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Order driven markets : from empirical properties to optimal trading

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1. Introduction

2. Empirical evidence of the order flow dependency structure

3. Modelling with Hawkes processes

4. Some current research projects
   - Dependencies: a recurrence time perspective
   - Market making in order-driven markets

Why a microscopic description of financial markets ?

- Classical mathematical modelling of financial assets
  - directly describes the price as a stochastic process
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Motivation

A description at the **order level** provides a much better understanding of financial markets

Why a microscopic description of financial markets ?

A single trading day generates as many data points as **100 years** of close-to-close data (daily returns)

Why a microscopic description of financial markets ?

New paradigm for high frequency finance

- volatility is observable
- order flow is observable
- agent strategies are (partially) observable

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  - **limit order**: specify a price at which one is willing to buy (sell) a certain number of shares
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Price dynamics

The price dynamics becomes a by-product of the order book dynamics

Limit order book evolution

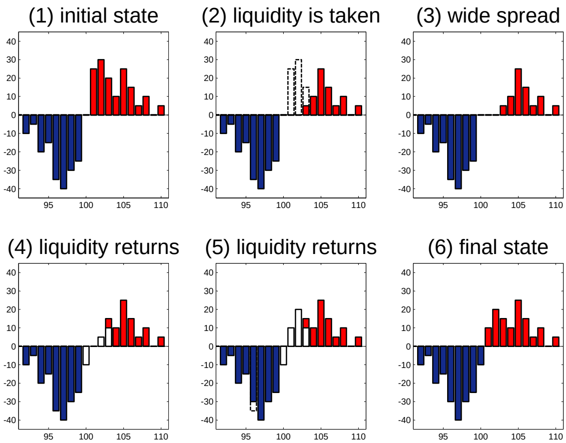

Figure: Dynamics of the order book

Motivations

$\Rightarrow$ To design micro-founded, realistic models of financial markets

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Today's questions of interest:

- dependencies;
- macroscopic behaviour (stationarity, ergodicity);
- price dynamics at high and low frequencies;
- trading strategies in a realistic framework;

Some notations

The limit order book is described as a (simple) point process;

- It is represented by a finite-size vector of quantities

$$\mathbf{X}(t) := (\mathbf{a}(t); \mathbf{b}(t)) := (a_1(t), \ldots, a_K(t); b_1(t), \ldots, b_K(t));$$

- $\mathbf{a}(t)$: ask side of the order book
- $\mathbf{b}(t)$: bid side of the order book
- ΔP: tick size
- q: unit volume
- $P = \frac{P^A + P^B}{2}$: mid-price
- $A(p), B(p)$: cumulative number of sell (buy) orders up to price level p

Limit order book representation

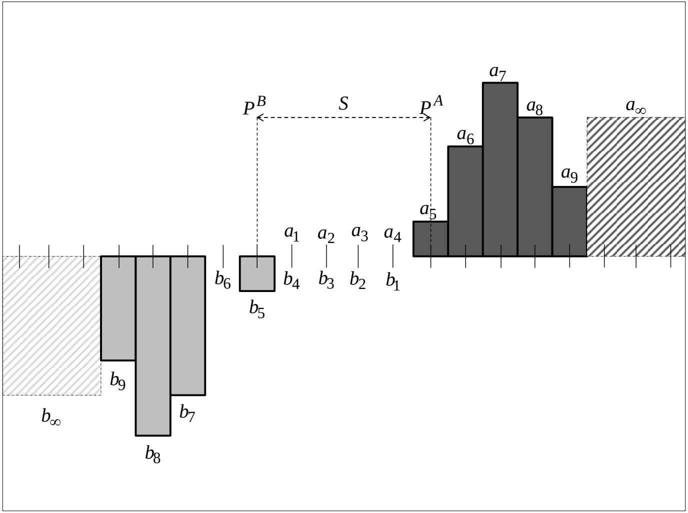

Figure: Order book notations

Plan

A coarser description of the order flow

In Abergel et al. (2016), order book events are clustered according to a coarser-grain description in order to identify some dependency structures - an approach similar to e.g. Muni Toke (2011)Cont et al. (2014)Eisler et al. (2012)

Notation	Definition
M^0_{buy}, M^0_{sell}	buy/sell market order that does not change the mid price
M^1_{buy}, M^1_{sell}	buy/sell market order that changes the mid price
L^0_{buy}, L^0_{sell}	buy/sell limit order that does not change the mid price
L^1_{buy}, L^1_{sell}	buy/sell limit order that changes the mid price
C^0_{buy}, C^0_{sell}	buy/sell cancellation that does not change the mid price
C^1_{buy}, C^1_{sell}	buy/sell cancellation that changes the mid price

Table: Event types definitions

Measuring dependencies in the order flow: passive orders

	L^0_{buy}	L^0_{sell}	C^0_{buy}	C^0_{sell}	M^0_{buy}	M^0_{sell}
$\|L^0_{buy}$	41.37	9.64	16.00	22.40	2.90	1.58
$\|L^0_{sell}$	9.61	41.79	21.95	16.12	1.61	2.96
$\|C^0_{buy}$	17.91	25.88	40.67	5.98	1.39	1.74
$\|C^0_{sell}$	25.18	17.98	6.04	41.30	1.79	1.42
$\|M^0_{buy}$	22.17	5.33	4.75	9.94	34.64	0.70
$\|M^0_{sell}$	5.60	21.14	10.61	5.01	0.72	34.32
$\|L^1_{buy}$	32.39	8.06	0.21	25.27	4.84	5.58
$\|L^1_{sell}$	7.65	29.94	26.04	0.22	5.63	5.62
$\|C^1_{buy}$	25.02	19.09	35.70	4.96	0.96	0.67
$\|C^1_{sell}$	21.48	23.28	5.42	34.70	0.76	1.16
$\|M^1_{buy}$	28.27	9.60	7.38	28.12	3.11	1.02
$\|M^1_{sell}$	11.83	23.05	33.36	7.24	1.04	3.13
$\|O$	22.82	22.93	19.80	20.03	2.99	3.00

Table: Conditional probabilities (in %) of occurrences per event type

Measuring dependencies in the order flow: aggressive orders

	L^1_{buy}	L^1_{sell}	C^1_{buy}	C^1_{sell}	M^1_{buy}	M^1_{sell}
$\|L^0_{buy}$	2.35	1.12	0.02	1.08	1.39	0.16
$\|L^0_{sell}$	1.02	2.29	1.05	0.02	0.15	1.44
$\|C^0_{buy}$	1.20	2.34	1.49	0.37	0.56	0.47
$\|C^0_{sell}$	2.08	1.27	0.37	1.49	0.51	0.60
$\|M^0_{buy}$	7.68	0.65	0.55	1.31	11.86	0.42
$\|M^0_{sell}$	0.53	7.19	1.48	1.10	0.42	11.88
$\|L^1_{buy}$	1.42	1.57	5.80	1.77	2.44	10.65
$\|L^1_{sell}$	1.39	1.36	1.42	5.39	12.37	2.96
$\|C^1_{buy}$	8.34	3.59	0.72	0.35	0.48	0.12
$\|C^1_{sell}$	3.20	7.88	0.63	0.75	0.18	0.57
$\|M^1_{buy}$	11.52	7.98	0.90	0.87	0.67	0.55
$\|M^1_{sell}$	6.79	9.34	1.05	1.81	0.66	0.70
$\|O$	2.07	2.12	0.85	0.88	1.27	1.26

Table: Conditional probabilities (in %) of occurrences per event type

Interpreting the results

M^0_{buy}: increases the probability of M^0_{buy}. This can be explained by *order splitting* - large orders are split into smaller pieces that are more easily executed - and the *momentum effect* - other participants following the move. The increase of the probability of M^1_{buy} and L^1_{buy} is also explained by the momentum effect.

L^1_{buy}: improves the offered price to buy the stock. The first major effect observed is a big increase in the probability of M^1_{sell} - this is the *market taking* effect. The second effect is a large increase in the probability of C^1_{buy} - the new liquidity is rapidly cancelled. This might reflect a market manipulation, where agents are posting fake orders.

M^1_{buy}: consumes all the offered liquidity at the best ask. This increases the probability of L^1_{sell} when some participants re-offer the liquidity at the same previous best ask price. It also increases the probability of L^1_{buy}, when a new consensus is concluded by the market participants at a higher price. This is the *market making* effect.

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The failures of zero-intelligence models

Elementary, ‘zero-intelligence ’models fail to capture the dependencies between various types of orders

- clustering of market orders;
- interplay between liquidity taking and providing;
- leverage effect,

that have just been identified.

Hawkes processes provide an *ad hoc* tool to describe the mutual excitations of the arrivals of different types of orders.
In D dimensions, a Hawkes process N_s^j has a stochastic intensity λ_t^j such that

$$\lambda_t^j = \lambda_0^j + \sum_{p=1}^{D} \int_0^t \phi_{jp}(t-s)dN_s^p \equiv \lambda_0^j + \sum_{p=1}^{D} \mu_t^{jp}. \tag{3.1}$$

A simplifying choice is the exponential kernel

$$\phi_{jp}(s) = \alpha_{jp} \exp(-\beta_{jp} s) \tag{3.2}$$

leading to markovian processes.
A classical result states that the process is stationary *iff* the spectral radius of the matrix

$$[\frac{\alpha_{jp}}{\beta_{jp}}] \tag{3.3}$$

is < 1, see Massoulié (1998).

$$
\begin{aligned}
\mathcal{L}F(\vec{a};\vec{b};\vec{\mu}) = &\lambda^{M^+}\left(F\left([a_i-(q-A(i-1))_+]_+; J^{M^+}(\vec{b}); \vec{\mu}+\Delta^{M^+}(\vec{\mu})\right)-F\right)\\
&+\sum_{i=1}^{K}\lambda_i^{L^+}\left(F\left(a_i+q; J^{L_i^+}(\vec{b}); \vec{\mu}+\Delta^{L_i^+}(\vec{\mu})\right)-F\right)\\
&+\sum_{i=1}^{K}\lambda_i^{C^+}a_i\left(F\left(a_i-q; J^{C_i^+}(\vec{b})\right)-F\right)\\
&+\lambda^{M^-}\left(F\left(J^{M^-}(\vec{a}); [b_i+(q-B(i-1))_+]_-; \vec{\mu}+\Delta^{M^-}(\vec{\mu})\right)-F\right)\\
&+\sum_{i=1}^{K}\lambda_i^{L^-}\left(F\left(J^{L_i^-}(\vec{a}); b_i-q; \vec{\mu}+\Delta^{L_i^-}(\vec{\mu})\right)-F\right)\\
&+\sum_{i=1}^{K}\lambda_i^{C^-}|b_i|\left(f\left(J^{C_i^-}(\vec{a}); b_i+q\right)-f\right)\\
&-\sum_{i,j=1}^{D}\beta_{ij}\mu_{ij}\frac{\partial F}{\partial \mu_{ij}}. \qquad (3.4)
\end{aligned}
$$

Large-time behaviour

In Abergel and Jedidi (2015), we identify the

Large-time behaviour for Hawkes-process driven LOB

- Under the usual stationarity conditions for the intensities, there exists a Lyapunov function $V = \sum |a_i| + \sum |b_i| + \sum \delta_{jk}\mu_{jk}$ and the LOB converges exponentially to its stationary distribution Π
- The rescaled, (deterministically) centered price converges to a Wiener process

The proofs rely on the assumption that the **proportional** cancellation rate remains bounded away from zero
Some extra care is required to prove that the solution to the Poisson equation is in $\mathbf{L}^2(\Pi(dx))$

A general approach to study price asymptotics

Over the past few years, several papers have addressed the question of long-time price and order book asymptotics Cont and de Larrard (2012)Abergel and Jedidi (2013)Abergel and Jedidi (2015)Horst and Paulsen (2015)Huang and Rosenbaum (2015)...

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Combining ergodic theory and martingale convergence

The ergodicity of the order book allows for a direct study of the asymptotic behaviour of the price, based on Foster-Lyapunov-type criteria Meyn and Tweedie (1993)Glynn and Meyn (1996) and the convergence of martingales Ethier and Kurtz (2005):

- the evolution of the price is $dP_t = \sum_{i=1}^{K'} F_i(X_t) dN_t^i$;
- the rescaled, centered price is $\tilde{P}_t^n \equiv \frac{P_{nt} - \int_0^{nt} \sum_{i=1}^{K'} F_i(X_s)\lambda_i ds}{\sqrt{n}}$
- its predictable quadratic variation is $< \tilde{P}^n, \tilde{P}^n >_t = \frac{\int_0^{nt} \sum_{i=1}^{K'} (F_i(X_s))^2 \lambda_i ds}{n}$
- ergodicity ensures the convergence of $\frac{\int_0^{nt} \sum_{i=1}^{K'} (F_i(X_s))^2 \lambda_i ds}{nt}$ as $n \to \infty$

Of course, some care is needed in order to characterize the asymptotic, **deterministic** drift, see Abergel and Jedidi (2015), but that's the idea...

Plan

Current projects

- Dependencies: a recurrence time perspective
  With M. Anane, X. Lu
- Optimal trading: stochastic control of order books *via* limit orders
  With C. Huré, H. Pham

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Inter-arrival times for ALV

As a by-product, a good model should reproduce empirical patterns at all scales (higher and lower frequencies) of inter-arrival times

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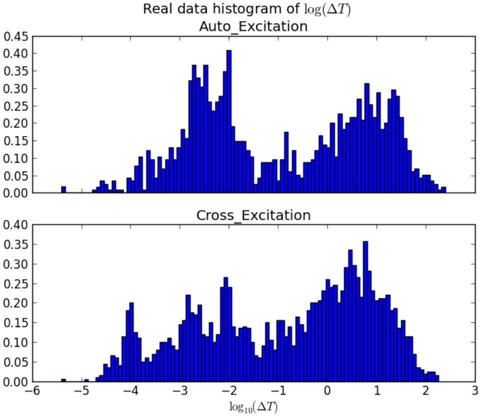
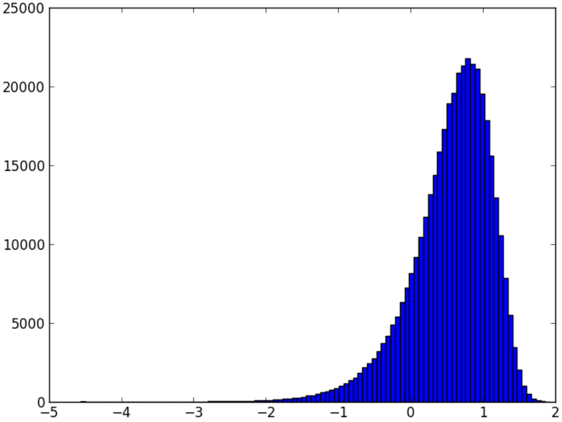
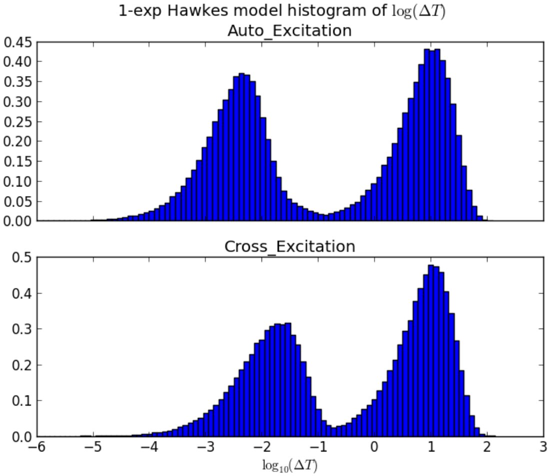


Such an analysis is performed in Abergel et al. (a)

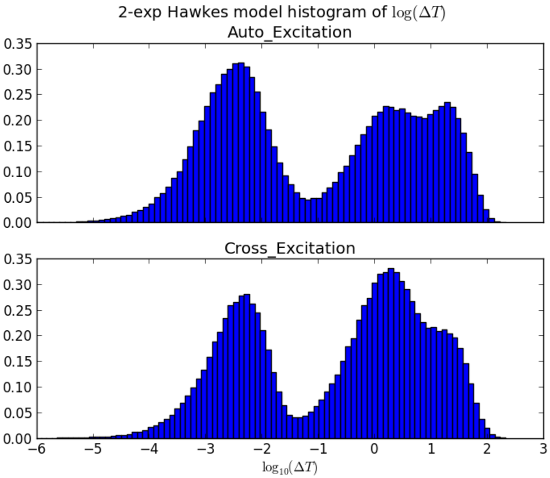
When compared to the data, Poisson arrival times (as in a zero-intelligence LOB models) fail to reproduce the observed phenomena



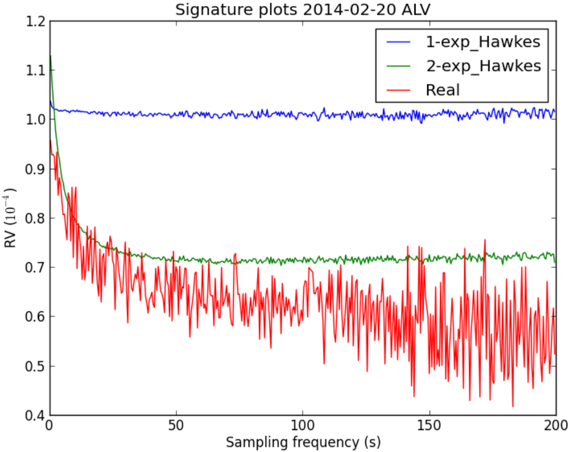
Hawkes inter-arrival times



Better Hawkes inter-arrival times



Better Hawkes inter-arrival times



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Surviving in order-driven markets I

The business model of high frequency trading:

- find α;
- fight against those trying to make the spread !

A citation from Hemmelgarn et al. (2015): '*The national FTT in place in Greece, France, and Italy contains specific exemptions for market making activities in view of their perceived positive influence on market liquidity. The main difficulty in dealing with market making is separating it from proprietary trading* '

In many markets, for many players, market making has become an almost mandatory, hardly profitable business. Optimal market making - or: optimal liquidity providing - is therefore a very important practical question.

In Abergel et al. (b), we study it from a theoretical and numerical point of view.

The theoretical framework

(Markovian) limit order book models with (Markovian) controls and possibly state-dependent intensities have an infinitesimal generator of the form

$$\mathcal{L}_\alpha f(\mathbf{k}) = \sum_{i=1}^{P} \lambda_i(t, \mathbf{k}, \alpha) \left(B_\alpha^i - Id \right) (f)(\mathbf{k}).$$

The Kolmogorov backward equation

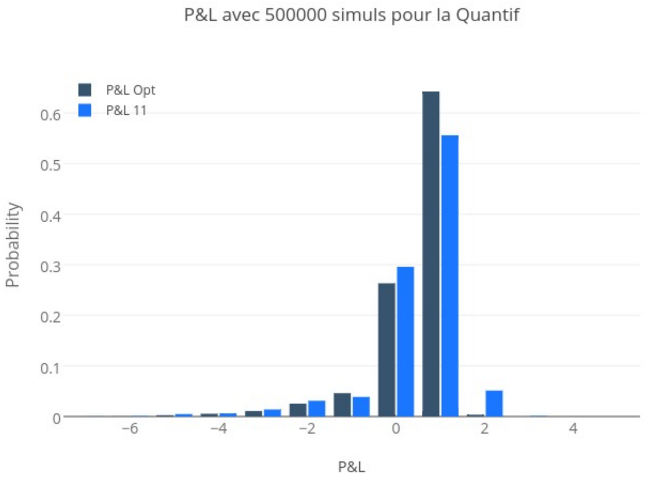
$$\begin{aligned} \frac{du}{dt} + \mathcal{L}_\alpha u &= 0,\ 0 \leqslant t \leqslant T, \\ u(T, z) &= \Phi(Z), \end{aligned}$$

leads, *via* dynamic programming, to the corresponding HJB equation

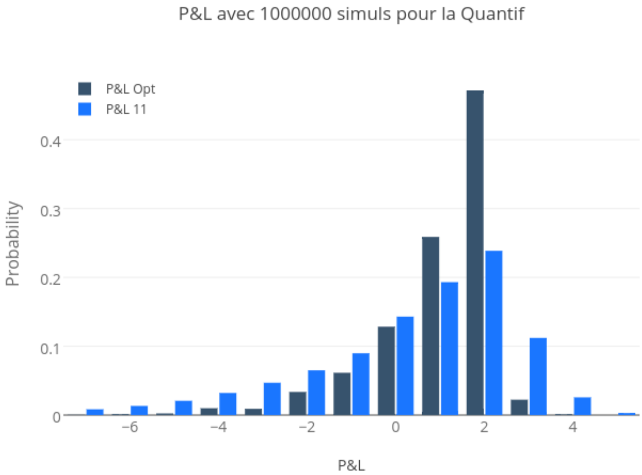
$$\begin{aligned} \frac{dv}{dt} + \sup_\alpha \left(\mathcal{L}_\alpha v \right) &= 0,\ 0 \leqslant t \leqslant T, \\ v(T, z) &= \Phi(Z), \end{aligned} \tag{4.1}$$

The main theoretical result in Abergel et al. (b) is the well-posedness of (4.1), and the associated verification theorem.

Optimal market making in a zero-intelligence model



Optimal market making in a model with state dependent intensities (inspired by Huang et al. (2015))



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