EFFECTIVE-MASS THEORY

Lecture #7

8

Landau-Levels
Franz-Keldysh eff
~~[illegible]~~

Ziman, Ch. 6 + Jones/March Vol. 2 A10.4

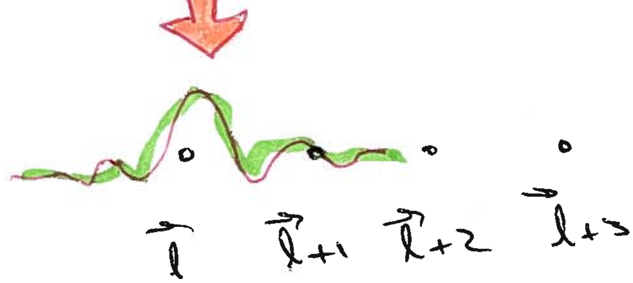
$$\hat{H} = \underbrace{H_0^{(0)}}_{\text{electron in periodic lattice}} + \underbrace{\hat{V}(\vec{r},t)}_{\text{external perturbation}}$$

$$\Psi = \sum_{n,\vec{l}} \overbrace{f_n(\hat{l},t)}^{\text{envelope function}} \underbrace{a_n(\vec{r}-\vec{l})}_{\text{WANNIER FUNCTION}}$$

lattice site

n labels a band

$$\boxed{a_n(\vec{r}-\vec{l}) = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{l}}\, \psi_{\vec{k},\nu}(\vec{r})}$$



$$\left[\mathcal{E}_n(-i\vec{\nabla}) f_n - i\hbar \frac{\partial f_n}{\partial t}\right]_{\vec{l}} +$$

EXACT

$$+ \sum_{n',l'} V_{nn'}(\vec{l},\vec{l}')\, f_{n'}(l',t) = 0$$

TO GET EFF- MASS EQ.:

ASSUMPTIONS

1. $V_{nn'}(\ell, \ell') \approx V(\vec{r})\Big|_{\vec{r}=\vec{\ell}}$

TRUE IF $\hat{V}$ IS SLOWLY VARYING BECAUSE OF ORTHOGONALITY OF WANNIER FUNCTIONS

2. $\mathcal{E}(\vec{k}) \approx \dfrac{\hbar^2 k^2}{2m^*}$

IGNORE HIGHER-ORDER TERMS

THEN

$$\left[-\frac{\hbar^2}{2m^*}\nabla^2 - i\hbar\frac{\partial}{\partial t}\right] f_n + \hat{V}(\vec{r}) f_n = 0$$

EFFECTIVE - MASS EQUATION

$$\Psi_n \approx \sum_{\vec{\ell},\vec{k}} \frac{f_n(\vec{\ell},t)}{\sqrt{N}} e^{-i\vec{k}\cdot\vec{\ell}}\, \psi_{\vec{k},n}(\vec{r})$$

$$\psi_n \approx \frac{1}{\sqrt{N}} \sum_{\vec{l},\vec{k}} f_n(\vec{l},t)\, e^{i\vec{k}(\vec{r}-\vec{l})}\, u_{\vec{k},n}(\vec{r})$$

SINCE $\hat{V}$ IS SLOWLY VARYING THE FOURIER COMPONENTS OF f_n ARE NON-ZERO FOR $|\vec{k}| \sim \frac{1}{l_{\hat{V}}}$

$$\Rightarrow \psi_n \approx \underbrace{f_n(\vec{r},t)}_{\text{ENVELOPE}}\, u_{\vec{k}=0,n}(\vec{r})$$

IF WE ARE INTERESTED IN VALUES OF $\vec{k} \neq 0$, SUCH THAT

$$E \approx E_0 + \frac{\hbar^2}{2m}(\vec{k}-\vec{k}_0)^2$$

$$\Rightarrow -\frac{\hbar^2}{2m^*}\left[e^{-i\vec{k}_0\cdot\vec{r}}\frac{\partial}{\partial\vec{r}}e^{i\vec{k}_0\cdot\vec{r}}\right]^2 f$$

$$-i\hbar\frac{\partial f}{\partial t} + \hat{V}(\vec{r})\, f = 0$$

$$\Rightarrow -\frac{\hbar^2}{2m}\nabla^2\left(e^{i\vec{k}_0\cdot\vec{r}}\underbrace{f}_{\text{SLOWLY VARYING}}\right) + \hat{V}\left(e^{i\vec{k}_0\cdot\vec{r}} f\right) = i\hbar\frac{\partial}{\partial t}\left(e^{i\vec{k}_0\cdot\vec{r}} f\right)$$

OPTICAL SELECTION RULES

- $\Psi_v = f_v(\vec{r})\, u_v(\vec{r})$

$f_{c,v}$: SLOWLY VARYING

$$\Psi_c = f_c(\vec{r})\, u_c(\vec{r})$$

$$\langle \Psi_c | \vec{A}_0 \cdot \vec{p} | \Psi_v \rangle \approx \langle f_c | f_v \rangle \underbrace{\langle u_c | \vec{A}_0 \cdot \vec{p} | u_v \rangle}_{\text{ALREADY DISCUSSED}}$$

$\boxed{\text{EXAMPLE \# 1}}$ LANDAU-LEVELS

- FOR CONSTANT MAGNETIC FIELD, PARALLEL TO $\hat{z}$

TAKE $A_x = -B_0 y \quad A_y = A_z = 0$

(alternatively, $\vec{A} = \frac{1}{2}\vec{B}_0 \times \vec{r}$)

ORIGINAL PROBLEM:

$$\hat{H} = \frac{1}{2\underbrace{m}_{\text{BARE MASS}}}\left[\left(\hat{P}_x + \frac{eB_0}{c} y\right)^2 + P_y^2 + P_z^2\right] + \underbrace{U(\vec{r})}_{\text{PERIODIC POTENTIAL}}$$

- $$H^*_{EFF} = \frac{1}{2m^*}\left[\left(\hat{P}_x + \frac{eB_0}{c} y\right)^2 + P_y^2 + P_z^2\right]$$

SOLUTIONS

$$f = e^{i/\hbar\left[P_x^{(0)} x + P_z^{(0)} z\right]} \chi(y)$$

$$\frac{1}{2m^*}\left[\left(P_x^{(0)} + \frac{eB_0}{c} y\right)^2 - \hbar^2 \frac{\partial^2}{\partial y^2} + P_z^{(0)2}\right]\chi$$

$$= E\chi$$

TAKE $y_0 = -\frac{P_x^{(0)} c}{eB_0}$

$$\omega_c = \frac{|e| B_0}{m^* c}$$

$$\Downarrow$$

$$-\frac{d^2\chi}{dy^2} + \frac{2m^*}{\hbar^2}\left[\left(E - \frac{P_z^{(0)2}}{2m^*}\right) - \frac{1}{2} m^* \omega_c^2\right.$$

$$\left.\underbrace{(y - y_0)^2}\right]\chi = 0$$

HARMONIC OSCILLATOR CENTERED @ $y = y_0$

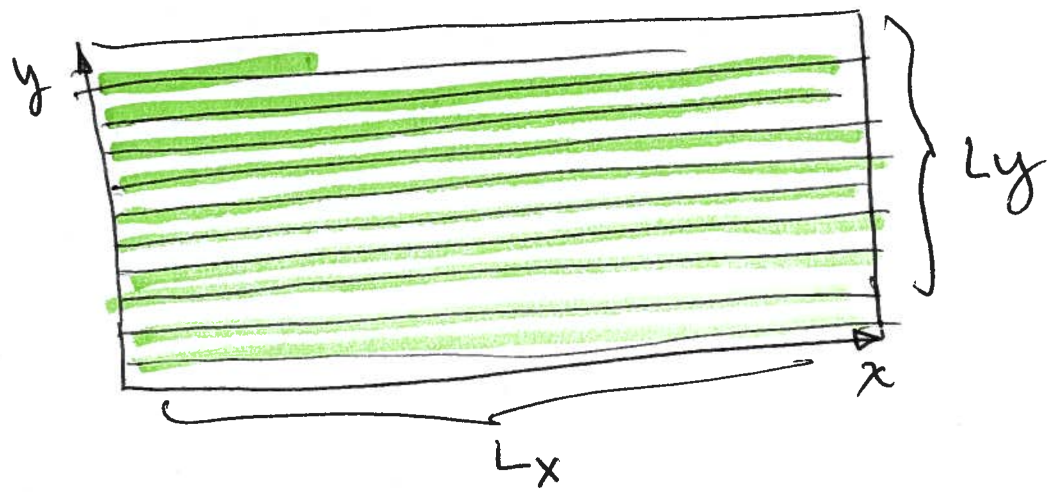
$$\Rightarrow E = \left(n + \tfrac{1}{2}\right)\hbar\omega_c + \frac{P_z^{(0)2}}{2m^*}$$

$$\chi_n(y) = \mathbb{C}\, e^{-\frac{(y-y_0)^2}{2a_c^2}} \underbrace{H_n\left(\frac{y-y_0}{a_c}\right)}_{\text{HERMITE POLYNOMIAL}}$$

$$a_c = \sqrt{\frac{\hbar}{m\omega_c}}$$

CYCLOTRON LENGTH

FOR A FIXED n, SAY, $n=0$ (& FIXED $P_x^{(0)}$)

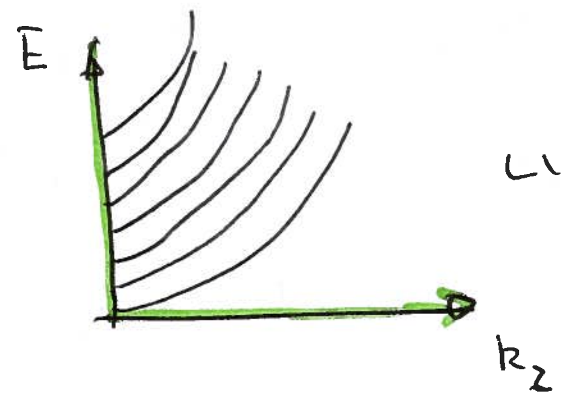


ENORMOUS DEGENERACY

$$\#_{\text{STATES}} = \frac{L_x L_y}{\hbar} m^* \omega_c$$

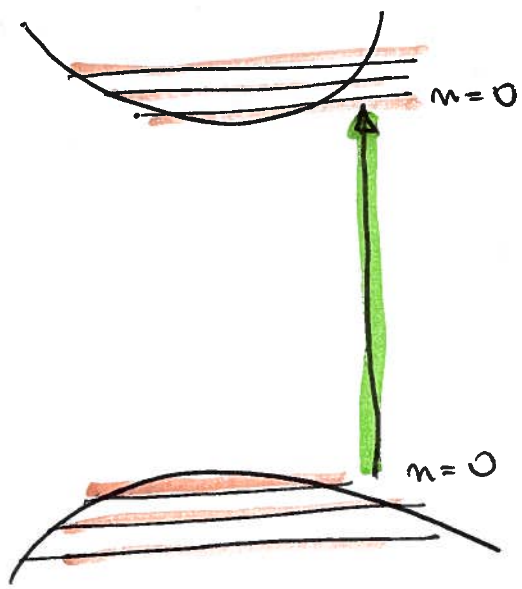
$$= \frac{L_x L_y |e| B_0}{\hbar c} \quad \text{(INDEPENDENT OF MASS)}$$

$$\left\{ \frac{P_x^{(0)}}{\hbar} L_x \le 1 \qquad y_0 \le L_y \right\}$$



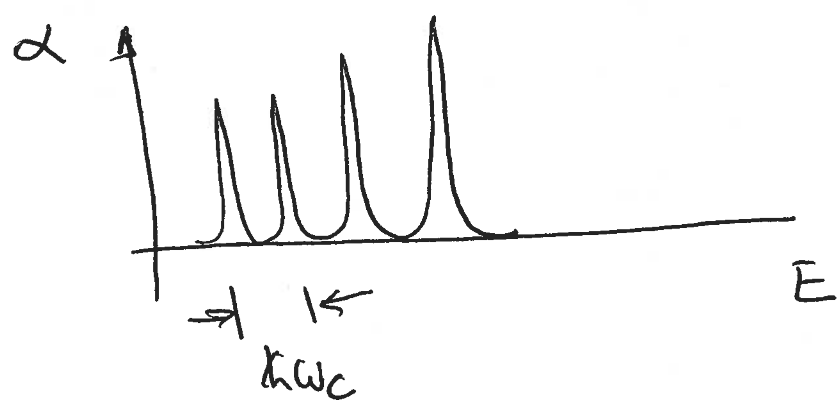
LIKE 1D PROBLEM !

OPTICAL SELECTION RULES:



ALL QUANTUM NUMBERS OF ENVELOPE FUNCTION MUST BE THE SAME

SAME y_0 !



NOTE:
BECAUSE V.b IS NOT PARABOLIC, THINGS ARE MORE COMPLICATED FOR III-V COMPOUNDS

FRANZ-KELDYSH

ELECTRIC FIELD F_E

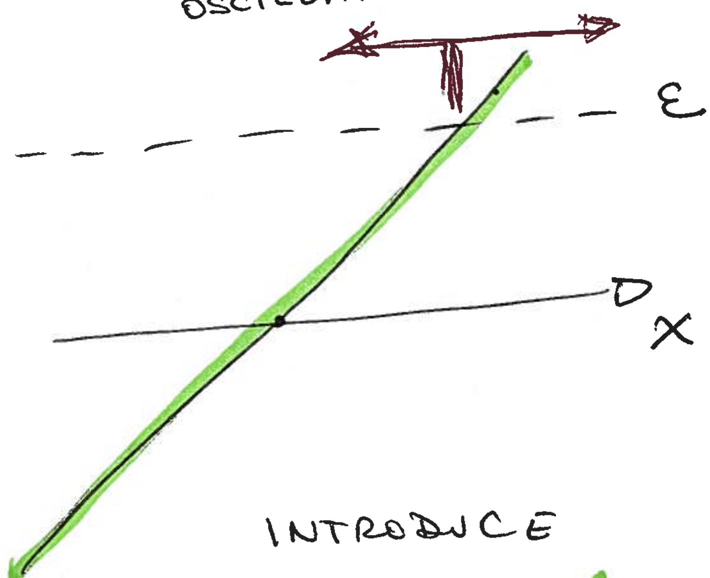
$$\alpha(\hbar\omega) \propto \begin{cases} \exp\left[-\frac{4\sqrt{2m^*}}{3|e|F_E\hbar}(E_g-\hbar\omega)^{3/2}\right] \\ \sin\left[\frac{2\sqrt{2m^*}}{3|e|F_E\hbar}(\hbar\omega-E_g)^{3/2}+\frac{\pi}{4}\right] \end{cases}$$

OSCILLATIONS FOR $\hbar\omega \geq E_g$

EFFECTIVE HAMILTONIAN

$$-\frac{\hbar^2}{2m^*}\frac{d^2\Psi}{dx^2} - e|F_E|x\Psi = \varepsilon\Psi$$

OSCILLATES EXPONENTIAL DECAY



INTRODUCE

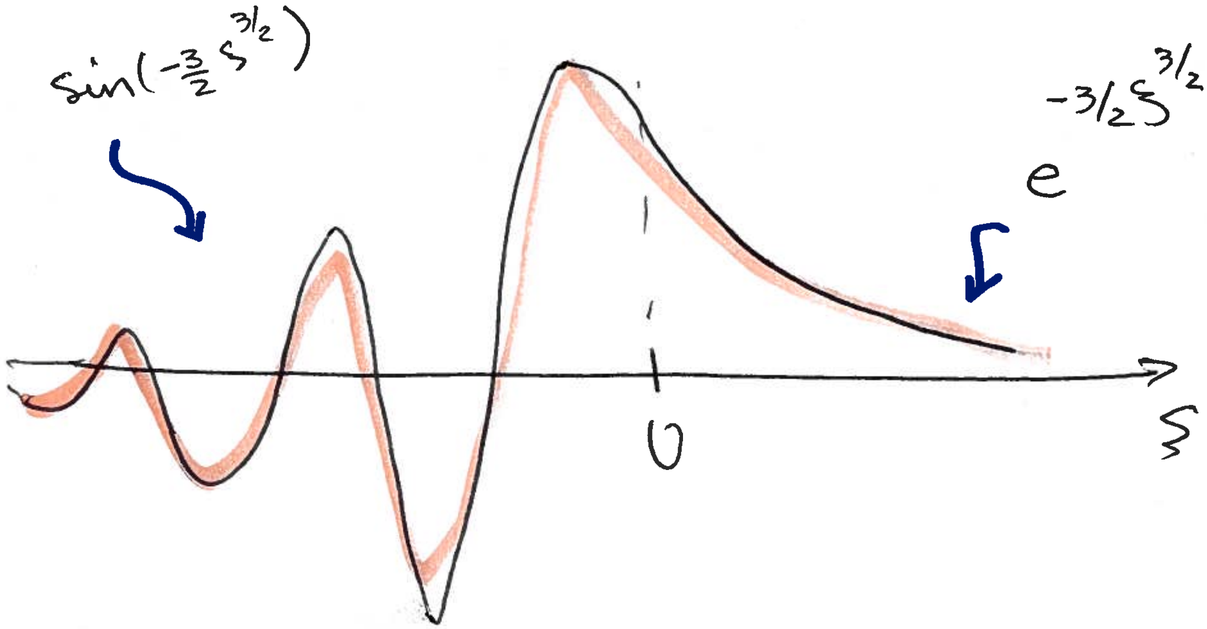
$$\xi = -\left(x + \frac{\varepsilon}{|e|F_E}\right)/\ell$$

$$-\frac{\hbar^2}{2m^*\ell^2}\frac{d^2\Psi}{d\xi^2} + e|F_E|\ell\xi\Psi = 0$$

$$\ell^3 \equiv \frac{\hbar^2}{2m^*|e|F_e}$$

$$\Rightarrow \ddot{\Psi} = \xi \Psi$$

AIRY FUNCTION



FOR LARGE $|\xi|$, SOLUTIONS ARE OF THE FORM

$$\Psi \propto \xi^{-1/4} e^{a|\xi|^{3/2}} \qquad a \equiv \begin{cases} -3/2 \\ \pm i\,3/2 \end{cases}$$

HOW DOES THIS WORK?

IF $\frac{\mathcal{E}}{eF_e} \gg$ length of sample

$\Rightarrow |\psi(\mathcal{E})|^2$ PROBABILITY OF FINDING AND ELECTRON OF ENERGY $\mathcal{E}$

~~ALTERNATIVELY~~ THUS,

$$\alpha \propto \sum_{\mathcal{E}} |\langle u_c | \vec{p}.\vec{A} | u_v \rangle|^2 \underbrace{\langle \psi_c | \psi_v \rangle_{\mathcal{E}}}_{\substack{\text{ORTHOGONAL} \\ \text{IF } \mathcal{E}_v \neq \mathcal{E}_c}} \times \delta(\hbar\omega + \mathcal{E} - E_G)$$

SINCE NOTHING CHANGES IN THE OTHER DIRECTIONS AND THE DENSITY OF STATES IS CONSTANT IN TWO DIMENSIONS, WE GET

$$\alpha \propto |\langle u_c | \vec{P}.\vec{A} | u_v \rangle|^2 \cdot \underbrace{\psi^2(E_g - \hbar\omega)}_{e^{-4/3\left[\frac{E_g - \hbar\omega}{eE\ell}\right]^{3/2}}}$$

A BETTER TREATMENT USES

$$E = -\frac{1}{c}\frac{\partial A}{\partial t} \quad \text{WITH} \quad \underbrace{A = -ctE_0}$$

$\Rightarrow$ L.V. KELDYSH, JETP $\underline{34}$, 788 (1958)

END OF DIRECT GAP ABSORPTION