Order driven markets: from empirical properties to optimal trading

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Latin American School and Workshop on Data Analysis and Mathematical Modelling of Social Sciences

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2  Empirical evidence of the order flow dependency structure

3  Modelling with Hawkes processes

4  Some current research projects
   - Dependencies: a recurrence time perspective
   - Market making in order-driven markets
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   - Dependencies: a recurrence time perspective
   - Market making in order-driven markets
Why a microscopic description of financial markets?

- Classical mathematical modelling of financial assets
  - directly describes the price as a stochastic process
  - imposes drastic limitations on trading strategies and agent behaviour
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- The information contained in high-frequency financial data allows one to
  - relate the price evolution to the microstructure of the market
  - explore the strategies of financial agents
Why a microscopic description of financial markets?

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- directly describes the price as a stochastic process
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The information contained in high-frequency financial data allows one to
- relate the price evolution to the microstructure of the market
- explore the strategies of financial agents

Motivation

A description at the order level provides a much better understanding of financial markets
Why a microscopic description of financial markets?

A single trading day generates as many data points as **100 years** of close-to-close data (daily returns)
Why a microscopic description of financial markets?

New paradigm for high frequency finance

- volatility is observable
- order flow is observable
- agent strategies are (partially) observable
The limit order book is the list, at a given time, of all buy and sell limit orders, with their corresponding prices and volumes.

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3 main types of orders:
- **limit order**: specify a price at which one is willing to buy (sell) a certain number of shares.
- **market order**: immediately buy (sell) a certain number of shares at the best available opposite quote(s).
- **cancellation order**: cancel an existing limit order.

Price dynamics becomes a by-product of the order book dynamics.
What is a limit order book?

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**Price dynamics**

The price dynamics becomes a by-product of the order book dynamics.
Limit order book evolution

Figure: Dynamics of the order book
Motivations

⇒ To design micro-founded, realistic models of financial markets
To design micro-founded, realistic models of financial markets

Today’s questions of interest:
- dependencies;
- macroscopic behaviour (stationarity, ergodicity);
- price dynamics at high and low frequencies;
- trading strategies in a realistic framework;
Some notations

The limit order book is described as a (simple) point process;

- It is represented by a finite-size vector of quantities

\[ X(t) := (a(t); b(t)) := (a_1(t), \ldots, a_K(t); b_1(t), \ldots, b_K(t)); \]

- \( a(t) \): ask side of the order book
- \( b(t) \): bid side of the order book
- \( \Delta P \): tick size
- \( q \): unit volume
- \( P = \frac{P_A + P_B}{2} \): mid-price
- \( A(p), B(p) \): cumulative number of sell (buy) orders up to price level \( p \)
Figure: Order book notations
Plan

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A coarser description of the order flow

In Abergel et al. (2016), order book events are clustered according to a coarser-grain description in order to identify some dependency structures - an approach similar to e.g. Muni Toke (2011) Cont et al. (2014) Eisler et al. (2012)

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
</tr>
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<tbody>
<tr>
<td>$M^0_{buy}, M^0_{sell}$</td>
<td>buy/sell market order that does not change the mid price</td>
</tr>
<tr>
<td>$M^1_{buy}, M^1_{sell}$</td>
<td>buy/sell market order that changes the mid price</td>
</tr>
<tr>
<td>$L^0_{buy}, L^0_{sell}$</td>
<td>buy/sell limit order that does not change the mid price</td>
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<tr>
<td>$L^1_{buy}, L^1_{sell}$</td>
<td>buy/sell limit order that changes the mid price</td>
</tr>
<tr>
<td>$C^0_{buy}, C^0_{sell}$</td>
<td>buy/sell cancellation that does not change the mid price</td>
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<tr>
<td>$C^1_{buy}, C^1_{sell}$</td>
<td>buy/sell cancellation that changes the mid price</td>
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Table: Event types definitions
Measuring dependencies in the order flow: passive orders

<table>
<thead>
<tr>
<th>Event Type</th>
<th>$L^0_{buy}$</th>
<th>$L^0_{sell}$</th>
<th>$C^0_{buy}$</th>
<th>$C^0_{sell}$</th>
<th>$M^0_{buy}$</th>
<th>$M^0_{sell}$</th>
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<tr>
<td>$L^0_{buy}$</td>
<td>41.37</td>
<td>9.64</td>
<td>16.00</td>
<td>22.40</td>
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<td>1.58</td>
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<td>$L^0_{sell}$</td>
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<td>21.95</td>
<td>16.12</td>
<td>1.61</td>
<td>2.96</td>
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<tr>
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<td>5.98</td>
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<td>1.74</td>
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<td>22.93</td>
<td>19.80</td>
<td>20.03</td>
<td>2.99</td>
<td>3.00</td>
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Table: Conditional probabilities (in %) of occurrences per event type
Measuring dependencies in the order flow: aggressive orders

<table>
<thead>
<tr>
<th></th>
<th>$L_0^{buy}$</th>
<th>$L_0^{sell}$</th>
<th>$C_0^{buy}$</th>
<th>$C_0^{sell}$</th>
<th>$M_0^{buy}$</th>
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<td>$C_1^{sell}$</td>
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<td>$M_1^{buy}$</td>
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<td>$M_1^{sell}$</td>
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<td>$L_1^{buy}$</td>
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<tr>
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<td>1.39</td>
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<td>$C_1^{sell}$</td>
<td>3.20</td>
<td>7.88</td>
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<td>$M_1^{buy}$</td>
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<td>7.98</td>
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<tr>
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<td>0.85</td>
<td>0.88</td>
<td>1.27</td>
<td>1.26</td>
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</tbody>
</table>

**Table**: Conditional probabilities (in %) of occurrences per event type
Interpreting the results

$M^0_{buy}$ : increases the probability of $M^0_{buy}$. This can be explained by order splitting - large orders are split into smaller pieces that are more easily executed - and the momentum effect - other participants following the move. The increase of the probability of $M^1_{buy}$ and $L^1_{buy}$ is also explained by the momentum effect.

$L^1_{buy}$ : improves the offered price to buy the stock. The first major effect observed is a big increase in the probability of $M^1_{sell}$ - this is the market taking effect. The second effect is a large increase in the probability of $C^1_{buy}$ - the new liquidity is rapidly cancelled. This might reflect a market manipulation, where agents are posting fake orders.

$M^1_{buy}$ : consumes all the offered liquidity at the best ask. This increases the probability of $L^1_{sell}$ when some participants re-offer the liquidity at the same previous best ask price. It also increases the probability of $L^1_{buy}$, when a new consensus is concluded by the market participants at a higher price. This is the market making effect.
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Elementary, ‘zero-intelligence ’ models fail to capture the dependencies between various types of orders

- clustering of market orders;
- interplay between liquidity taking and providing;
- leverage effect,

that have just been identified.
Hawkes processes provide an ad hoc tool to describe the mutual excitations of the arrivals of different types of orders. In $D$ dimensions, a Hawkes process $N^i_s$ has a stochastic intensity $\lambda^i_t$ such that

$$\lambda^i_t = \lambda^i_0 + \sum_{p=1}^{D} \int_0^t \phi_{jp}(t-s) dN^p_s \equiv \lambda^i_0 + \sum_{p=1}^{D} \mu^i_p. \quad (3.1)$$

A simplifying choice is the exponential kernel

$$\phi_{jp}(s) = \alpha_{jp} \exp(-\beta_{jp}s) \quad (3.2)$$

leading to markovian processes. A classical result states that the process is stationary iff the spectral radius of the matrix

$$\left[ \begin{array}{c} \alpha_{jp} \\ \beta_{jp} \end{array} \right] \quad (3.3)$$

is $< 1$, see Massoulié (1998).
\[ \mathcal{L}F(\vec{a}; \vec{b}; \vec{\mu}) = \lambda^{M^{+}}(F\left([a_i - (q - A(i - 1))_+; J^{M^{+}}(\vec{b}); \vec{\mu} + \Delta^{M^{+}}(\vec{\mu})\right) - F) \\
+ \sum_{i=1}^{K} \lambda^{L^{+}}_i (F(a_i + q; J^{L^{+}}(\vec{b}); \vec{\mu} + \Delta^{L^{+}}(\vec{\mu})) - F) \\
+ \sum_{i=1}^{K} \lambda^{C^{+}}_i a_i (F(a_i - q; J^{C^{+}}(\vec{b})) - F) \\
+ \lambda^{M^{-}}(F(J^{M^{-}}(\vec{a}); [b_i + (q - B(i - 1))_+; \vec{\mu} + \Delta^{M^{-}}(\vec{\mu})) - F) \\
+ \sum_{i=1}^{K} \lambda^{L^{-}}_i (F(J^{L^{-}}(\vec{a}); b_i - q; \vec{\mu} + \Delta^{L^{-}}(\vec{\mu})) - F) \\
+ \sum_{i=1}^{K} \lambda^{C^{-}}_i |b_i|(f(J^{C^{-}}(\vec{a}); b_i + q) - f) \\
- \sum_{i,j=1}^{D} \beta_{ij} \mu_{ij} \frac{\partial F}{\partial \mu_{ij}}. \] (3.4)
In Abergel and Jedidi (2015), we identify the

**Large-time behaviour for Hawkes-process driven LOB**

- Under the usual stationarity conditions for the intensities, there exists a Lyapunov function $V = \sum |a_i| + \sum |b_i| + \sum \delta_{jk} \mu_{jk}$ and the LOB converges exponentially to its stationary distribution $\Pi$.
- The rescaled, (deterministically) centered price converges to a Wiener process.

The proofs rely on the assumption that the **proportional** cancellation rate remains bounded away from zero. Some extra care is required to prove that the solution to the Poisson equation is in $L^2(\Pi(dx))$. 
A general approach to study price asymptotics


Combining ergodic theory and martingale convergence


\[ \frac{dP}{t} = \sum_{K_i} F_i(X_t) dN_i(t) \]

the rescaled, centered price is

\[ \tilde{P}_n = P_n - \int_0^n \sum_{K_i} F_i(X_s) \lambda_i ds \]

its predictable quadratic variation is

\[ \langle \tilde{P}_n, \tilde{P}_n \rangle = \int_0^n \sum_{K_i} (F_i(X_s))^2 \lambda_i ds \]

ergodicity ensures the convergence of

\[ \int_0^n \sum_{K_i} F_i(X_s)^2 \lambda_i ds \]

as \( n \to \infty \)

Of course, some care is needed in order to characterize the asymptotic, deterministic drift, see Abergel and Jedidi (2015), but that's the idea...
A general approach to study price asymptotics

Over the past few years, several papers have addressed the question of long-time price and order book asymptotics: Cont and de Larrard (2012), Abergel and Jedidi (2013), Abergel and Jedidi (2015), Horst and Paulsen (2015), Huang and Rosenbaum (2015),... 

Combining ergodic theory and martingale convergence

The ergodicity of the order book allows for a direct study of the asymptotic behaviour of the price, based on Foster-Lyapunov-type criteria: Meyn and Tweedie (1993), Glynn and Meyn (1996) and the convergence of martingales: Ethier and Kurtz (2005):

- the evolution of the price is $dP_t = \sum_{i=1}^{K'} F_i(X_t) dN_i$;
- the rescaled, centered price is $\tilde{P}_t^n \equiv \frac{P_{nt} - \int_0^{nt} \sum_{i=1}^{K'} F_i(X_s) \lambda_i ds}{\sqrt{n}}$;
- its predictable quadratic variation is $\langle \tilde{P}_t^n, \tilde{P}_t^n \rangle_t = \frac{\int_0^{nt} \sum_{i=1}^{K'} (F_i(X_s))^2 \lambda_i ds}{nt}$;
- ergodicity ensures the convergence of $\int_0^{nt} \sum_{i=1}^{K'} (F_i(X_s))^2 \lambda_i ds$ as $n \to \infty$.

Of course, some care is needed in order to characterize the asymptotic, deterministic drift, see Abergel and Jedidi (2015), but that's the idea...
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Current projects

- Dependencies: a recurrence time perspective
  With M. Anane, X. Lu

- Optimal trading: stochastic control of order books via limit orders
  With C. Huré, H. Pham
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As a by-product, a good model should reproduce empirical patterns at all scales (higher and lower frequencies) of inter-arrival times.
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Such an analysis is performed in Abergel et al. (a)
When compared to the data, Poisson arrival times (as in a zero-intelligence LOB models) fail to reproduce the observed phenomena.
Hawkes inter-arrival times

1-exp Hawkes model histogram of $\log(\Delta T)$

Auto Excitation

Cross Excitation

$\log_{10}(\Delta T)$
Better Hawkes inter-arrival times

Signature plots 2014-02-20 ALV

1-exp_Hawkes
2-exp_Hawkes
Real

RV ($10^{-4}$) vs. Sampling frequency (s)
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The business model of high frequency trading:

- find $\alpha$;
- fight against those trying to make the spread!

A citation from Hemmelgarn et al. (2015): ‘The national FTT in place in Greece, France, and Italy contains specific exemptions for market making activities in view of their perceived positive influence on market liquidity. The main difficulty in dealing with market making is separating it from proprietary trading’

In many markets, for many players, market making has become an almost mandatory, hardly profitable business. Optimal market making - or: optimal liquidity providing - is therefore a very important practical question.

In Abergel et al. (b), we study it from a theoretical and numerical point of view.
(Markovian) limit order book models with (Markovian) controls and possibly state-dependent intensities have an infinitesimal generator of the form

\[ \mathcal{L}_\alpha f(k) = \sum_{i=1}^{P} \lambda_i(t, k, \alpha) \left( B^i_\alpha - \text{Id} \right)(f)(k). \]

The Kolmogorov backward equation

\[ \frac{du}{dt} + \mathcal{L}_\alpha u = 0, \quad 0 \leq t \leq T, \]

\[ u(T, z) = \Phi(Z), \]

leads, via dynamic programming, to the corresponding HJB equation

\[ \frac{dv}{dt} + \sup_{\alpha} (\mathcal{L}_\alpha v) = 0, \quad 0 \leq t \leq T, \]

\[ v(T, z) = \Phi(Z), \quad (4.1) \]

The main theoretical result in Abergel et al. (b) is the well-posedness of (4.1), and the associated verification theorem.
Some numerical results

Optimal market making in a zero-intelligence model

P&L avec 500000 simuls pour la Quantif

Probability

P&L

P&L Opt
P&L 11
Some numerical results

Optimal market making in a model with state dependent intensities (inspired by Huang et al. (2015))

![Histogram of P&L](image)


Abergel, F., Huré, C., and Pham, H. Optimal market making strategies in order-driven markets.


F. Abergel Order driven markets : from empirical properties to optimal trading 37/38
