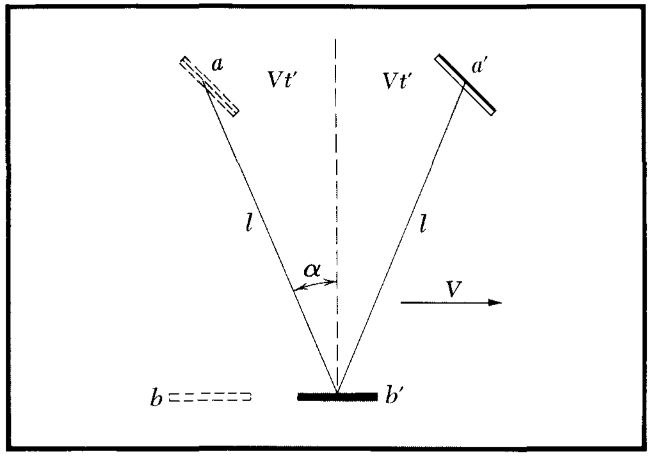


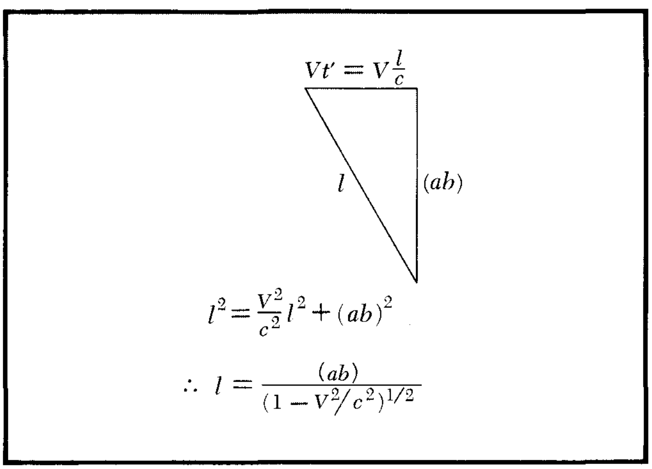
FIG. 10.15 *(cont'd)* (*e*) Thus the time to go from a to c' and back to a' is

$$\Delta t(ac'a') = \frac{(ac')}{c - V} + \frac{(ac')}{c + V}$$

where (ac') denotes the distance between a and c'.

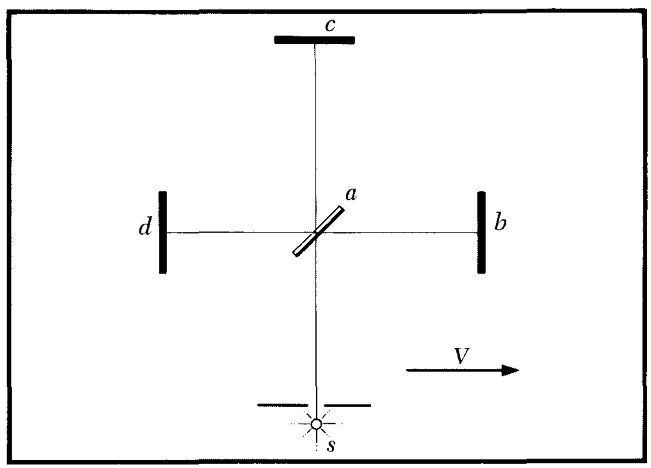


(*f*) What is the time $\Delta t(ab'a') = 2t'$ to go from a to b' and back to a'? In the galilean frame S at rest in the ether, the interferometer has velocity V to the right; light has speed c.



(*g*) $\Delta t(ab'a') = 2t' = 2(ab)/\sqrt{c^2 - V^2}$. To terms of the order V^2/c^2, this time is the same as

$$\frac{2(ab)\sqrt{1 + \frac{V^2}{c^2}}}{c}$$



(*h*) Thus even if $(ab) = (ac)$, the galilean transformation leads us to expect a shift in the interference pattern if the interferometer changes its velocity with respect to the ether. None was observed. Here the apparatus is shown turned through 90° to repeat the test with the motion parallel to ab instead of ac.

by the mirrors b and c, along ba and ca. ba is partly transmitted along ad, and ca is partly reflected along ad. If then the paths ab and ac are equal, the two rays interfere along ad. Suppose now, the ether being at rest, that the whole apparatus moves in the direction sc with the velocity of the earth in its orbit; the directions and distances traversed by the rays will be altered thus:—The ray sa is reflected along ab' [as in Fig. 10.15f]; it is returned along $b'a'$, where the angle $ab'a'$ is twice the aberration angle, or 2α, and goes to the focus of the telescope, whose direction is unaltered. The transmitted ray goes along ac', is returned along $c'a'$ [as in Fig. 10.15e] and is reflected at a', making $c'a'd'$ [not shown] equal $90° - \alpha$, and therefore still coinciding with the first ray. It may be remarked that the rays $b'a'$ and $c'a'$ do not now meet exactly in the same point a', though the difference is of the second order; this does not affect the validity of the reasoning. Let it now be required to find the difference in the two paths $ab'a'$ and $ac'a'$.

Let c = velocity of light
V = velocity of the earth in its orbit
D = distance ab or ac
T = time light occupies to pass from a to c'
T' = time light occupies to return from c' to a'

Then

$$T = \frac{D}{c - V} \qquad T' = \frac{D}{c + V}$$

The whole time of going and coming is

$$T + T' = 2D\frac{c}{c^2 - V^2}$$

and the distance traveled in this time is

$$2D\frac{c^2}{c^2 - V^2} \approx 2D\left(1 + \frac{V^2}{c^2}\right)$$

neglecting terms of the fourth order. The length of the other path is evidently

$$2D\sqrt{1 + \frac{V^2}{c^2}}$$

or to the same degree of accuracy,

$$2D\left(1 + \frac{V^2}{2c^2}\right)$$

The difference is therefore

$$D\frac{V^2}{c^2}$$

If now the whole apparatus be turned through 90°, the difference will be in the opposite direction, hence the displacement of the interference fringes should be $2D(V^2/c^2)$. Considering only the velocity of the earth in its orbit, this would be $2D \times 10^{-8}$. If, as was the case in the first experiment, $D = 2 \times 10^6$ waves of yellow light, the displacement to be expected would be 0.04 of the distance between the interference fringes.

In the first experiment one of the principal difficulties encountered was that of revolving the apparatus without producing distortion; and another was its extreme sensitiveness to vibration. This was so great that it was impossible to see the interference fringes except at brief intervals when working in the city, even at two o'clock in the morning. Finally, as before remarked, the quantity to be observed, namely, a displacement of something less than a twentieth of the distance between the interference fringes may have been too small to be detected when masked by experimental errors.

The first named difficulties were entirely overcome [in the second experiment] by mounting the apparatus on a massive stone floating on mercury; and the second by increasing, by repeated reflection, the path of the light to about ten times its former value.

. . . Considering the motion of the earth in its orbit only, this displacement should be

$$2D\frac{V^2}{c^2} = 2D \times 10^{-8}$$

The distance D was about eleven meters, or 2×10^7 wavelengths of yellow light; hence the displacement to be expected was 0.4 fringe [if the earth were traveling through an ether]. The actual displacement was certainly less than the twentieth part of this, and probably less than the fortieth part [see Fig. 10.16]. But since the displacement is proportional to the square of the velocity, the relative velocity of the earth and the ether is probably less than one-sixth the earth's orbital velocity, and certainly less than one-fourth.

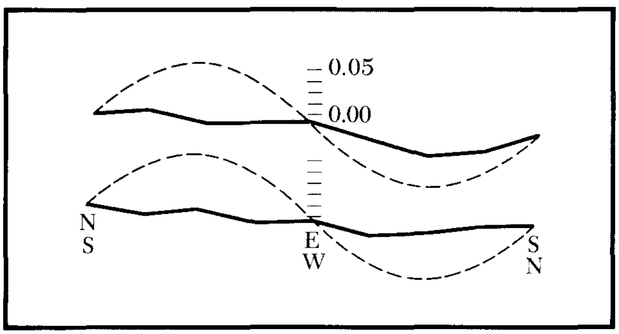


FIG. 10.16 "The results of the observations are expressed graphically [in the figure]. The upper is the curve for the observations at noon, and the lower that for the evening observations. The dotted curves represent *one-eighth* of the theoretical displacements. It seems fair to conclude from the figure that if there is any displacement due to the relative motion of the earth and the luminiferous ether, this cannot be much greater than 0.01 of the distance between the fringes." [A. A. Michelson and E. W. Morley, *Am. J. Sci.*, 34:333 (1887).] The vertical axis is the displacement of the fringes; the horizontal axis refers to the orientation of the interferometer relative to an east-west line.

The experimental results of Michelson and Morley were contrary to what we would expect, based on the galilean transformation. The experiments have since been repeated (with variations) with different wavelengths of light, with starlight, with extremely monochromatic light from a modern laser, at high altitudes, under the earth's surface, on different continents, and at different seasons over a period of some 80 yr. We can say that the change in c (the ether drift) is zero to a precision which is best expressed by saying that the speeds of light upstream and downstream are equal within a variation of less than 10^3 cm/s, or of 1 part in 1000 of the earth's orbital velocity about the sun.

Invariance of c The null result of the Michelson–Morley experiment suggests that the effects of the ether are undetectable. The result also suggests that the speed of light is independent of the motion of the source or of the observer. The experimental evidence on the latter point is quite good, but could be improved. The work by Sadeh quoted in Chap. 11 shows that the velocity of γ-rays is constant within ± 10 percent, independent of the velocity of the source, for source velocities of the order of $\frac{1}{2}c$. We conclude from all the experimental evidence that *a spherical wave front of light emitted from a point source in one inertial frame will appear as spherical to an observer in any other inertial frame.*

We noted in an earlier section that the speed of electromagnetic waves is independent of frequency over the range 10^8 to 10^{22} cps. Careful measurements also show that c is independent of the intensity of the light and also of the presence of other electric and magnetic fields. Our discussions have been limited entirely to electromagnetic waves traveling in free space.

DOPPLER EFFECT

The doppler effect or doppler shift relates the measured frequency of a wave to the relative velocities of the source, the medium, and the receiver. It is familiar, for sound, to anyone who has listened to an automobile approaching and then receding; or to those "older" people who have stood on a railroad platform and listened while a whistling train passed by. When the source is approaching, the number of waves emitted in 1 s will reach the receiver in less than 1 s because the source is closer when the last wave is emitted than when the first. Therefore, the frequency is higher. Vice versa, when the source is receding the frequency is lower. The same type of argument applies to a fixed source and moving receiver. The relations for sound are given by

$$\nu_R = \nu_T \frac{1 + v_R/\mathcal{V}}{1 - v_S/\mathcal{V}} \tag{10.6}$$

where $\mathcal{V}$ is the velocity of the sound wave in the medium, e.g., air, considered at rest, v_S is the velocity of the source considered positive when it is moving toward the receiver, v_R is the velocity of the receiver considered positive when it is moving toward the source, ν_T is the frequency of the source (transmitter) measured by an observer at rest with respect to the source, and ν_R is the frequency measured by the receiver.

Note that if $v_S \ll \mathcal{V}$ (assume $v_R = 0$),

$$\nu_R = \nu_T\left(1 + \frac{v_S}{\mathcal{V}}\right) \tag{10.7}$$

and

$$\frac{\nu_R - \nu_T}{\nu_T} = \frac{\Delta\nu}{\nu} = \frac{v_S}{\mathcal{V}} \tag{10.8}$$

In the case of light similar effects are present though we shall see some essential differences. In explaining and analyzing the doppler effect for sound we must consider the medium bearing the sound waves and the motion of source or receiver relative to the medium. In the case of light we must not understand the doppler effect in this way since the Michelson–Morley experiment result does not permit us to consider a medium (i.e., the ether). The doppler effect provides some interesting tests of special relativity and also some important results, particularly for astronomy. We shall treat the doppler effect correctly for light in Chap. 11.

EXAMPLE

The Recessional Red Shift Spectrographic analysis of light received from distant galaxies shows that certain prominent spectral lines identified in spectroscopic studies in the laboratory are shifted very significantly toward the red, or low-frequency, end of the visible spectrum. This shift may be interpreted as a doppler shift arising from the velocity of recession of the source. It is also known that the velocities calculated from these doppler shifts are directly proportional to the distances of the sources from us determined by independent means.

This is an extraordinary and provocative observational fact. The simplest nonrelativistic explanation of the distance-velocity relation is known as the "big-bang" theory, according to which the universe was formed from an explosion about 10^{10} yr ago. The fastest-moving products of the original explosion now form the outermost regions of the universe. Thus the greater the radial velocity of matter (relative to us), the farther it is from us and the greater is its red shift. There also are more sophisticated explanations of the recessional red shift. None is proved (see Fig. 10.17).

A pair of easily recognizable absorption lines in the spectrum of potassium (the K and H lines) are prominent in the spectra of many stars. These lines occur near wavelength[1] 3950 Å in laboratories on earth. We assume that laboratory observers moving in the rest frame of any star would measure the same wavelength. In light coming from a nebula in the constellation Boötes we observe these same lines at a wavelength of 4470 Å, a shift toward the red of 4470 − 3950 = 520 Å. This is a relative shift of

[1] angstrom ≡ 10^{-8} cm ≡ 1 Å

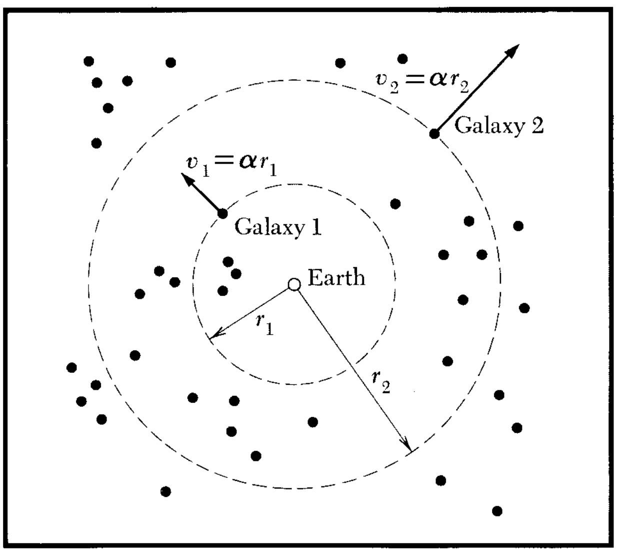
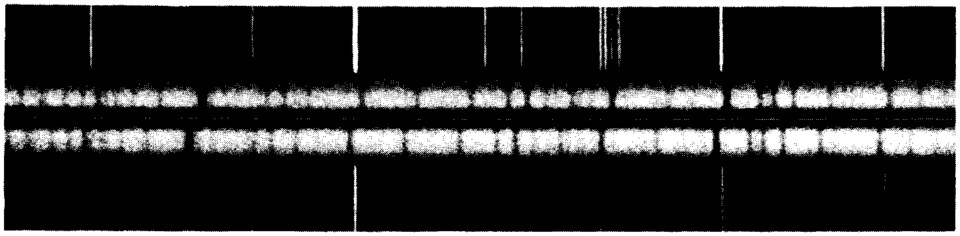


FIG. 10.17 The doppler effect observed in light from distant stars indicates that the galaxies are receding from us with a velocity proportional to their distance from earth. Galaxies 1 and 2 are assumed to have their distances r_1 and r_2 measured by other means, their velocities v_1 and v_2 by the doppler effect.



Laboratory reference spectrum
Star approaching
Star receding
Laboratory reference spectrum

FIG. 10.18 Two spectrograms (taken at different times) of the binary star α^1 Geminorum. Only one of the two stars in this binary emits enough light to be detected. Notice that the spectral lines from the star are shifted, with respect to the laboratory reference lines, in different directions corresponding to two phases of motion of the star. In one phase the star is moving toward the earth and the frequency of the light is increased; in the other phase the star is moving away from the earth and the frequency is decreased. (*Lick Observatory photograph*)

$$\frac{\Delta\lambda}{\lambda} = \frac{520}{3950} = 0.13$$

We observe that, by using Eq. (10.8) with $\mathcal{V}$ equal to c (as will be justified for light waves in Chap. 11), and differentiating[1] $\nu = c/\lambda$ with c constant,

$$\frac{\Delta\nu}{\nu} = -\frac{\Delta\lambda}{\lambda} \qquad \text{or} \qquad \frac{\Delta\lambda}{\lambda} \approx \frac{v_S}{c} \tag{10.9}$$

We infer from Eqs. (10.8) and (10.9) that the nebula is receding from us with a relative speed $|v| \approx 0.13c$, which is really quite fast. For *higher speeds* we need to use one or another relation for the doppler shift as modified by the theory of relativistic models of the universe.[2] Also, the approximate expressions in Eqs. (10.8) and (10.9) suitable for either sound or light at speeds low compared to the speeds of sound or light, respectively, must be replaced by the correct expressions for light.

Similar observations on large numbers of galaxies can be combined with independent estimates of their distances to obtain an amazing empirical result: The relative velocity of a galaxy at distance r from us may be represented by the relation

$$v = \alpha r \tag{10.10}$$

where the constant α is empirically determined to be about $1.6 \times 10^{-18}\ \text{s}^{-1}$. (The estimation of galactic distances is a complex subject for which an astronomy text must be consulted.) The reciprocal of α has the dimensions of time:

$$\frac{1}{\alpha} \approx 6 \times 10^{17}\ \text{sec} \approx 2 \times 10^{10}\ \text{yr} \tag{10.11}$$

It is the time, beginning with the "big bang," taken by the star to

[1]Note a little computational trick: Suppose that $y = Ax^n$, where A, n are constants, and we want to find dy/y in terms of dx/x. We take the natural logarithm of both sides to form $\log y = \log A + n \log x$. We then take differentials of both sides to obtain $dy/y = n\, dx/x$. Here we have used the relation $d \log x/dx = 1/x$.

[2]See G. C. McVitties, *Physics Today*, p. 70 (July 1964).

reach its present distance. When we multiply $1/\alpha$ by c, we obtain a length:

$$\frac{c}{\alpha} \approx (3 \times 10^{10})(6 \times 10^{17}) \approx 2 \times 10^{28} \text{ cm} \qquad (10.12)$$

The time [Eq. (10.11)] is loosely called the *age of the universe;* the length [Eq. (10.12)] is loosely called the *radius of the universe.* The real significance of these quantities is not known at present, although several different cosmological models have been proposed to account for the form of the relations.

THE ULTIMATE SPEED

We have seen that electromagnetic waves in free space can only travel with the speed c. Can the speed of anything exceed the speed limit c?

Consider the motion of charged particles in an accelerator. Can particles be accelerated to travel faster than c? We have not as yet in this course encountered directly any principle which prevents the acceleration of charged particles to arbitrarily high velocities (see Fig. 10.19).

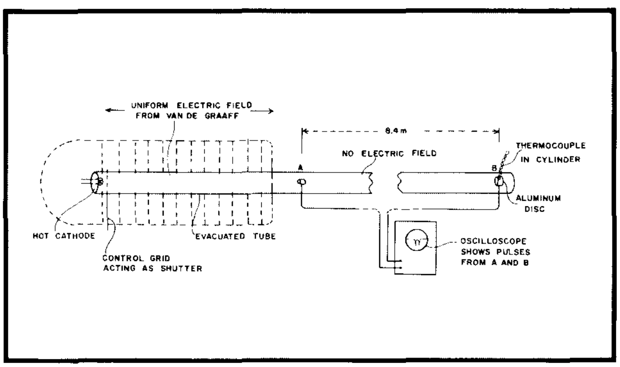


FIG. 10.19 The general arrangement of the ultimate speed experiment. The electrons are accelerated in a uniform field on the left and timed between A and B by the oscilloscope.

The following experiment[1] illustrates the proposition that a particle cannot be accelerated to a speed greater than c. Pulses of electrons are accelerated by successively larger electrostatic fields in a Van de Graaff accelerator, after which the electrons drift with constant velocity through a field-free region. Their time of flight, and hence their velocity over a measured distance AB, is measured directly, and the kinetic energy (which is turned to heat at the target at the end of the path) is measured by means of a calibrated thermocouple.

In the experiment the accelerating potential Φ is known with good precision. The kinetic energy of an electron is

$$K = eEL = e\,\Phi$$

where L is the distance over which acceleration occurs and $\Phi = EL$ is the difference in electric potential between the ends of the accelerating path. If $\Phi = 10^6$ V, the electron after acceleration has an energy of 1×10^6 eV (1 MeV). Now $10^6 \text{ V} \approx 10^6/300$ statvolts, so that the kinetic energy acquired by an electron is

[1] This experiment was performed by W. Bertozzi in connection with the PSSC film "The Ultimate Speed." Our account draws directly from chap. A-3 of the PSSC Advanced Topics Program. See *Am. J. Phys.*, 32:551 (1964).

$$\frac{(4.80 \times 10^{-10})(10^6)}{300} \approx 1.60 \times 10^{-6} \text{ erg}$$

If N electrons per second travel in the beam, the power delivered to the aluminum target at the end of the beam should be $1.60 \times 10^{-6}\, N$ erg/s. This agrees exactly with the direct thermocouple determination of the power absorbed by the target. This result confirms that the electrons deliver to the target the kinetic energy acquired during their acceleration. Further, on the basis of nonrelativistic mechanics, we expect that

$$K = \tfrac{1}{2}mv^2 \tag{10.13}$$

so that a graph of v^2 against the kinetic energy K should be a straight line. For energies greater than about 10^5 eV, however, the linear relation between v^2 and K does *not* hold experimentally. Instead, the velocity is observed to approach the limiting value 3×10^{10} cm/s at higher energies. So when the measured velocity is compared with the velocity calculated from Eq. (10.13) it is found to be less than Eq. (10.13) predicts. In fact the graph of v^2 against K bends over as shown in Fig. 10.20, approaching the value 9×10^{20} cm^2/s^2. The experimental results may be summarized: The electrons absorb the expected energy from the accelerating field, but their velocity does not increase without limit. Our only recourse in understanding this fact is to assume that m in Eq. (10.13) is not constant as K becomes large. We shall deal with this problem in Chap. 12. Many other experiments suggest, as this one does, that c is the upper limit to the velocity of particles. Thus we believe firmly that c is the maximum signaling speed with either particles or electromagnetic waves: *c is the ultimate speed.*

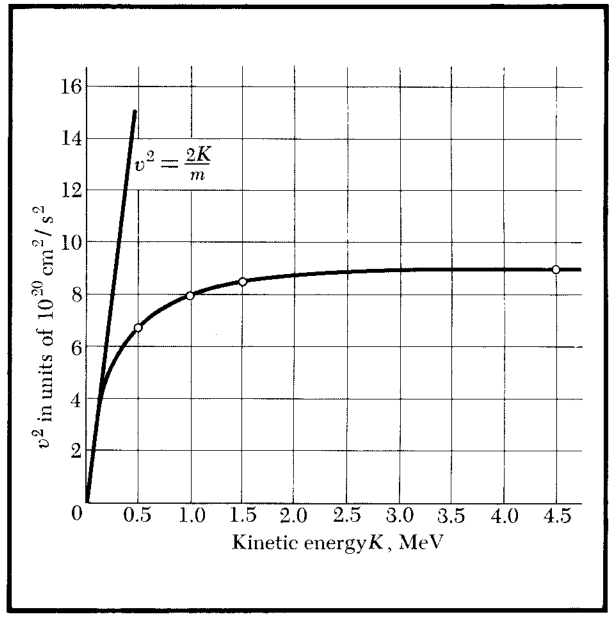


FIG. 10.20 Graph of v^2 against kinetic energy. Open dots are experimental points.

CONCLUSIONS

We are now prepared to study special relativity in Chap. 11, with the knowledge from experiment that

1. c is invariant among inertial frames, that is, frames of reference moving with uniform velocity with respect to each other.
2. c is the maximum speed at which energy can be transmitted.
3. The absolute velocity of a frame of reference has no meaning. Only relative velocities can be experimentally determined.

4 Simple galilean transformations do not provide satisfactory explanations of phenomena involving high speeds.

5 The newtonian formula for kinetic energy, $\frac{1}{2}mv^2$, fails when v approaches c.

We have reviewed only a very small fraction of the experiments that support the special theory of relativity, which is now very solidly established. Physicists place as much reliance on this theory as any other part of physics. Our next endeavor must be to formulate it precisely and to understand some of its major consequences.

PROBLEMS

1. *Doppler shift.* A space navigator wishes to determine his velocity of approach as he nears the moon. He sends a radio signal of frequency $\nu = 5000$ Mc/s and compares this frequency with its echo, observing a difference of 86 kc/s. Calculate the velocity of the space vehicle relative to the moon. (The nonrelativistic expression for the doppler effect is sufficiently accurate for many purposes.) *Ans.* 2.6×10^5 cm/s.

2. *Recessional red shift.* A spectral line appearing at a wavelength of 5000 Å in the laboratory is observed at 5200 Å in the spectrum of light coming from a distant galaxy.
(*a*) What is the recessional velocity of the galaxy? *Ans.* 1.2×10^9 cm/s.
(*b*) How far away is the galaxy? *Ans.* 8×10^{26} cm.

3. *Speed of light.* In Michelson's celebrated measurement of the speed of light, an octagonal reflecting prism rotating about the axis of the prism reflected a beam of light from a distant light source and back to an observer near the source. The timing provides that the transit time of the light equal one-eighth of the period of rotation of the octagonal prism. The one-way distance was $L = 35.410 \pm 0.003$ km and the frequency of rotation of the prism was $\nu = 529$ cps to an accuracy of 3×10^{-5} cps.
(*a*) Calculate the speed of light from these data. (A fractional correction of the order of 10^{-5} for atmospheric effects had to be applied.)
(*b*) The angle between any two adjacent prism faces was $135° \pm 0.1''$. Estimate the overall precision of the measurement of c.

4. *Eclipses of Io.* Jupiter's satellite Io moves in an orbit of radius 4.21×10^{10} cm with an average period of 42.5 h. Roemer observed that the period varied regularly during the year, with a period of variation of about 1 yr. The maximum deviation of the period from the average was 15 s, at times approximately 6 months apart. Neglect the orbital travel of Jupiter.
(*a*) Estimate the distance the earth travels in one period of Io's motion about Jupiter. *Ans.* 4.5×10^{11} cm.
(*b*) When does Io's period appear to be greatest?
(*c*) Use the preceding result and the data provided to estimate the velocity of light.
(*d*) Estimate the accumulated delay in the 6 months following the point of zero delay when the earth is closest to Jupiter.

5. *Stellar parallax and aberration.* Stellar parallax was predicted by Aristarchus of Samos (ca. 200 B.C.) and it was finally observed for certain by Bessel in 1838. A notably unsuccessful attempt was made by Bradley, who discovered instead the aberration of starlight. During the course of a year the apparent position of a star shifts between extremes by approximately 40″ of arc due to aberration.
(*a*) What would be the distance in parsecs of a star with a parallax of 20″? The nearest known star is α Centauri at a distance of about 1.3 parsecs. *Ans.* 0.05 parsec.
(*b*) Show that the apparent annual motion from aberration of stars near the ecliptic is a straight line whose ends

subtend a 40″ angle. The ecliptic is the plane of the earth's orbit.

6. *Rotation of galaxies.* In 1916, before the great distances of the nebulae (galaxies) were known, the spiral M101 was reported to rotate like a solid body with a period of 85,000 yr. The observed angular diameter is 22′. Calculate the maximum possible distance of the nebula if the above period is correct, supposing that the extremities of the nebula are not to move faster than c. (Recent measurements of stars in M101 place it at a distance of 8.5×10^{24} cm. It is apparent that the rotation period reported in 1916 was underestimated.)

7. *Variable stars.* The 200-in. Mt. Palomar telescope can barely resolve individual stars in galaxies at a distance of 3×10^{25} cm. One method for calibrating distances of this order of magnitude involves observation of the periods in the luminosity of certain Cepheid-type variable stars. A Cepheid-type star is a gravitationally unstable star that exhibits periodic pulsations in which its radius may change by perhaps 5 to 10 percent. The period of a Cepheid is related to its average luminosity. The temperature of the star changes with the same period as the radius, so that one observes periodic variations in brightness. Periods as short as a few hours have been found. A Cepheid whose intrinsic luminosity is 2×10^4 times that of the sun has a period of 50 days in our galaxy.

(*a*) Estimate from the distance-velocity relation [Eq. (10.10)] the radial velocity for a galaxy at a distance of 3×10^{25} cm.

(*b*) What would we expect to observe for the period of this Cepheid in a galaxy at the distance cited above?

Ans. 50.08 days.

8. *Novae.* Occasionally a star is seen to experience an explosion in which a portion of its outer layers is thrown out with high velocity. Such a star is called a *nova.* A recent nova was observed visually to have a surrounding shell after its outburst. The angular diameter of the shell was found to increase by 0.3″/yr. The spectrum of the nova is a normal stellar spectrum with superimposed broad emission lines, the widths (in wavelengths) of which remain constant at 10 Å (in the vicinity of a wavelength of 5000 Å), though the lines are dimming. The width is to be interpreted as a measure of the doppler shift between the parts of the shell advancing toward us and receding from us. Estimate the distance to the nova, if the shell is optically thin (so that we receive as much light from the far hemisphere as from the near).

Ans. 1.3×10^{21} cm.

9. *Velocities of galaxies.* Measured radial velocities of galaxies relative to the earth are not isotropic over the sky. Nonisotropy results from the motion of the sun (orbital velocity) with respect to the center of our galaxy, and from our galaxy's own motion with respect to the local extragalactic standard of rest. Let us examine all galaxies at a particular distance, say, 3.26×10^7 light yr.

(*a*) What is the mean radial velocity of these galaxies?
Ans. The mean velocity of the galaxies as calculated from the velocity-distance relation is 494 km/s.

(*b*) Where in their spectra will be the average location of the Hα line of hydrogen? (In the laboratory, $\lambda_{H\alpha} = 6.563 \times 10^{-5}$ cm.)
Ans. The Hα line will be, on the average, at 6.574×10^{-5} cm.

In our sample we find that in a certain direction the velocities are 300 km/s larger than the average and in just the opposite direction they are this much too small.

(*c*) What is the velocity of the sun in this frame of reference?
Ans. 300 km/s.

(*d*) Is that necessarily the orbital velocity of the sun around the center of our galaxy?
Ans. No, for it can include any motion of our galaxy as a whole in this reference frame.

(*e*) Assuming that this is the orbital velocity, estimate the mass of our galaxy, taking all the mass to be at its center and the orbit of the sun to be circular (the distance to the center of the galaxy is 3500 light yr). Compare with the mass of 8×10^{44} g quoted for the mass of the galaxy and explain the difference.
Ans. 4.5×10^{43} g. This is less than that usually quoted because much of the mass of our galaxy is not at the center—in fact, much mass lies exterior to the sun, where it would not affect the sun's motion or be detectable in this way.

10. *Rotation of stars.* The sun is seen from its surface features to rotate slowly, with a period of 25 days at the equator. Some stars, however, rotate far faster. How can this be determined in view of the fact that the stars are too distant to be seen except as points of light?

FURTHER READING

HPP, "Project Physics Course," chaps. 16 (sec. 6) and 20 (sec. 1), Holt, Rinehart and Winston, New York, 1970.

A. A. Michelson, "Studies in Optics," The University of Chicago Press, Chicago, 1927; paperback reprint, 1962.

CONTENTS

11

Special Relativity: The Lorentz Transformation

BASIC ASSUMPTIONS

The null result of the Michelson–Morley experiment to detect the drift of the earth through an ether and the other results discussed in Chap. 10 can only be understood by making a revolutionary change in our thinking; the new principle we need is simple and clear:

> The speed of light is independent of the motion of the light source or receiver.

That is, the speed of light is the same in all reference frames in uniform motion with respect to the source. To this new assumption must be added our earlier assumption:

> Space is isotropic and uniform. The fundamental laws of physics are identical for any two observers in uniform relative motion.

All the vast consequences of the special theory of relativity follow from these assumptions.

Electromagnetic waves or photons are not unique in having a velocity independent of the motion of the source. Physicists believe, with strong evidence, that there are other particles, notably neutrinos and antineutrinos, that have velocities equal to c. We shall, however, discuss photons because it is easier to carry out experiments with them.

Consider first a light wave spreading out from a point source. The wave front (surface of equal phase) will be a sphere if viewed in the reference frame in which the source is at rest. But according to our new principle the wave front must also be a sphere when viewed in a reference frame in uniform motion with respect to the source; otherwise we could tell from the shape of the wave front that the source is moving. The fundamental assumption that the speed of light is independent of the motion of the source demands that we be unable to tell from the shape of the wave front whether or not the source is in uniform motion.

LORENTZ TRANSFORMATION

In Chap. 4 we introduced the galilean transformation in order to understand how phenomena would look from two different points of view. We shall use the same ideas here with two different frames of reference S and S′, moving with uniform velocity V with respect to each other. We wish to find a transformation of coordinates, and possibly of the time also, such as the galilean transformation [Eq. (4.14)] relating the coordinates and time in one frame of reference to the coordinates and time in another frame of reference in such a way as to be consistent with the relativity assumptions. If we assume that in the frame S a light source is at the origin, the equation of a spherical wave front emitted at $t = 0$ is

$$x^2 + y^2 + z^2 = c^2t^2 \tag{11.1}$$

In the frame of reference S' in which the coordinates are x', y', z', and t', the equation of the spherical wave front must be

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \tag{11.2}$$

The speed of light c is the same in both Eqs. (11.1) and (11.2).

We can try the galilean transformation to see whether it gives results in agreement with Eqs. (11.1) and (11.2).

$$x' = x - Vt \qquad y' = y \qquad z' = z \qquad t' = t \tag{11.3}$$

When we substitute Eq. (11.3) in Eq. (11.2) we obtain directly

$$x^2 - 2xVt + V^2t^2 + y^2 + z^2 = c^2t^2$$

This result is certainly not in agreement with Eq. (11.1). Thus the galilean transformation fails, and we must attempt to find some other transformation. It must reduce to the galilean transformation when the velocity V becomes very small compared with the velocity of light c.

Let us try

$$x' = \alpha x + \epsilon t \qquad y' = y \qquad z' = z \qquad t' = \delta x + \eta t$$

We know that for $x' = 0$, $dx/dt = V$; and for $x = 0$, $dx'/dt' = -V$. The algebra leads to

$$V = -\frac{\epsilon}{\alpha} \qquad -V = \frac{\epsilon}{\eta}$$

or

$$\alpha = \eta$$

When we write, repeating Eq. (11.2),

$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

we get

$$\alpha^2x^2 + 2\alpha\epsilon xt + \epsilon^2t^2 + y^2 + z^2 = c^2(\delta^2x^2 + 2\delta\alpha xt + \alpha^2t^2)$$

This is to be compared to Eq. (11.1), and we see that consistency is possible if

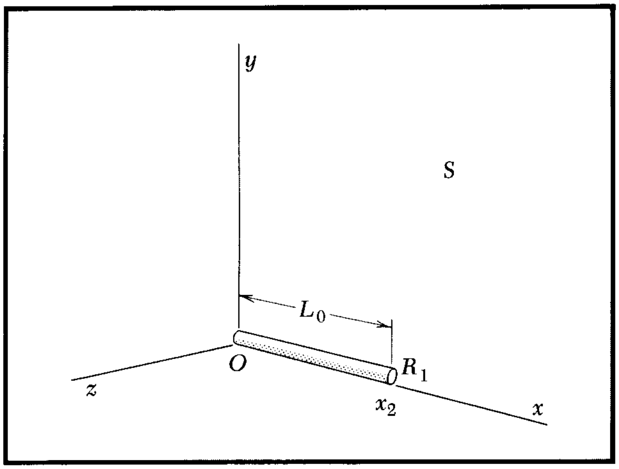
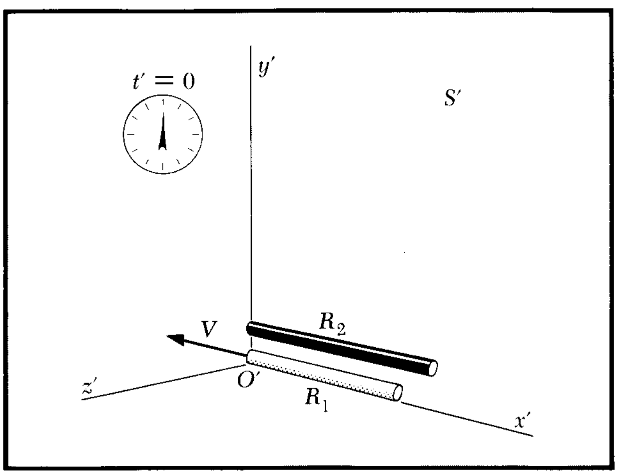


FIG. 11.1 (*a*) Consider a rigid rod R_1 of length L_0 in its rest frame S.



(*b*) The Lorentz transformation tells us that R_1, which has speed V in S', will be measured to have length $L' = L_0\sqrt{1 - V^2/c^2}$ in S'. Note that $x_1 = x_1' = 0$ in the figure.

$$2\alpha\epsilon = 2c^2\delta\alpha$$
$$\alpha^2 - c^2\delta^2 = 1$$

and

$$c^2\alpha^2 - \epsilon^2 = c^2$$

Eliminating ϵ by using $\epsilon = -V\alpha$, we get

$$\alpha = \frac{1}{(1 - V^2/c^2)^{\frac{1}{2}}} \qquad \epsilon = \frac{-V}{(1 - V^2/c^2)^{\frac{1}{2}}}$$
$$\delta = \frac{-V/c^2}{(1 - V^2/c^2)^{\frac{1}{2}}} \qquad \eta = \frac{1}{(1 - V^2/c^2)^{\frac{1}{2}}}$$

Our transformation is then

$$x' = \frac{x - Vt}{(1 - V^2/c^2)^{\frac{1}{2}}} \qquad y' = y \qquad z' = z$$
$$t' = \frac{t - (V/c^2)x}{(1 - V^2/c^2)^{\frac{1}{2}}} \tag{11.4}$$

This is the *Lorentz transformation*.[1] It is linear in x and t; it reduces to the galilean transformation for $V/c \rightarrow 0$; when substituted in Eq. (11.2) it gives

$$x^2 + y^2 + z^2 = c^2t^2$$

exactly as required. That is,

$$x'^2 + y'^2 + z'^2 = c^2t'^2$$

is *invariant* under a Lorentz transformation. The form of the equation describing the wave front is the same in all frames moving with uniform relative velocity. Equation (11.4) is the unique solution to all our difficulties. It is a good shorthand way to remember many important results in relativity. We shall discuss several of them below with the help of the Lorentz transformation.

It is usually convenient to make use of the standard notation used in relativity:

$$\beta \equiv \frac{V}{c} \tag{11.5}$$

That is, β (Greek beta) is the velocity measured in a natural

[1] This transformation has a long history. It was first used by J. Larmor to explain the null result of the Michelson–Morley experiment, in his "Aether and Matter," pp. 174–176, Cambridge University Press, New York, 1900. Larmor claims accuracy only to order v^2/c^2; in fact, his results are exact.

system of units in which $c = 1$. It is also convenient to introduce γ (Greek gamma):

$$\gamma \equiv \frac{1}{(1-\beta^2)^{\frac{1}{2}}} \equiv \frac{1}{(1-V^2/c^2)^{\frac{1}{2}}} \tag{11.6}$$

Note that $\gamma \geq 1$. The Lorentz transformation Eq. (11.4) then becomes

$$x' = \gamma(x - \beta ct) \qquad y' = y \qquad z' = z \qquad t' = \gamma\left(t - \frac{\beta x}{c}\right) \tag{11.7}$$

and the reader can prove (Prob. 2) that the inverse transformation is

$$x = \gamma(x' + \beta ct') \qquad y = y' \qquad z = z' \qquad t = \gamma\left(t' + \frac{\beta x'}{c}\right) \tag{11.8}$$

Length Contraction Consider a rod (see Fig. 11.1*a*) lying along the x axis and at rest in reference frame S. Because the rod is at rest in S, the position coordinates of its ends x_1 and x_2 are independent of time. Thus

$$L_0 = x_2 - x_1$$

is called the *rest length* or *proper length* of the rod. Also consider a rod (see Fig. 11.2*a*) lying along the x' axis and at rest in reference frame S'. For the same reason

$$L_0 = x_2' - x_1'$$

is called the *rest length* or *proper length* of the rod in S'.

We now wish to determine the lengths of these rods when viewed from a moving reference frame. First look at the rod in Fig. 11.1*a* from the reference frame S' which moves with velocity $V\hat{\mathbf{x}}$ with respect to the rod at rest in S. (See Fig. 11.1*b* and note that the rod R_2 from Fig. 11.2*a* is at rest in S'.) We determine the length of the rod as viewed from S' by determining at a given time t' the positions x_1' and x_2' that coincide with the ends of the rod. The important point here is that the time t' is the same for x_1' and x_2'. To say this another way, the distance between positions x_1' and x_2' in S' which coincide

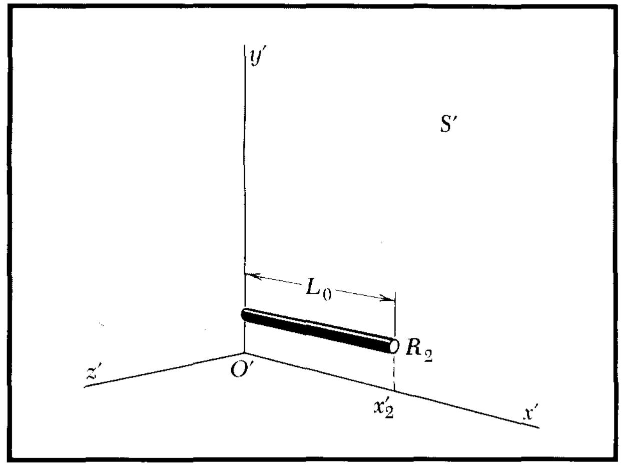
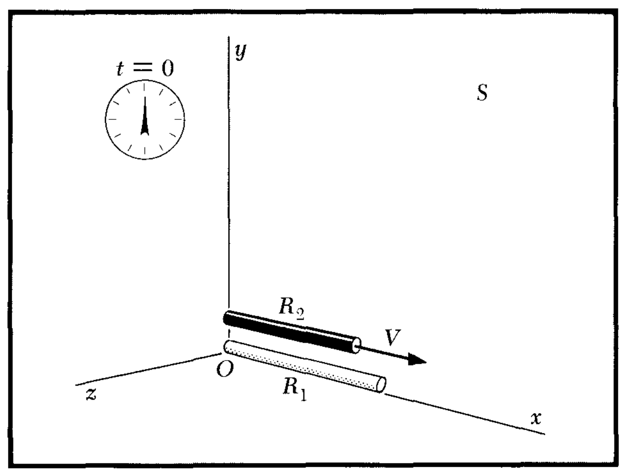


FIG. 11.2 (*a*) Consider a similar rigid rod R_2 of length L_0, as measured in its rest frame S'.



(*b*) The Lorentz transformation also tells us that R_2, which has speed V in S, will be measured to have length $L = L_0\sqrt{1 - V^2/c^2}$ in S. Note that $x_1' = x_1 = 0$ in the figure.

simultaneously (in S') with the endpoints of the rod is the natural definition of the length L in the moving frame S'.

From the Lorentz transformation, Eq. (11.8), we have

$$x_1 = \gamma(x_1' + Vt_1')$$
$$x_2 = \gamma(x_2' + Vt_2')$$
$$x_2 - x_1 = L_0 = \gamma(x_2' - x_1') + \gamma V(t_2' - t_1')$$

Now letting $t_2' = t_1'$ as we saw was necessary for the measurement in S', we get

$$L_0 = \gamma(x_2' - x_1') = \gamma L$$

or

$$\boxed{L = \frac{L_0}{\gamma} = L_0(1 - \beta^2)^{\frac{1}{2}}} \tag{11.9}$$

by using our definition $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$. In other words, the measurement in the moving frame gives a shorter length than the measurement in the stationary frame.

Alternatively we look at the rod in Fig. 11.2*a* (at rest in S') from the reference frame S which moves with velocity $-V\hat{\mathbf{x}}'$ with respect to the rod at rest in S'. (See Fig. 11.2*b* and note that the rod R_1 from Fig. 11.1*a* is at rest in S.) The procedure is the same, but now the time t is the same for the determination of the endpoints x_1 and x_2. From the Lorentz transformation, Eq. (11.7), we have

$$x_1' = \gamma(x_1 - Vt_1)$$
$$x_2' = \gamma(x_2 - Vt_2)$$
$$x_2' - x_1' = L_0 = \gamma(x_2 - x_1) - \gamma V(t_2 - t_1)$$

and letting $t_2 = t_1$, we get

$$L_0 = \gamma(x_2 - x_1) = \gamma L$$
$$L = L_0(1 - \beta^2)^{\frac{1}{2}}$$

The measurement of the moving rod again gives a length shorter than the measurement of the stationary rod.

This is the famous Lorentz–Fitzgerald contraction of a rod moving parallel to its length with respect to the observer. One may worry at this point whether the rod has "actually contracted." Of course nothing physical has happened to the rod, but the process of measurement in the moving frame has given a different result. For a discussion of the figures of rapidly moving objects as photographed with a camera, see the excellent review by Weisskopf.[1] It has been shown, for example,

[1] V. F. Weisskopf, *Physics Today*, **13**:24–27 (Sept. 1960).

by calculation of trajectories that a moving sphere will photograph as a sphere and not as an ellipsoid.

In the foregoing discussion we have emphasized that the observer makes his measurement of length by *simultaneously* recording the positions of the ends of the rod in his own reference frame. That is what we required of the observer in the moving frame S' when he measured the length of the rod stationary in S with the result L_0/γ [Eq. (11.9)]. It is essential that we recognize that this act of simultaneously registering the endpoints at time t' in S' does *not* transform into simultaneous events at the endpoints x_1 and x_2 in S; on the contrary the Lorentz equations indicate a time interval

$$t_2 - t_1 = \frac{\beta(x_2 - x_1)}{c}$$

in S for the registering of the two endpoints that was done simultaneously in S'. We will presently see that for a rod lying on the y axis, the question of simultaneity does not arise, but for a rod along the x axis, the matter of simultaneity is all important.[1]

This is illustrated by a different example. We can easily synchronize a series of clocks in S, the frame in which the rod is at rest. Let the clocks at $x = 0$ and $x = L_0$ (at each end of the meter stick) each emit at $t = 0$ a directional flash of light in the y direction. These two flashes are received in S' by two of a series of counters spaced along the x' axis. How far apart are the two counters which were triggered? From Eq. (11.7) we have, for the location of the two counters,

$$x_1' = 0 \cdot \gamma - c \cdot 0 \cdot \beta\gamma = 0$$
$$x_2' = L_0\gamma - c \cdot 0 \cdot \beta\gamma = L_0\gamma$$

so that their distance apart is

$$x_2' - x_1' = L_0\gamma = \frac{L_0}{(1 - \beta^2)^{\frac{1}{2}}} \tag{11.10}$$

This does not agree with Eq. (11.9)! But we have done a *different* experiment and obtained a different result. Our earlier experiment was based on the natural definition of length in S', using the requirement of simultaneity in S'. That earlier experiment involved comparing $\Delta x'$ with Δx when $\Delta t' = 0$,

[1] The reader is referred to Taylor and Wheeler, "Space-Time Physics—An Introduction," pp. 64–66, W. H. Freeman and Company, San Francisco, 1965.

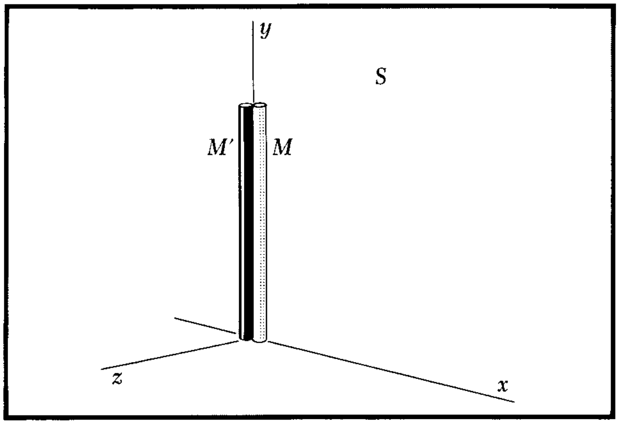
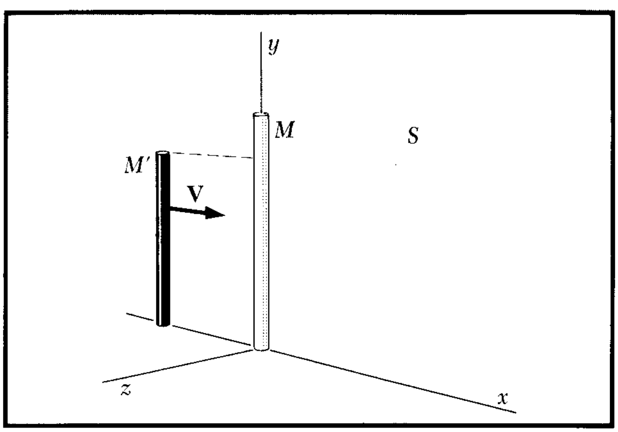


FIG. 11.3 (*a*) Suppose we have two identical rods M' and M at rest in S.



(*b*) Suppose M' appears shorter to observer in S when it moves relative to S.

whereas the second experiment involved comparing $\Delta x'$ with Δx when $\Delta t = 0$.

We have learned indirectly from the result in Eq. (11.10) of the second experiment that two events simultaneous in S are not, in general, simultaneous in S′. Thus from Eq. (11.7) we see that two events *simultaneous* ($\Delta t = 0$) in S, which are separated by Δx in space, will be separated in S′ in both space and time:

$$\Delta x' = \gamma\,\Delta x \qquad c\,\Delta t' = -\beta\gamma\,\Delta x$$

Measurement of Length Perpendicular to Relative Velocity

Contrary to the measurement of the distance in the direction of the relative velocity, we see from the Lorentz transformation, Eq. (11.7), that

$$y' = y \qquad z' = z$$

These relations are equivalent to the statement that the measurement of the length of a meter stick is independent of its velocity *if* the meter stick moves perpendicular to its length.

How would we verify this statement experimentally? We can take a meter stick and move it at uniform velocity past another meter stick which is at rest. There is no problem in making the origins of both meter sticks cross exactly. Then the 1-m mark of each will also cross exactly, or, if the motion changes the length, we can arrange for the 1-m mark of the shorter stick to make a scratch on the longer stick (see Fig. 11.3*a* to *c*). This provides a definite physical record of the length.

Let S be the rest frame of one meter stick and S′ the rest frame of the other. Suppose the motion does change the apparent length. Then if the laws of physics are to remain the same for an observer on S as for an observer on S′, it is necessary that the stick which appeared the shorter to an observer on S should appear the longer to an observer on S′. But this reversal of the roles is incompatible with our physical record that one meter stick is shorter than the other. Therefore, the lengths must be equal when viewed from S and S′ (see Fig. 11.3*d* and *e*). This discussion merely confirms that $y = y'$ and $z = z'$.

These results concerning the measurements of lengths parallel and perpendicular to the relative velocity imply that the measurements of angles involving x coordinates will be different in the two frames. This is true, and the reader can work out for himself the relations between the trigonometric functions of the angles in the two frames. (See Prob. 5 at the end

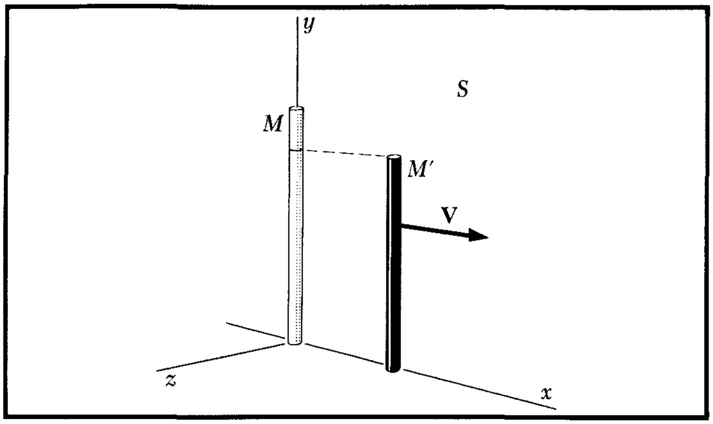
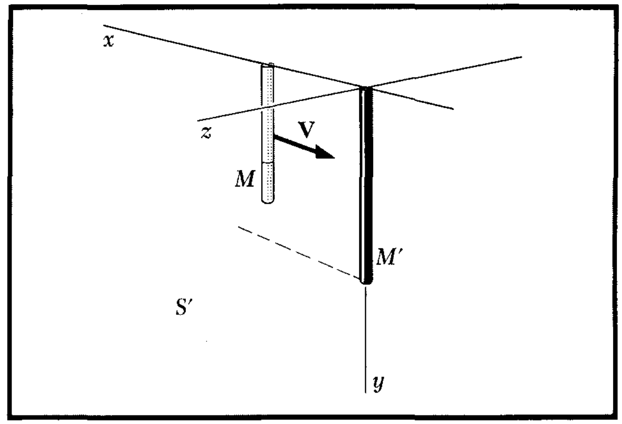


FIG. 11.3 (*cont'd*) (*c*) Then we could arrange it so that the end of M' leaves a scratch on M as it passes by.



(*d*) The scratch is a physical result of an experiment, and it must be observed in another frame, for example, upside down in the rest frame of M'. But now M must appear *shorter* than M' since M is moving and M' is at rest.

of the chapter.) Remember that the important point here is to determine in which frame the measurements of the ends of the length are simultaneous.

Time Dilation of Moving Clocks Used in the ordinary sense, the word *dilate* means *enlarge beyond normal size;* in connection with a clock, it means to lengthen an interval of time. We now consider a clock which is at rest in reference frame S.

The result of the measurement of a time interval in the frame in which the clock is *at rest* is denoted as

$$\tau = t_2 - t_1$$

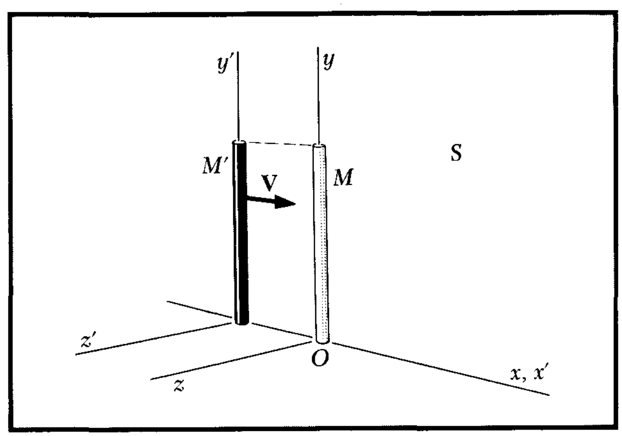
and is called the *proper time*. Then using the Lorentz transformation [Eq. (11.7)], we get

$$t_2' = \gamma\left(t_2 - \frac{\beta x_2}{c}\right) \qquad t_1' = \gamma\left(t_1 - \frac{\beta x_1}{c}\right)$$

or

$$t_2' - t_1' = \gamma(t_2 - t_1) = \gamma\tau = \frac{\tau}{(1-\beta^2)^{\frac{1}{2}}} \tag{11.11}$$

where we have set $x_2 - x_1 = 0$; the clock stays at the same place in S. This is the time interval measured by a clock at rest in S′ moving with velocity $V\hat{\mathbf{x}}$ with respect to the frame



(*e*) Thus we have a contradiction, which is resolved only if M' and M have the same length even when one of them is moving. Thus $y' = y$. By a similar argument $z' = z$.

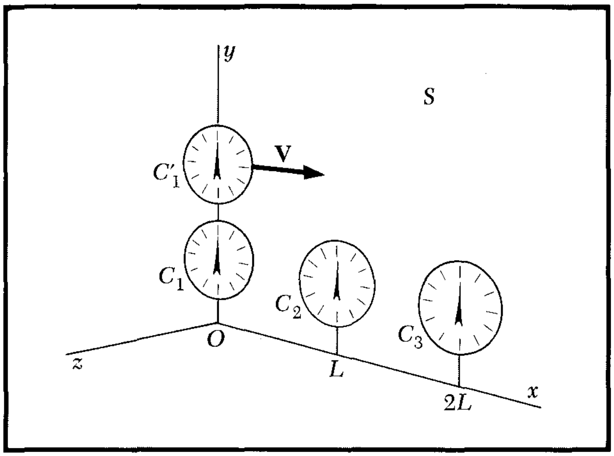
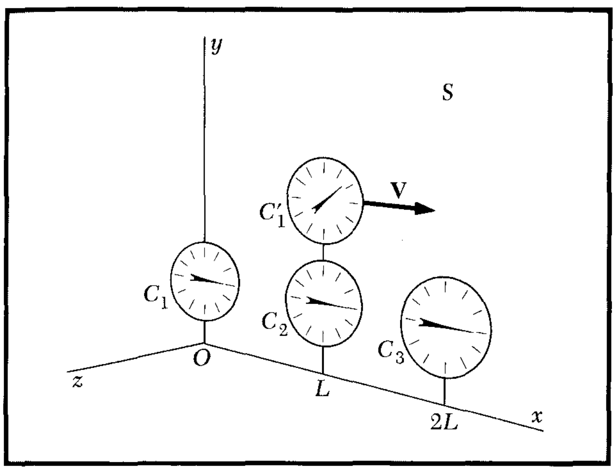


FIG. 11.4 (*a*) Clocks C_1, C_2, C_3 are at rest in S, spaced at equal intervals L along the x axis, and all synchronized. Clock C'_1 has velocity V with respect to S. Suppose $t' = 0$ when $t = 0$, as shown.



(*b*) The Lorentz transformation yields $t' = (t - xV/c^2)\ \gamma = t\sqrt{1 - V^2/c^2}$ since $x = L = Vt$. To the observer in S, the moving clock C'_1 runs slow.

S of the original clock. The time interval measured in S' is longer than the time interval measured in the frame S. If, however, we perform the experiment pictured in Fig. 11.4*a* and *b* we find that the measurement in S of a time interval in S' is longer than the clock in S' shows it to be.

The conclusion we must accept is this: Consider two reference frames S and S' in constant relative motion. Each frame has an observer with his own synchronized clocks held at rest in that frame. If two events occur at a fixed location in S separated by the time interval Δt as measured by the S observer, the time interval measured by the S' observer will be longer; it will be $\Delta t' = \gamma\, \Delta t$. Conversely, for two events at a fixed position in S' separated by time $\Delta t'$, the observer in S will measure a longer interval; he will measure $\Delta t = \gamma\, \Delta t'$ (see Fig. 11.5*a* and *b*).

This effect is called *time dilation*. Moving clocks appear to advance more slowly than clocks at rest. This is not easy to understand in an intuitive way and it may take you a long time to feel content with time dilation. The root of the apparent paradox is the invariance of c, and a straightforward problem illustrates how time dilation is forced upon us by this constancy of the speed of light. Let us construct a standard clock in the reference frame S (see Fig. 11.6). The clock can be used to measure the time τ needed for a light pulse to travel a fixed distance L from a source at rest to a mirror at rest, and back again. The light path is along the y axis. Thus

$$\tau = \frac{2L}{c} \tag{11.12}$$

This time can be read on a dial or it can be printed out on a piece of paper. Observers in any frame can look at the printed record of the flight time of the pulse and they will all agree that a clock in the rest frame S recorded the time τ. But what do their own clocks, not in S, record?

An observer in a frame S' (moving uniformly in the x direction with respect to S) (see Fig. 11.6) can also time the light-reflection experiment while it is carried out in S. The observer in S' will do this by using a set of synchronized clocks at rest in S'. We start two clocks at rest in S' at the same time (synchronized) by flashing a light source located midway between them; each starts from zero at the instant when the flash reaches it. The procedure may be extended to other clocks. We can also synchronize any number of clocks in one reference frame by synchronizing them when they are close together in

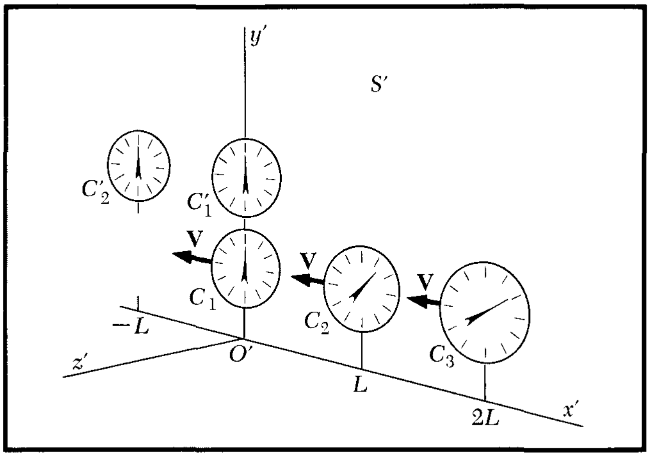
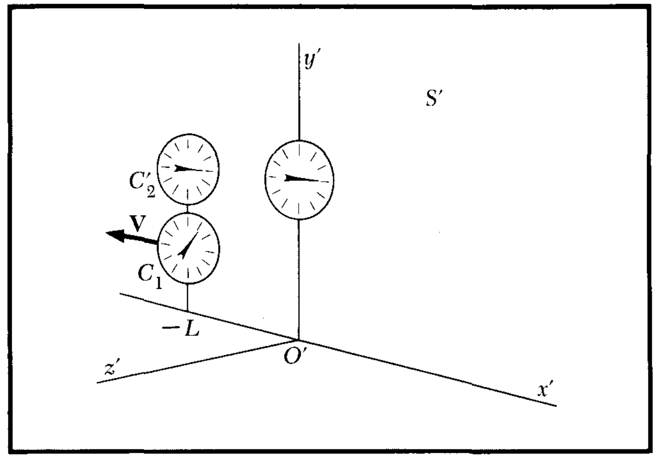


FIG. 11.5 (*a*) In S' clocks C'_1, C'_2, etc., are at rest, separated by distance L, and synchronized. To the observer in S', clocks C_1, C_2, C_3 are *not synchronized!* What do they read?



(*b*) To the observer in S', it is the *moving clock* C_1 which runs slow! Where are clocks C_2, C_3 and what do they read at this instant?

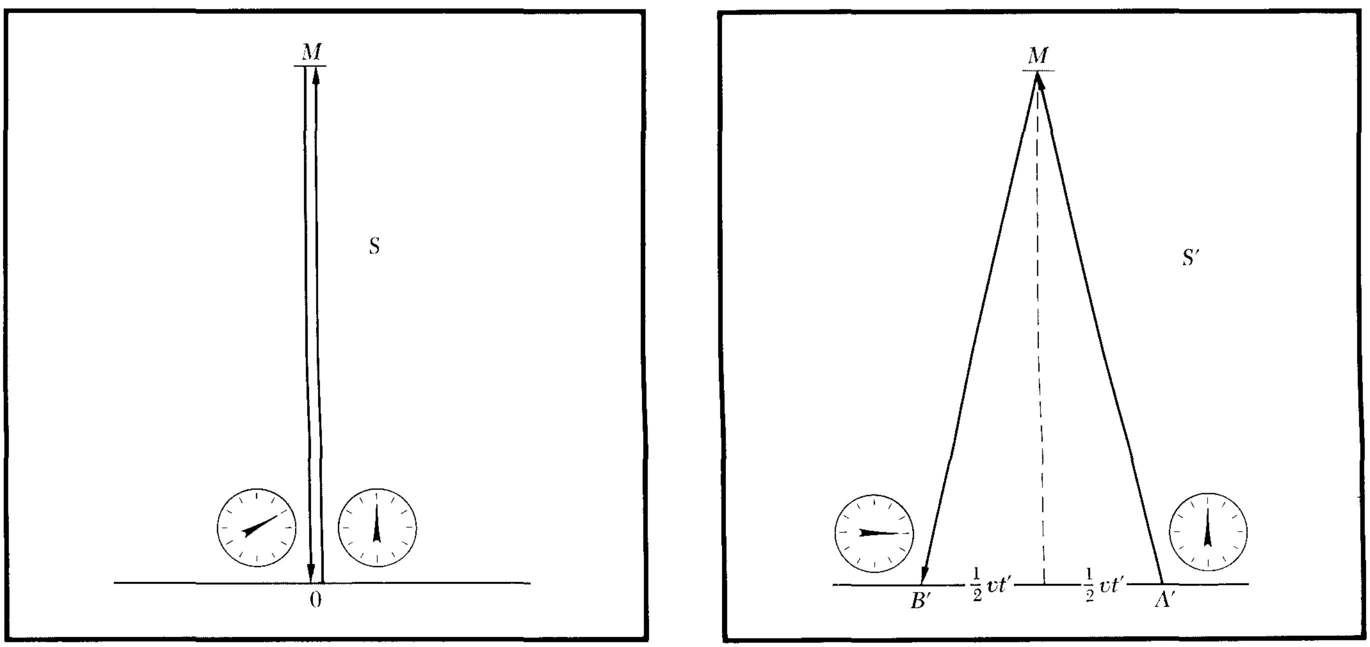


FIG. 11.6 View of path of light in frames S and S'. Point A' is coincident with 0 at time light is emitted. In S', light travels from A' to mirror M to B'.

space and then separating them slowly until they take up the desired positions.

We can read any clock in S' and be certain that all other clocks at rest in S' will read the same time. In particular we read whatever clock in S' is closest in space to the single clock in S which is used for the reflection experiment. One clock in S' will be closest and will be read when the light pulse starts out in S; another clock in S' will be closest and will be read when the light pulse returns and is recorded by the clock in S.

The path traversed by the light in S is $2L$. But the path as viewed from S' is longer because the apparatus in S has moved relative to S' by $V \cdot \frac{1}{2}t'$ along the x axis during the outbound passage of the light pulse from the source to the mirror and by another $V \cdot \frac{1}{2}t'$ during the inbound passage (see Fig. 11.6). Here t' is the time as observed in S'. The distance in S' traveled by the pulse is

$$2[L^2 + (\tfrac{1}{2}Vt')^2]^{\frac{1}{2}}$$

because the pulse travels always with the speed c, this distance must equal ct'. Thus

$$(ct')^2 = 4L^2 + (Vt')^2$$

or

$$t' = \frac{2L}{(c^2 - V^2)^{\frac{1}{2}}} = \frac{2L}{c}\frac{1}{(1-\beta^2)^{\frac{1}{2}}}$$

or, by reference to Eq. (11.12),

$$t' = \frac{\tau}{(1-\beta^2)^{\frac{1}{2}}} \tag{11.13}$$

exactly the same as Eq. (11.11). Thus the clock in S will seem to the timekeepers in S' to run slowly, because the S clock has printed out a time τ less than the time t'.

We see that the time-dilation effect does not involve mysterious processes in the interior of atoms; the effect arises in the measurement process. The clock at rest in S reads the proper time τ when viewed by an observer at rest in S. But when we view from S' a time interval which is τ in S, we see a longer time t' because of the longer light path. Any kind of clock will behave in the same way. In particular if τ is the decay half-life of mesons or of radioactive matter as measured in the frame S in which the particles are at rest, then

$$t' = \frac{\tau}{(1-\beta^2)^{\frac{1}{2}}} \tag{11.14}$$

is the decay half-life observed in the frame S' in which the particles are moving with velocity β. This is illustrated in Fig. 11.7*a* to *g*, which refers to the following example.

EXAMPLE

Lifetime of π^+ Mesons It is known that a π^+ meson decays into a μ^+ meson and a neutrino. The π^+ meson in a frame in which it is at rest has a mean life before decaying of about 2.5×10^{-8} s.† If a beam of π^+ mesons is produced with a velocity $\beta \approx 0.9$, what is the lifetime of the beam as viewed from the laboratory reference frame? A π^+ meson is a positively charged unstable particle with mass about $273m$, where m is the mass of the electron. The μ^+ meson has a mass of about $207m$; the neutrino has zero rest mass.

The proper lifetime τ of the π^+ meson is 2.5×10^{-8} s. If $\beta \approx 0.90$, then $\beta^2 \approx 0.81$ and the expected lifetime in the laboratory frame will be, from Eq. (11.14),

$$t' \approx \frac{2.5 \times 10^{-8}}{(1-0.81)^{\frac{1}{2}}} \approx 5.7 \times 10^{-8}\ \text{s}$$

Thus on the average, before decaying, the particle will travel over twice as far as we would expect nonrelativistically from the product of the velocity times the proper lifetime.

Experiments on the lifetime of π^+ mesons (positive pions) are reported by R. P. Durbin, H. H. Loar, and W. W. Havens, Jr., *Phys. Rev.*, **88**:179 (1952). The results are in good agreement with the predicted time dilation for the appropriate velocity. Beams of π^+ mesons have been produced with

$$\beta = 1 - (5 \times 10^{-5})$$

their mean life in the beam is 2.5×10^{-6} s, or 100 times the proper lifetime of π^+ mesons at rest.

Consider a beam of π^+ mesons traveling with a velocity nearly equal to c. If the relativistic time-dilation effect did not exist, they would traverse a mean distance equal to $(2.5 \times 10^{-8}\ \text{s})(3 \times 10^{10}\ \text{cm/s}) \approx 700$ cm before decaying. They actually travel much farther than this, because of time dilation. The hydrogen bubble chamber at the Lawrence Berkeley Laboratory was about 100 m from the pion source in the Bevatron. The distance the pions travel before decay is of the order of $(2.5 \times 10^{-6})(3 \times 10^{10}) \approx 10^5$ cm, or about 100 times the distance they would travel before decay without the time-dilation effect. The design of apparatus for high-energy experiments in particle physics takes advantage of the long

† If N_0 is the number of radioactive particles present at time $t = 0$, the number left after time t is $N_0 e^{-\lambda t}$. The mean life is $1/\lambda$; λ is the decay constant.

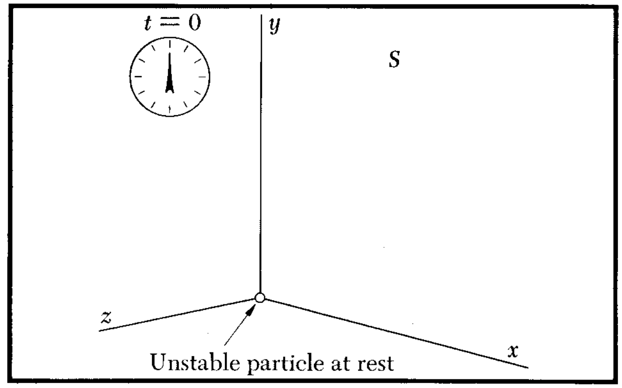
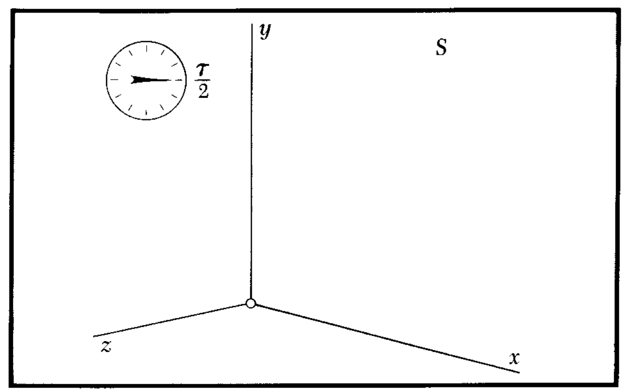
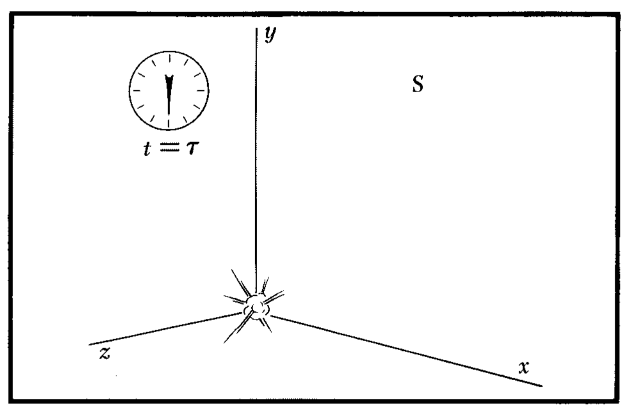


FIG. 11.7 (*a*) Another example of time dilation: An unstable particle is at rest in S. We begin to observe it at $t = 0$.



(*b*) Time elapses.



(*c*) The particle decays at time $t = \tau$.

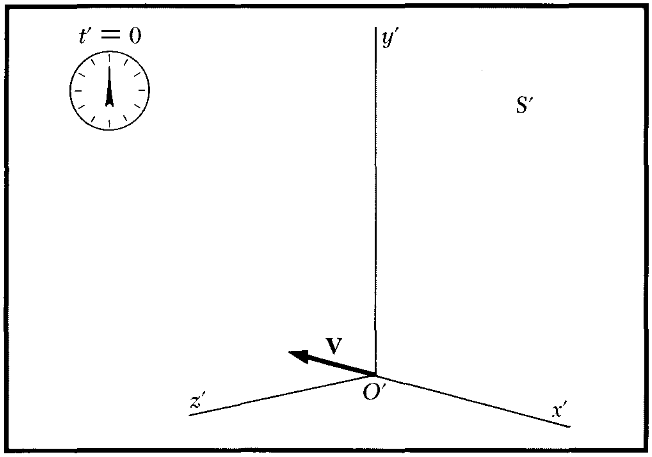
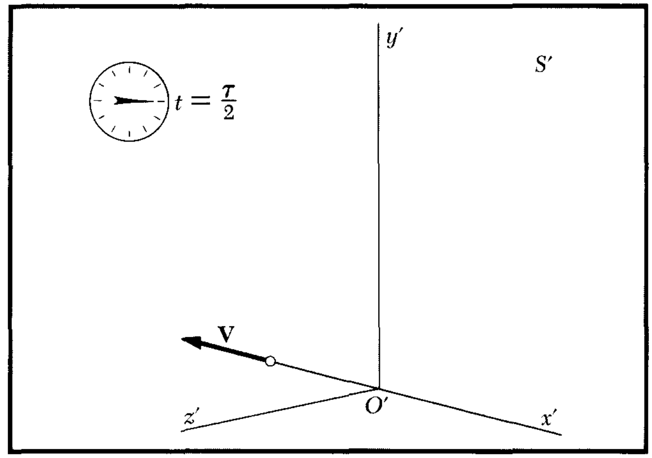
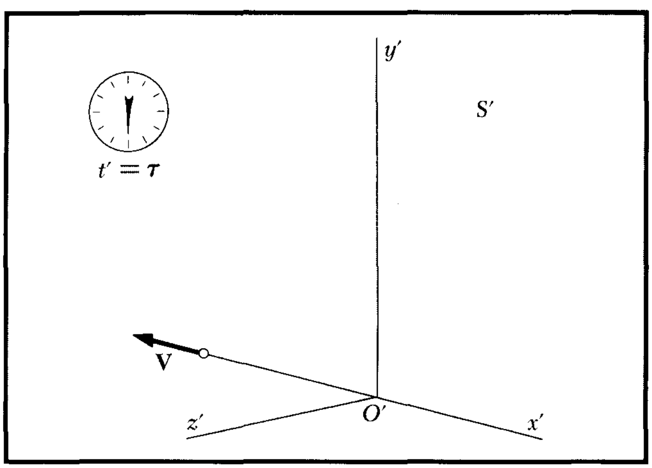


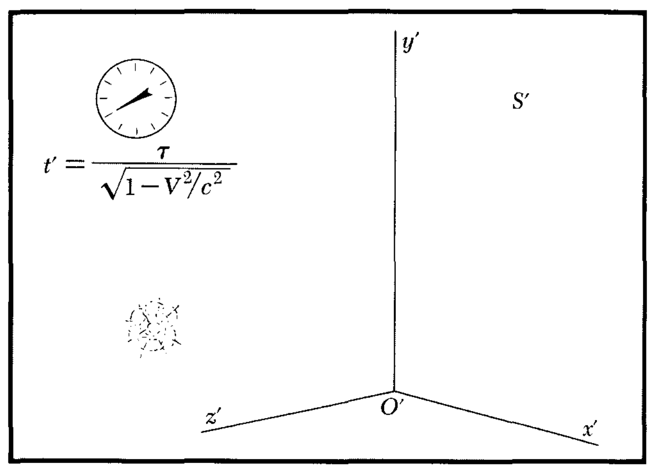
FIG. 11.7 (*cont'd*) (*d*) The same phenomenon observed from S'. Now the particle has speed V. We begin to observe it at $t' = 0 = t$.



(*e*) Time elapses.



(*f*) But at $t' = \tau$ the particle has not yet decayed!



(*g*) The particle decays at $t' = \tau(1 - V^2/c^2)^{-1/2}$ according to an observer in S'.

decay distance due to relativity. It has been said that almost every high-energy physicist tests special relativity every day. He uses the Lorentz transformation with the same confidence that physicists in the nineteenth century used Newton's laws.

We repeat that there is nothing mysterious about the clocks. If there is anything mysterious about special relativity, it is the constancy of the speed of light. Granted that, everything else follows directly and fairly simply. Every new situation must be analyzed carefully, however. The field is rich in apparent paradoxes. Perhaps the most famous of these is the twin paradox.[1]

These two effects, length contraction and time dilation, are the most famous effects predicted by special relativity and verified by experiment. However, there are many more effects that have been thoroughly verified by experiment and we give some of them below. First we shall discuss transformations of velocities. In the galilean transformation we saw that velocities in the x direction simply add, and so we might expect that when the velocities approach the speed of light they would also add. However, we have seen in Chap. 10 that the speed of light is the greatest possible speed, and therefore we must change our conception derived from the galilean transformation of how velocities add.

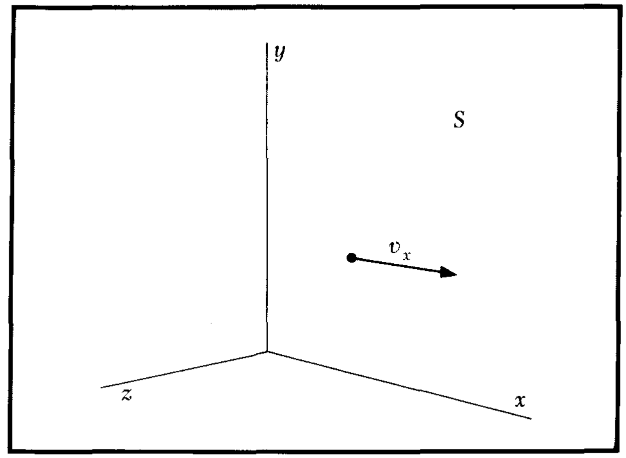


FIG. 11.8 (*a*) Suppose a particle has velocity v_x in S.

Velocity Transformation Suppose the S' reference frame moves with uniform velocity $V\hat{\mathbf{x}}$ relative to the S reference frame. A particle moves with uniform velocity components v_x, v_y, v_z relative to the S frame. What are the velocity components v_x', v_y', v_z' of the particle relative to the S' frame (see Fig. 11.8*a* and *b*)?

From Eq. (11.7) we have

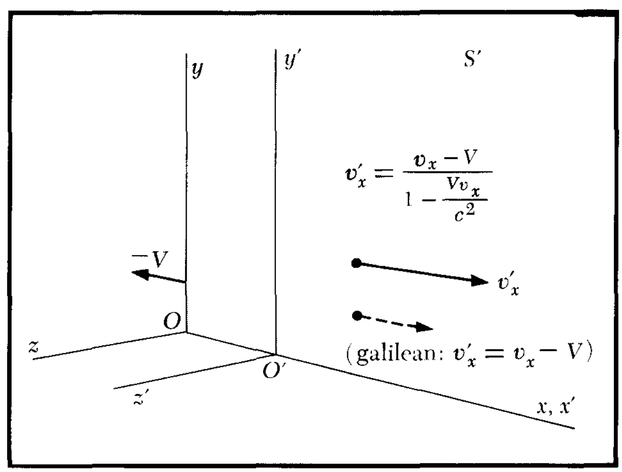
$$x' = \gamma(x - \beta ct) \qquad t' = \gamma\left(t - \frac{\beta x}{c}\right)$$

whence

$$dx' = \gamma\, dx - \gamma\beta c\, dt \qquad dt' = \gamma\, dt - \frac{\gamma\beta\, dx}{c}$$

Thus

$$v_x' = \frac{dx'}{dt'} = \frac{\gamma\, dx - \gamma\beta c\, dt}{\gamma\, dt - \gamma\beta\, dx/c} = \frac{v_x - \beta c}{1 - v_x\beta/c}$$



(*b*) Then in S' the Lorentz transformation predicts $v_x' = (v_x - V)/(1 - v_xV/c^2)$. The galilean transformation would predict $v_x' = v_x - V$.

[1]This problem has recently been raised again. See M. Sachs, *Physics Today*, **24**:23 (September 1971) and a group of letters in *Physics Today*, **25**:9 (January 1972).

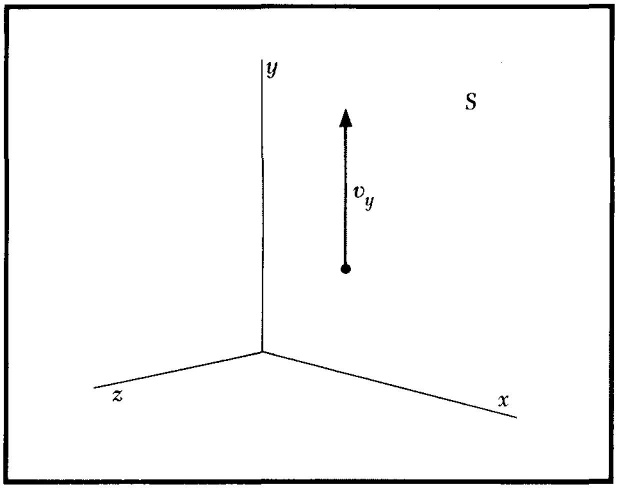


FIG. 11.9 (*a*) A particle has velocity v_y in the y direction in S.

or

$$v'_x = \frac{v_x - V}{1 - v_x V/c^2} = \frac{v_x - V}{1 - \beta v_x/c} \tag{11.15}$$

This result may be compared with the galilean result $v'_x = v_x - V$ as in Chap. 4. Similarly, because $y = y'$ and $z = z'$ (see Fig. 11.9*a* and *b*),

$$v'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma\, dt - \gamma\beta\, dx/c} = \frac{v_y}{1 - v_x V/c^2}\left(1 - \frac{V^2}{c^2}\right)^{\frac{1}{2}} = \frac{v_y}{\gamma(1 - \beta v_x/c)} \tag{11.16}$$

and

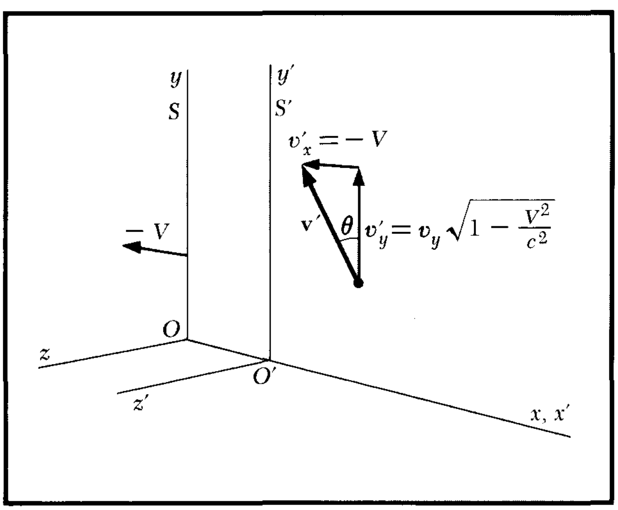
$$v'_z = \frac{v_z}{1 - v_x V/c^2}\left(1 - \frac{V^2}{c^2}\right)^{\frac{1}{2}} = \frac{v_z}{\gamma(1 - \beta v_x/c)} \tag{11.17}$$

The inverse transformations follow from Eq. (11.8) or by solving Eqs. (11.15) to (11.17) for the unprimed velocity components.

$$v_x = \frac{v'_x + V}{1 + v'_x V/c^2} = \frac{v'_x + V}{1 + \beta v'_x/c}$$

$$v_y = \frac{v'_y}{1 + v'_x V/c^2}\left(1 - \frac{V^2}{c^2}\right)^{\frac{1}{2}} = \frac{v'_y}{\gamma(1 + \beta v'_x/c)} \tag{11.18}$$

$$v_z = \frac{v'_z}{1 + v'_x V/c^2}\left(1 - \frac{V^2}{c^2}\right)^{\frac{1}{2}} = \frac{v'_z}{\gamma(1 + \beta v'_x/c)}$$



(*b*) Then it has the components shown in S', according to the Lorentz transformation

$$|\tan\theta| = \frac{V}{v_y\sqrt{1 - V^2/c^2}}$$

Note that for $V \ll c$, these reduce to the galilean transformation.

Suppose that the particle is a photon, and $v_x = c$ in S. From Eq. (11.15) we see that (Fig. 11.10)

$$v'_x = \frac{c - V}{1 - cV/c^2} = c$$

The velocity of the photon is also c in the frame S'. The Lorentz transformation was designed to produce this result, and it is a reassuring check that we obtain c in both reference frames.

If $v_y = c$ and $v_x = 0$, then (see Fig. 11.11)

$$v'_x = -V \qquad \text{and} \qquad v'_y = c\left(1 - \frac{V^2}{c^2}\right)^{\frac{1}{2}}$$

so that

$$\frac{v'_x}{v'_y} = -\frac{V}{c(1 - V^2/c^2)^{\frac{1}{2}}}$$

and

$$\sqrt{v'^2_x + v'^2_y} = c$$

EXAMPLE

Velocity Addition Suppose that two particles are traveling opposite to each other with velocity $v'_x = \pm 0.9c$ as observed in the S' system. What is the velocity of one particle with respect to the other, that is, as measured by the other? To solve this problem, let S be the reference frame in which the $-0.9c$ particle is at rest. Then the velocity of S' relative to S is $V = 0.9c$ so that the particle which in S' has velocity $v'_x = +0.9c$ has a velocity in S [see Eq. (11.18)]

$$v_x = \frac{v'_x + V}{1 + v'_x V/c^2} \approx \frac{1.8c}{1 + (0.9)^2} = \frac{1.80}{1.81}c = 0.994c$$

Notice that the relative velocity of the two particles is less than c.

If a photon is traveling at velocity $+c$ in S', and S' is traveling relative to S at velocity $+c$, the photon as viewed from S is traveling only at velocity $+c$, and not at $+2c$. This result is contained in Eq. (11.18). The fact of an ultimate speed is a consequence of the structure of the velocity-addition equations which we have derived from the Lorentz transformation. Note further that there is *no* frame in which a photon (light quantum) is at rest.

D. Sadeh has carried out [*Phys. Rev. Letters,* **10**:271 (1963)] a beautiful experiment which shows that the velocity of γ-rays is constant (±10 percent), independent of the velocity of the source, for a source velocity close to $\frac{1}{2}c$ compared with a source at rest.† We quote from his paper:

> In our experiments we used the annihilation in flight of positrons. In the annihilation the center-of-mass system of the positron and electron moves with a velocity close to $\frac{1}{2}c$, and two gamma rays are emitted. In the case of annihilation at rest, the two gamma rays are emitted at an angle of 180° and their velocity is c. In the case of annihilation in flight, the angle is smaller than 180° and depends on the energy of the positron. If the velocity of the gamma ray adds on to the velocity of the center of mass according to classical vector addition, and not according to the Lorentz transformation, then the gamma ray traveling with a component of motion in the direction of the positron flight will have a velocity greater than c, and that having a component in the opposite direction will have a velocity smaller than c. If it is found that the two gamma rays reach

† For a more detailed consideration, see J. G. Fox, J. Opt. Soc. Am. **57**:967 (1967).

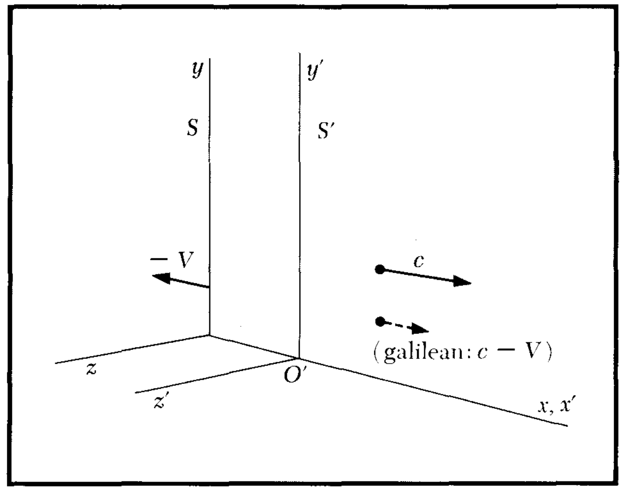


FIG. 11.10 As we know, if $v_x = c$, $v'_x = c$ also, according to the Lorentz transformation. *This* was built into our theory from the beginning.

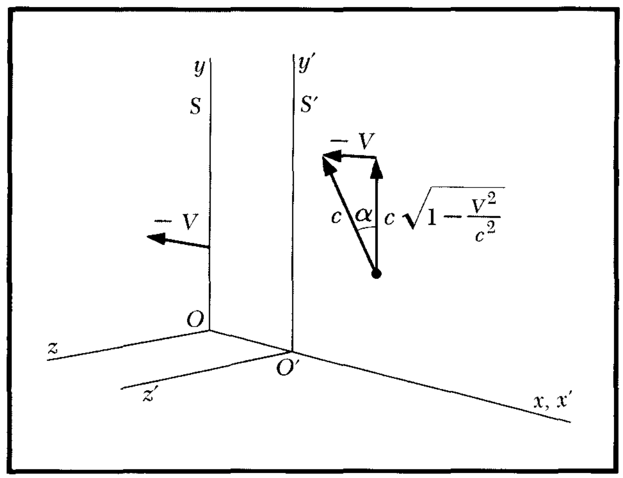


FIG. 11.11 In particular if $v_y = c$, the resultant has magnitude c in S'. Thus

$$|\tan \alpha| = \frac{V}{c\sqrt{1 - V^2/c^2}}$$

This is the relativistic theory of aberration.

the counters at the same time for equal distances between the counters and the point of annihilation, this would prove that even for a moving source the two gamma rays travel with the same velocity.

EXAMPLE

Aberration of Light We saw in Eq. (10.1) that for a star directly overhead (when the earth's velocity v_e is perpendicular to the line of observation) the tilt angle, or aberration, of the telescope is given by

$$\tan \alpha = \frac{v_e}{c} \tag{11.19}$$

This result was derived using a nonrelativistic argument. Now consider the problem relativistically as an exercise in the use of the Lorentz transformation.

Suppose that in reference frame S shown in Fig. 11.11 a star located at rest at O is observed by receiving light rays from it emitted along the y axis. What will be the trajectory in S′ of these rays that move along the y axis in S? In S their velocity components are $v_x = 0$, $v_y = c$, $v_z = 0$. The velocity components in S′ may be obtained by using Eqs. (11.15) to (11.17). Thus

$$v_x' = -V \qquad v_y' = \frac{c}{\gamma} \qquad v_z' = 0$$

So the direction of these rays in S′ is at the angle given by

$$\tan \alpha = \frac{-v_x'}{v_y'} = \frac{\gamma V}{c} = \beta\gamma = \frac{V/c}{\sqrt{1 - V^2/c^2}}$$

or

$$\sin \alpha = \frac{V}{c} = \beta \tag{11.20}$$

This is the correct result; it agrees within the accuracy of measurement with the nonrelativistic result of Eq. (11.19) only because V/c for the earth's motion is small, being approximately 10^{-4}.

EXAMPLE

Longitudinal Doppler Effect Consider two pulses of light sent out at $t = 0$ and $t = \tau$ by a transmitter at rest at $x = 0$ in reference frame S. Reference frame S′ moves with velocity $V\hat{\mathbf{x}}$ with respect to S. The initial pulse is received at $x' = 0$ in S′ at time $t' = 0$. The point in S′ which coincides with $x = 0$ at $t = \tau$ is given by the Lorentz transformation, Eq. (11.7),

$$x' = \frac{x - Vt}{(1 - \beta^2)^{\frac{1}{2}}} = \frac{-V\tau}{(1 - \beta^2)^{\frac{1}{2}}}$$

taking $x = 0$. The corresponding time in S′ is

$$t' = \frac{t - Vx/c^2}{(1-\beta^2)^{\frac{1}{2}}} = \frac{\tau}{(1-\beta^2)^{\frac{1}{2}}}$$

The time needed for the second pulse to travel in S' from $-V\tau/(1-\beta^2)^{\frac{1}{2}}$ to the origin is

$$\Delta t' = \frac{\tau V/c}{(1-\beta^2)^{\frac{1}{2}}}$$

so that the total time in S' between the reception at $x' = 0$ of the two pulses is

$$t' + \Delta t' = \tau \frac{1 + V/c}{(1-\beta^2)^{\frac{1}{2}}} = \tau \sqrt{\frac{1+\beta}{1-\beta}}$$

The time between the two signals can equally well be interpreted as the elapsed time between two successive nodes of a light wave. The frequency is the reciprocal of the period of the wave, so that

$$\nu' = \nu \sqrt{\frac{1-\beta}{1+\beta}} \tag{11.21}$$

Here ν' is the frequency as received in S', and ν is the frequency as transmitted in S. If the receiver is receding from the source, then $\beta = V/c$ is positive and ν' is less than ν. If the receiver is approaching the source, we take β to be negative and ν' is greater than ν. In terms of wavelength, $\lambda = c/\nu$ and $\lambda' = c/\nu'$, so that

$$\lambda' = \lambda \sqrt{\frac{1+\beta}{1-\beta}} \tag{11.22}$$

Equation (11.21) describes the relativistic longitudinal doppler effect for light waves in a vacuum. The frequency shift agrees to order β with the nonrelativistic result, Eq. (10.7) derived in Chap. 10.† The term of order β^2 in the series expansion of Eq. (11.21) has been confirmed experimentally by Ives and Stilwell.

H. E. Ives and G. R. Stilwell [*J. Opt. Soc. Am.*, **28**:215 (1938); **31**:369 (1941)] have carried out spectroscopic experiments on beams of hydrogen atoms in excited electronic states. The atoms were accelerated as molecular hydrogen ions H_2^+ and H_3^+ in an intense electric field. Atomic hydrogen was formed as a breakup product of the ions. The velocity of the atoms was of the order of $\beta = 0.005$. Ives and Stilwell looked for a shift in the *average* wavelength of a particular spectral line emitted by the hydrogen atoms. The average was taken over the forward and backward directions with respect to the line of flight of the atoms. From Eq. (11.22) we have, using $\beta_{\text{fwd}} = -\beta_{\text{bkwd}}$, the average wavelength

$$\frac{1}{2}(\lambda_{\text{fwd}} + \lambda_{\text{bkwd}}) = \frac{1}{2}\lambda_0\left(\sqrt{\frac{1-\beta}{1+\beta}} + \sqrt{\frac{1+\beta}{1-\beta}}\right)$$
$$= \frac{\lambda_0}{(1-\beta^2)^{\frac{1}{2}}} \tag{11.23}$$

† The reader should show that for $\beta \ll 1$, $\sqrt{(1+\beta)/(1-\beta)} \cong 1 + \beta$.

Thus there is a shift of order β^2 in the mean position of the displaced lines, with respect to the wavelength λ_0 emitted from an atom at rest. In their 1941 paper Ives and Stilwell report an observed shift of 0.074 Å in the average wavelength, as compared with the value 0.072 Å calculated from Eq. (11.23) for a value of β deduced from the accelerating potential applied to the original ions. This is an excellent confirmation of the theory of the relativistic doppler effect.

The *transverse* doppler effect applies to observations made at right angles to the direction of travel of the light source, which is usually an atom. In the nonrelativistic approximation there is no transverse doppler effect. A transverse doppler effect for light waves is predicted by relativity theory; the frequencies must be related as the inverse of the times in Eq. (11.11), so that

$$\nu' = (1 - \beta^2)^{\frac{1}{2}}\nu$$

where ν is the frequency in the frame in which the atom is at rest, and ν' is the frequency as observed in a frame moving with velocity $V(=\beta c)$ with respect to the atom.

Accelerated Clocks The special theory of relativity describes and relates measurements which are independent of the detailed structure of real bodies. It makes no prediction about the dynamical effects of acceleration, such as the stresses induced by acceleration. If such stresses are absent or may be ignored, the theory does give us an unambiguous description of the effect of acceleration on clock rates. The result is as if at each instant an accelerated clock had a different velocity, with a rate to be calculated using in Eq. (11.11) the appropriate instantaneous velocity.

If this prediction is correct, two consequences follow:

1 If the speed is constant but the direction varies, Eq. (11.11) holds without change. The frame of the clock is noninertial.
2 If the speed is constant except for brief moments of acceleration or deceleration (moments negligibly short in comparison with the total time), then Eq. (11.11) will still describe accurately the relation between the proper time and the stationary laboratory time.

A fast charged particle in a constant magnetic field experiences an acceleration perpendicular to its motion, but the speed never changes. If the particle is unstable, the measured half-life should be exactly the same as if it moved with the same speed in a straight line with no magnetic field present. This forecast is confirmed by experiments on the μ^- meson, which decays with a proper mean life of 2.2×10^{-6} s into an electron and neutrinos. The same proper lifetime is observed for μ^- mesons which are free or spiraling in a magnetic field or allowed to come to rest. It is believed that the special theory of relativity gives a good description of the circular (accelerated) motion of particles in a magnetic field.

PROBLEMS

1. *Lorentz invariant.* Verify from Eq. (11.7) that

$$x^2 - c^2t^2 = x'^2 - c^2t'^2$$

Note that if we write $x_1 \equiv x$; $x_4 \equiv ict$, then $x^2 - c^2t^2 \equiv x_1^2 + x_4^2$. Here $i = \sqrt{-1}$.

2. *Lorentz transformation.* Given Eq. (11.7), demonstrate Eq. (11.8).

3. *Change of volume.* Show that if L_0^3 is the rest volume of a cube, then

$$L_0^3(1 - \beta^2)^{\frac{1}{2}}$$

is the volume viewed from a reference frame moving with uniform velocity β in a direction parallel to an edge of the cube.

4. *Simultaneity.* Show from the Lorentz transformation that two events simultaneous ($t_1 = t_2$) at different positions ($x_1 \neq x_2$) in reference frame S are not in general simultaneous in reference frame S′.

5. *Change of angle.* Calculate in S′ the length and angle with the x' axis of a rod of length L_0 and angle θ with the x axis in S. S′ moves with velocity $V\hat{\mathbf{x}}$ with respect to S.

6. *Addition of velocities.* Show that if in the S′ frame we have $v_y' = c \sin\theta$ and $v_x' = c\cos\theta$, then in the S frame

$$v_x^2 + v_y^2 = c^2$$

The S′ frame moves with velocity $V\hat{\mathbf{x}}$ with respect to the S frame.

7. *π^+ mesons*

(*a*) What is the mean life of a burst of π^+ mesons traveling with $\beta = 0.73$? (The proper mean lifetime τ is 2.5×10^{-8} s.) *Ans.* 3.6×10^{-8} s.

(*b*) What distance is traveled at $\beta = 0.73$ during one mean life? *Ans.* 800 cm.

(*c*) What distance would be traveled without relativistic effects? *Ans.* 550 cm.

(*d*) Answer parts (*a*) to (*c*) again, but for $\beta = 0.99$.

8. *μ mesons.* The proper mean life of the μ meson is approximately 2×10^{-6} s. Suppose that a large burst of μ mesons produced at some height in the atmosphere travels downward at $v = 0.99c$. The number of collisions in the atmosphere on the way down is small.

(*a*) If 1 percent of those in the original burst survive to reach the earth's surface, estimate the original height. [In the μ meson frame of reference the number of particles which survive to a time t is given by $N(t) = N(0)e^{-t/\tau}$.] *Ans.* 2×10^6 cm.

(*b*) Calculate this distance of travel as measured by the μ meson.

9. *Two events.* Consider two inertial frames S and S′. Let S′ move with velocity $V\hat{\mathbf{x}}$, with respect to S. At a point x_1' an event takes place at time t_1'. At x_2' another event takes place at time t_2'. The origins coincide at time $t = t' = 0$. Find the corresponding times and distances in S.

10. *π^+ mesons.* A burst of 10^4 π^+ mesons travels in a circular path of radius 20 m at a speed $\beta = 0.99c$. The proper mean life of the π^+ meson is 2.5×10^{-8} s.

(*a*) How many survive when the burst returns to the point of origin?

(*b*) How many mesons would be left in a burst that had remained at rest at the origin for this same period of time?

11. *Recessional velocity of galaxy.* We stated in Chap. 10 that red-shift data on distant galaxies gave a velocity of recession proportional to distance, in the nonrelativistic region:

$$V = \alpha r \qquad \alpha \approx 1.6 \times 10^{-18}\ \text{s}^{-1}$$

Calculate the recession velocity of a galaxy at a distance of 3×10^9 light yr. Is this velocity relativistic? *Ans.* 4.5×10^9 cm/s.

12. *Galactic velocities.* We observe a galaxy receding in a particular direction at a speed $V = 0.3c$, and another receding

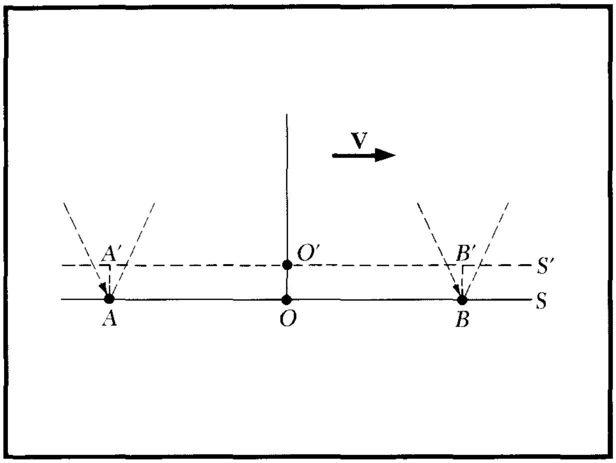


FIG. 11.12

in the opposite direction with the same speed. What speed of recession would an observer in one of these galaxies observe for the other galaxy?

13. *Simultaneity.* Consider the sources of two events to be located at rest at the points A and B, equal distances from the observer O in the frame S. Assume that at the particular instant of time (as determined by observer O in S) at which the two events occur, a second observer O' and his associated reference frame S', moving with a velocity $V\hat{\mathbf{x}}$ with respect to S, coincide with O and his frame S (see Fig. 11.12).

(*a*) Assume $V/c = \frac{1}{3}$. Sketch the positions of the two frames and the points A, A', B, B' when the signal from B arrives at the observer O'. Has this signal arrived at the observer O? Why?

(*b*) Sketch the positions of S and S' as both signals arrive at O.

(*c*) Sketch the positions of S and S' as the signal from A arrives at O'.

(*d*) Assume that the two events are recorded physically at the points A', B'; for example, on photographic plates. Show under the assumptions of this problem that the distances $A'O'$ and $B'O'$ are equal.

(*e*) Show that the two events are not simultaneous as viewed by O'. The constancy of the velocity of light under all circumstances is implicitly assumed in the definition of simultaneity. To make this dependence clear consider the following. Let the two events at A and B be the simultaneous radiation of pulses of sound as observed by O, an observer at rest with respect to the medium in which the sound is propagated. Let O' be an observer moving with a velocity V one-third that of sound.

(*f*) Use the galilean transformation to show that the velocity of the sound pulses toward O' from A and B are *not* the same as observed by O'.

(*g*) Show that even though the two signals arrive at O' at different times, the fact that the pulses have traveled with different velocities compensates for this fact and that the two events are inferred to be simultaneous, even by the observer O'.

14. *Relativistic doppler shift.* Protons are accelerated through a potential of 20 kV, after which they drift with constant velocity through a region where neutralization to H atoms and associated light emission takes place. The H_β emission ($\lambda = 4861.33$ Å for an atom at rest) is observed in a spectrometer. The optical axis of the spectrometer is parallel to the motion of the ions. The spectrum is doppler-shifted because of the motion of the ions in the direction of observed emission. The apparatus also contains a mirror which is placed so as to allow superposition of the spectrum of light emitted in the reverse direction. Recall that $1\ \text{Å} \equiv 10^{-8}$ cm.

(*a*) What is the velocity of the protons after acceleration? *Ans.* 2×10^8 cm/s.

(*b*) Calculate the first-order doppler shifts, depending on v/c, appropriate to the forward and backward directions, and indicate the appearance of the relevant part of the spectrum on a diagram.

(*c*) Now consider the second-order, or v^2/c^2, effect which arises from relativistic considerations. Show that the second-order shift is $= \frac{1}{2}\lambda(v^2/c^2)$, and evaluate this numerically for this problem. Notice that it is the same for both $+\mathbf{v}$ and $-\mathbf{v}$ motions. *Ans.* 0.10 Å.

FURTHER READING

In recent years a number of books on relativity have been published, some as paperbacks. The following constitute a selection.

J. A. Wheeler and E. F. Taylor, "Space-Time Physics—An Introduction," W. H. Freeman and Company, San Francisco, 1965. Highly recommended. Not a textbook; excellent for self-study.

A. P. French, "Special Relativity," W. W. Norton and Company, Inc., New York, 1968. One of the MIT Introductory Physics Series.

C. Kacser, "Introduction to the Special Theory of Relativity," Co-Op Paperback, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1967.

R. S. Shankland, "Conversations with Albert Einstein," *Am. J. Phys.*, **31**:47 (1963).

"Special Relativity Theory," selected reprints published for A.A.P.T., American Institute of Physics, 335 East 45th St., New York, 1962. This contains excellent discussion of the famous twin paradox; see especially the papers by Darwin, Crawford, and McMillan.

M. Born, "Einstein's Theory of Relativity," E. P. Dutton & Co., Inc., New York, 1924; reprint, Dover Publications, Inc., New York, 1962. A patient, full, and clear discussion of the special theory.

H. A. Lorentz, A. Einstein, H. Minkowski, and H. Weyl, "The Principle of Relativity: A Collection of Original Memoirs," translated by W. Perrett and G. B. Jeffery, Methuen & Co., Ltd., London, 1923; reprint, Dover Publications, Inc., New York, 1958.

W. Pauli, "Theory of Relativity," translated by G. Field, Pergamon Press, New York, 1958. A translation of an excellent monograph in German (Relativitätstheorie, published by B. G. Teubner, Leipzig, 1921). The contents of Part I are not difficult.

E. Whittaker, "History of the Theories of Aether and Electricity," 2 vols., Harper & Row, Publishers, Incorporated, New York, paperback reprint, 1960.