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A Collective Description of Electron Interactions: III. Coulomb Interactions in a Degenerate Electron Gas

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The behavior of the electrons in a dense electron gas is analyzed quantum-mechanically by a series of canonical transformations. The usual Hamiltonian corresponding to a system of individual electrons with Coulomb interactions is first re-expressed in such a way that the long-range part of the Coulomb interactions between the electrons is described in terms of collective fields, representing organized "plasma" oscillation of the system as a whole. The Hamiltonian then describes these collective fields plus a set of individual electrons which interact with the collective fields and with one another via short-range screened Coulomb interactions. There is, in addition, a set of subsidiary conditions on the system wave function which relate the field and particle variables. The field-particle interaction is eliminated to a high degree of approximation by a further canonical transformation to a new representation in which the Hamiltonian describes independent collective fields, with n' degrees of freedom, plus the system of electrons interacting via screened Coulomb forces with a range of the order of the inter electronic distance. The new subsidiary conditions act only on the electronic wave functions; they strongly inhibit long wavelength electronic density fluctuations and act to reduce the number of individual electronic degrees of freedom by n'. The general properties of this system are discussed, and the methods and results obtained are related to the classical density fluctuation approach and Tomonaga's one-dimensional treatment of the degenerate Fermi gas.

I.

IN this paper we wish to develop a collective description of the behavior of the electrons in a dense electron gas which will be appropriate when a quantum-mechanical treatment of the electronic motion is required, as is the case for the electrons in a metal. Our collective description is based on the organized behavior of the electrons brought about by their long-range Coulomb interactions, which act to couple together the motion of many electrons. In the first paper of this series[1] hereafter referred to as I, we developed a collective description of the organized behavior in an electron gas due to the transverse electromagnetic interactions between the electrons. This was done by means of a canonical transformation to a set of transverse collective coordinates which were appropriate for a description of this organized behavior. Here we shall develop an analogous canonical transformation to a set of longitudinal collective coordinates which are appropriate for a description of the organization brought about by the Coulomb interactions.

In the preceding paper[2] hereafter referred to as II, we developed a detailed physical picture of the electronic behavior (due to the Coulomb interactions). Although the electron gas was treated classically, we shall see that most of the conclusions reached there are also appropriate (with certain modifications) in the quantum domain. Let us review briefly the physical picture we developed in II, since we shall have occasion to make frequent use of it in this paper.

We found that, in general, the electron gas displays both collective and individual particle aspects. The primary manifestations of the collective behavior are organized oscillation of the system as a whole, the so-called "plasma" oscillation, and the screening of the field of any individual electron within a Debye length by the remainder of the electron gas. In a collective oscillation, each individual electron suffers a small periodic perturbation of its velocity and position due to the combined potential of all the other particles. The cumulative potential of all the electrons may be quite large since the long range of the Coulomb interaction permits a very large number of electrons to contribute to the potential at a given point. The screening of the electronic fields may be viewed as arising from the Coulomb repulsion, which causes the electrons to stay apart, and so leads to a deficiency of negative charge in the immediate neighborhood of a given electron. The collective behavior of the electron gas is decisive for phenomena involving distances greater than the Debye length, while for smaller distances the electron gas is best considered as a collection of individual particles which interact weakly by means of a screened Coulomb force.

These conclusions were reached by analyzing the behavior of the electrons in terms of their density fluctuations. It was found that these density fluctuations could be split into two approximately independent components, associated with collective and individual particle aspects of the electronic motion. The collective component is present only for wavelengths greater than the Debye length and represents the "plasma" oscillation. It may be regarded as including the effects of the long range of the Coulomb force which leads to the simultaneous interaction of many particles. The individual particles component is associated with the random thermal motion of the electrons and shows no collective behavior; it represents a collection of individual electrons surrounded by co-moving clouds of

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[1] D. Bohm and D. Pines, Phys. Rev. **82**, 625 (1951).

[2] D. Pines and D. Bohm, Phys. Rev. **85**, 338 (1952).

charge which act to screen their fields as described above. The individual particles component thus includes the effects of the residual short-range screened Coulomb force, which leads only to two-body collisions.

A quantum-mechanical generalization of the density fluctuation method is quite straightforward and is sketched briefly in Appendix I. However, we do not choose to adopt this point of view, because although it is quite useful in establishing the existence of collective oscillations and describing certain related phenomena, it does not enable one to obtain a satisfactory over-all description of the electron gas. Quantum-mechanical calculations aimed at solving for the wave functions and the energy levels of the system are much more conveniently done in terms of a Hamiltonian formalism through the use of appropriate canonical transformations.

Our general approach in this series of papers has been to analyze the collective oscillatory motion first, since this is associated with the long-range aspects of the interaction which, in a sense, are responsible for the major complications in the many-electron problem. Once the collective motion is accounted for, we then investigate the aspects of the electronic behavior which are independent of the collective behavior, and which, if our method is successful, should turn out to be simple. Thus we are led to seek a canonical transformation to a representation in which the existence of the collective oscillations is explicitly recognized, and in which these oscillations are independent of the individual electronic behavior. In this representation, which we shall call the collective representation, we do not expect that the electron gas can be described entirely in terms of the collective coordinates which describe the organized oscillations, since we know that the gas also displays individual particle behavior. We shall see that in the collective representation, the individual electronic coordinates correspond to the electrons plus their associated screening fields, and that as might be anticipated from II, these screened electrons interact rather weakly via a screened Coulomb force.

In this paper we shall be primarily concerned with obtaining the canonical transformation to the collective representation. We shall discuss the approximations involved and, in a general way, the resultant wave functions of our electron system in the collective representation. Our development of a quantum-mechanical description of the electron assembly makes possible a treatment of the effects of electron interaction in metallic phenomena which utilizes at the outset the simplicity brought about by the organized oscillatory behavior. The detailed application of the collective description to the electrons in a metal is given in the following paper,[3] hereafter referred to as IV.

Historically, the first utilization of the 'plasma' aspects of the electron gas in a metal is due to Kronig and Korringa,[4] who treated the effect of electron-electron interaction on the stopping power of a metal for fast charged particles. However, their treatment is open to objection, in that they describe the electron gas as a classical fluid, with an artificially introduced coefficient of internal friction. A more satisfactory treatment of electron-electron interaction in the stopping power problem is due to Kramers[4] and Bohr.[4] The quantum treatment of this problem from the viewpoint of the collective description is given in Paper IV.

Tomonaga[5] has independently investigated the extent to which a degenerate Fermi gas can be described in terms of longitudinal oscillations. Tomonaga's treatment is, however, confined to a one-dimensional system, and as we shall see, there are certain essential difficulties associated with its generalization to a three-dimensional system which make the direct extension of this approach to three dimensions impossible. The relationship between our approach and that of Tomonaga is discussed in Appendix II.

II.

We consider an aggregate of electrons embedded in a background of uniform positive charge, whose density is equal to that of the electrons. The Hamiltonian for our system may be written

$$\sum_i \frac{p_i^2}{2m} + 2\pi e^2 \sum_{kij}{}' \frac{e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}}{k^2} - 2\pi n e^2 \sum_k{}' \frac{1}{k^2}, \qquad (1)$$

where the first term corresponds to the kinetic energy of the electrons, the second to their Coulomb interaction and the third to a subtraction of their self energy. The prime in the summations over k denotes a sum in which $k=0$ is excluded, and this takes into account the uniform background of positive charge, and hence the over-all charge neutrality of our system.[6] In obtaining (1) we have used the fact that the Coulomb interaction between the ith and jth electrons may be expanded as a Fourier series in a box of unit volume, and is $(e^2/|\mathbf{x}_i-\mathbf{x}_j|)=4\pi e^2\sum_k(1/k^2)e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}$. n is the total number of electrons and is numerically equal to the mean density (since we are working in a box of unit volume).

Instead of working directly with the Hamiltonian of Eq. (1), we shall find it convenient to introduce an equivalent Hamiltonian which is expressed in terms of the longitudinal vector potential of the electromagnetic field, $\mathbf{A}(\mathbf{x})$, where $\mathbf{A}(\mathbf{x})$ may be Fourier-analyzed as

$$\mathbf{A}(\mathbf{x}) = (4\pi c^2)^{\frac{1}{2}} \sum_k q_k \boldsymbol{\varepsilon}_k e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad (2)$$

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[3] D. Pines, following paper [Phys. Rev. **92**, 626 (1953)].

[4] R. Kronig and J. Korringa, Physica **10**, 406 (1943). See also H. A. Kramers, Physica **13**, 401 (1947); A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **24**, No. 19 (1948); and R. Kronig, Physica **14**, 667 (1949).

[5] S. Tomonaga, Prog. Theor. Phys. **5**, 544 (1950).

[6] We shall drop this prime in the remainder of this paper since we have no further occasion to make explicit use of the fact that the term with $k=0$ is excluded.

and $\boldsymbol{\varepsilon}_k$ denotes a unit vector in the $\mathbf{k}$ direction. The electric field intensity, $\mathbf{E}(\mathbf{x})$ is

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= -(4\pi)^{\frac{1}{2}}\textstyle\sum_k \dot{q}_k \boldsymbol{\varepsilon}_k e^{+i\mathbf{k}\cdot\mathbf{x}} \\ &= (4\pi)^{\frac{1}{2}}\textstyle\sum_k p_{-k}\boldsymbol{\varepsilon}_k e^{i\mathbf{k}\cdot\mathbf{x}}.\end{aligned} \tag{3}$$

To ensure that $\mathbf{A}(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})$ are real, we take

$$q_k = -q_{-k}^*, \quad p_k = -p_{-k}^*. \tag{4}$$

Our equivalent Hamiltonian is then given by

$$H = \sum_i \left[\mathbf{p}_i + \frac{e}{c}\mathbf{A}(\mathbf{x}_i)\right]^2 \Big/ 2m + \int [E^2(\mathbf{x})/8\pi] d\mathbf{x} - 2\pi n e^2 \sum_k \frac{1}{k^2}, \tag{5}$$

which using (2) and (3) may be shown to become

$$\begin{aligned}H = \sum_i \frac{p_i^2}{2m} &+ \frac{e}{m}(4\pi)^{\frac{1}{2}} \sum_{ik} \boldsymbol{\varepsilon}_k \cdot (\mathbf{p}_i - \hbar\mathbf{k}/2) q_k e^{i\mathbf{k}\cdot\mathbf{x}_i} \\ &+ (2\pi e^2/m) \sum_{ikl} \boldsymbol{\varepsilon}_k \cdot \boldsymbol{\varepsilon}_l q_k q_l e^{i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i} - \textstyle\sum_k \frac{1}{2} p_k p_{-k} \\ &- 2\pi n e^2 \textstyle\sum_k 1/k^2.\end{aligned} \tag{6}$$

This Hamiltonian, when used in conjunction with a set of subsidiary conditions acting on the wave function of our system,

$$\Omega_k \Phi = 0 \quad (\text{for all } k), \tag{7}$$

where

$$\Omega_k = p_{-k} - i\left(\frac{4\pi e^2}{k^2}\right)^{\frac{1}{2}} \sum_i e^{-i\mathbf{k}\cdot\mathbf{x}_i} \tag{8}$$

will lead to the correct electron equations of motion. Ω_k is proportional to the kth fourier component of $\operatorname{div}\mathbf{E}(\mathbf{x}) - 4\pi\rho(\mathbf{x})$, and hence these subsidiary conditions guarantee that Maxwell's equations are satisfied. It may easily be verified that the subsidiary condition operator Ω_k commutes with the Hamiltonian (6), so that if the subsidiary condition (7) is satisfied at some initial time, it will be true at all subsequent times.[7]

The equivalence of our Hamiltonian (6) with the Hamiltonian expressed by (1) may be seen by applying the unitary transformation[8] $\Phi = S\psi$, where,

$$S = \exp\left[-(1/h) \sum_{ki} (4\pi e^2/k^2)^{\frac{1}{2}} q_k e^{i\mathbf{k}\cdot\mathbf{x}_i}\right]. \tag{9}$$

With this transformation, we find

$$\begin{aligned}p_i &\rightarrow S^{-1}p_i S = p_i - (4\pi e^2)^{\frac{1}{2}} \textstyle\sum_k q_k \boldsymbol{\varepsilon}_k e^{i\mathbf{k}\cdot\mathbf{x}_i} \\ p_k &\rightarrow S^{-1}p_k S = p_k + i(4\pi e^2/k^2)^{\frac{1}{2}} \textstyle\sum_i e^{i\mathbf{k}\cdot\mathbf{x}_i},\end{aligned}$$

and

$$\begin{aligned}H \rightarrow \mathcal{H} = S^{-1}HS = \sum_i \frac{p_i^2}{2m} &+ 2\pi e^2 \sum_{ij} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)}}{k^2} \\ &- \sum_k \frac{p_k p_{-k}}{2} - i\sum_{ki}\left(\frac{4\pi e^2}{k^2}\right)^{\frac{1}{2}} p_k e^{-i\mathbf{k}\cdot\mathbf{x}_i} - 2\pi n e^2 \sum_k \left(\frac{1}{k^2}\right).\end{aligned}$$

The subsidiary condition (8) becomes

$$p_{-k}\psi = 0 \quad (\text{for all } k).$$

If we choose a ψ which is independent of q_k, we may satisfy the new subsidiary condition identically, the terms involving p_k in the Hamiltonian will drop out, and $\mathcal{H}$ is seen to be equivalent to (1). We note that the term $-2\pi n e^2 \sum_k (1/k^2)$ was included in (6) so that this Hamiltonian might be numerically equivalent to (1), as well as leading to equivalent equations of motion, since this term is just what is needed to cancel the terms with $i = j$ in the Coulomb energy.

The introduction of the longitudinal decrees of freedom, q_k, and the subsidiary conditions (7) provides a convenient means of introducing the concept of independent collective oscillation within the framework of the Hamiltonian formalism. The utility of this representation lies in the fact that (7) introduces in a simple way a relationship between the fourier components of the electronic density, $\rho_k = \sum_i e^{-i\mathbf{k}\cdot\mathbf{x}_i}$, and a set of field variables p_k. We shall see that there is, in consequence, a very close parallel between the behavior of the ρ_k, as analyzed in II, and the behavior of our field coordinates. In this representation we find that the field variables (just as did the ρ_k) oscillate with a frequency equal to the plasma frequency, provided we neglect a small coupling between the collective motion and the individual electronic behavior (characterized by their random thermal motion). Furthermore, just as we found it possible in II to find a purely oscillatory component of the density fluctuations, which is approximately independent of the individual electronic behavior, so we shall here be able to carry out a canonical transformation to a new set of field variables, which describe pure collective behavior and do not interact with the individual electrons to a good degree of approximation. In this section we shall analyze the approximate oscillatory behavior of the (q_k, p_k), while in the next section we carry out the canonical transformation to the pure collective coordinates.

Before beginning our analysis, we find it desirable to modify somewhat our Hamiltonian (6). We found in Paper II that in the classical theory there is a minimum wavelength λ_c (which classically is the Debye length), and hence a maximum wave vector k_c, beyond which organized oscillation is not possible. We may anticipate that in the quantum theory a similar (but not identical) limit arises, so that there is a corresponding limit on the extent to which we can introduce collective coordinates to describe the electron gas.

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[7] This may be contrasted with the customary gauge [corresponding to $\operatorname{div}\mathbf{A} = (1/c)\partial\varphi/\partial t$], in which the commutator of the subsidiary condition with H is proportional to the subsidiary condition itself, and is therefore zero only when the subsidiary condition is satisfied.

[8] See G. Wentzel, *Quantum Theory of Wave Fields* (Interscience Publishers, New York, 1949), p. 131.

Since this is the case, rather than introduce the full spectrum of longitudinal field coordinates (and associated subsidiary conditions) as we do in (6), we might as well confine our attention to only as many p_k and q_k as we expect to display collective behavior, i.e., (p_k, q_k) for $k<k_c$. The number of collective coordinates, n', will then correspond to the number of k values lying between $k=0$ and $k=k_c$, and so will be given by

$$n'=\frac{4\pi}{3}\frac{k_c^3}{(2\pi)^3}=\frac{k_c^3}{6\pi^2}. \tag{10}$$

One might expect that there is a natural upper limit to n', *viz.*, the total number of longitudinal degrees of freedom n (for a system of n electrons), since at most n independent longitudinal degrees of freedom may be introduced. In practice we find that n' is considerably less than this theoretical maximum.

The modification of (6) to include only terms involving (p_k, q_k) with $k<k_c$ may be conveniently carried out by applying a unitary transformation similar to (9), but involving only q_k for which $k>k_c$. Thus we take $\Phi=S\psi$ where,

$$S=\exp\Big[-(1/\hbar)\sum_{i,k>k_c}(4\pi e^2/k^2)^{\frac{1}{2}}q_k e^{i\mathbf{k}\cdot\mathbf{x}_i}\Big], \tag{9a}$$

and where ψ is chosen to be independent of all q_k with wave numbers greater than k_c. We then obtain for our Hamiltonian

$$\begin{aligned}H={}&\sum_i\frac{p_i^2}{2m}+(4\pi)^{\frac{1}{2}}\frac{e}{m}\sum_{ik<k_c}\boldsymbol{\varepsilon}_k\cdot(\mathbf{p}_i-\hbar\mathbf{k}/2)q_k e^{i\mathbf{k}\cdot\mathbf{x}_i}\\&+(2\pi e^2/m)\sum_{\substack{ik<k_c\\ l<k_c}}\boldsymbol{\varepsilon}_k\cdot\boldsymbol{\varepsilon}_l q_k q_l e^{i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i}-\sum_{k<k_c}\frac{p_k p_{-k}}{2}\\&+2\pi e^2\sum_{\substack{k>k_c\\ i\neq j}}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}}{k^2}-2\pi ne^2\sum_{k<k_c}\frac{1}{k^2},\end{aligned} \tag{11}$$

with the associated set of subsidiary conditions:

$$\Omega_k\psi=0 \quad (k<k_c). \tag{12}$$

We shall find it convenient, in dealing with this Hamiltonian, to split up the third term into two parts. That part for which $\mathbf{k}+\mathbf{l}=0$ is independent of the electron coordinates and is given by

$$\frac{2\pi ne^2}{m}\sum_{k<k_c}q_k q_{-k}=\frac{\omega_p^2}{2}\sum_{k<k_c}q_k q_{-k}, \tag{13}$$

where we have introduced ω_p, the so-called plasma frequency, defined by

$$\omega_p=(4\pi ne^2/m)^{\frac{1}{2}}. \tag{14}$$

The remaining part, for which $\mathbf{k}+\mathbf{l}\neq 0$, we shall denote by U where,

$$U=\frac{2\pi e^2}{m}\sum_{\substack{ik<k_c\\ l<k_c\\ l\neq -k}}\boldsymbol{\varepsilon}_k\cdot\boldsymbol{\varepsilon}_l q_k q_l e^{i(\mathbf{k}+\mathbf{l})\cdot\mathbf{x}_i}. \tag{15}$$

U is much smaller than (13), for it always depends on the electron coordinates, and since these are distributed over a wide variety of positions, there is a strong tendency for the various terms entering into U to cancel. Let us for the time being neglect U, a procedure which we have called the random phase approximation in our earlier papers, and which we shall presently justify.

With this approximation we see that the third and fourth terms in our Hamiltonian (11) reduce to

$$H_{\text{osc}}=-\tfrac{1}{2}\sum_{k<k_c}(p_k p_{-k}+\omega_p^2 q_k q_{-k}) \tag{16}$$

the Hamiltonian appropriate to a set of harmonic oscillators, representing collective fields, with a frequency ω_p. The first term in (11) represents the kinetic energy of the electrons, while the second term,

$$H_{\text{I}}=(4\pi)^{\frac{1}{2}}\frac{e}{m}\sum_{ik<k_c}\boldsymbol{\varepsilon}_k\cdot\left(\mathbf{p}_i-\frac{\hbar\mathbf{k}}{2}\right)q_k e^{i\mathbf{k}\cdot\mathbf{x}_i}, \tag{17}$$

represents a simple interaction between the electrons and the collective fields, which is linear in the field variables. The fifth term,

$$H_{\text{s.r.}}=2\pi e^2\sum_{\substack{k>k_c\\ i\neq j}}\frac{e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}}{k^2}, \tag{18}$$

represents the short-range part of the Coulomb interaction between the electrons. If we carry out the indicated summation, we find

$$H_{\text{s.r.}}=\tfrac{1}{2}\sum_{i\neq j}\frac{e^2}{|\mathbf{x}_i-\mathbf{x}_j|}\left\{1-\frac{2}{\pi}\operatorname{Si}(k_c|\mathbf{x}_i-\mathbf{x}_j|)\right\}, \tag{19}$$

where

$$\operatorname{Si}(y)=\int_0^y dx\frac{\sin x}{x}.$$

$\operatorname{Si}(y)=\pi/2$ for $y=2$ and oscillates near $\pi/2$ for larger values of y, so that $H_{\text{s.r.}}$ describes screened electron interaction with a range $\sim k_c$. A plot of $H_{\text{s.r.}}$ is given in Fig. 1.

Thus we see that in using (11) we have redescribed the long-range part of the Coulomb interactions between the electrons in terms of the collective oscillations (16), which interact with the electrons via H_{I}, (17). Our problem has now been reduced to one quite analogous to that encountered in I, *viz.*, a set of particles interacting with collective fields; the only new complications are the short-range interaction $H_{\text{s.r.}}$, and the subsidiary conditions on the system wave function. We shall see that as was the case in I with the transverse collective oscillations, the coupling between the fields and particles described by H_{I} is not very strong, so that it is possible to obtain a good qualitative understanding of the behavior of the system by neglecting this term. In this section, we shall make this approximation, and then investigate to what extent it applies, while in Sec. III we will give a more accurate treatment which includes the effects of the electron-field interaction.

If we neglect H_{I}, we may write the stationary state wave function as

$$\psi=\psi_{\text{osc}}\chi(\mathbf{x}_i\cdots\mathbf{x}_n). \tag{20}$$

ψ_{osc} represents the wave functions of the collective fields, and may be written as a product of harmonic oscillator wave functions like

$$h_n(p_k)\exp[-(|p_k|^2/2\hbar\omega_p)],$$

where h_n is the nth Hermite polynomial, and we are using the *momentum* representation of the oscillator wave functions. $\chi(x_i\cdots x_n)$ represents the eigenfunction for a set of particles interacting through $H_{\text{s.r.}}$ For the lowest state, we then get

$$\psi_0=[\exp-\{\sum_{k<k_c}|p_k|^2/2\hbar\omega_p\}]\chi_0(\mathbf{x}_1\cdots\mathbf{x}_n), \tag{21}$$

where $\chi_0(\mathbf{x}_1\cdots\mathbf{x}_n)$ is the lowest state electron wave function.

In general χ_0 will be quite complex. However, just because the long-range part of the Coulomb potential is included in the oscillator energy, the remaining part $H_{\text{s.r.}}$ is considerably reduced in effectiveness. In fact it will often be of so short a range that for many purposes the free particle wave functions will constitute an adequate approximation. In this case, the lowest state wave function is

$$\psi_0=\{\exp[-\sum_{k<k_c}|p_k|^2/2\hbar\omega_p]\}D_0(\mathbf{x}_1\cdots\mathbf{x}_n), \tag{22}$$

where D_0 is the usual Slater determinantal wave function composed of the free electron wave functions appropriate to the ground state of the individual electrons. Our wave function ψ_0 then satisfies the exclusion principle.

Let us now consider the effects of the subsidiary conditions (12). In the representation in which p_k and $\mathbf{x}_i$ are diagonal these reduce to n' algebraic relations. We can view these relations in either of the following ways:

(a) They permit us to eliminate the p_k in terms of the $\mathbf{x}_i$.
(b) They permit us to eliminate n' of the $\mathbf{x}_i$ in terms of the p_k.

Let us begin with the first way. Our wave function (22) then becomes

$$\psi_0=\exp[-\tfrac{1}{2}(\sum_{ij}F(\mathbf{x}_i-\mathbf{x}_j)/\hbar\omega_p)D_0(\mathbf{x}_1\cdots\mathbf{x}_n), \tag{23}$$

where

$$F(\mathbf{x}_i-\mathbf{x}_j)=2\pi e^2\sum_{k<k_c}e^{i\mathbf{k}\cdot(\mathbf{x}_i-\mathbf{x}_j)}/k^2 \tag{24}$$

represents the long-range part of the Coulomb potential. $F(\mathbf{x}_i-\mathbf{x}_j)=e^2/|\mathbf{x}_i-\mathbf{x}_j|$ for $|\mathbf{x}_i-\mathbf{x}_j|\gg 1/k_c$ but approaches a constant $4\pi e^2\sum_{k<k_c}(1/k^2)$, when $|\mathbf{x}_i-\mathbf{x}_j|\ll 1/k_c$. Thus in (23) we have the usual free electron wave function D_0 modified by a factor which describes long-range electron correlation, such that the probability that two electrons are found a given distance apart is less than that calculated by neglecting the Coulomb interactions or by including the short-range interaction $H_{\text{s.r.}}$ In fact in consequence of this correlation term, each electron tends to keep apart from the others, in a manner quite similar to that obtained in the classical treatment of II. A similar result has been obtained by Tomonaga in his one-dimensional treatment.

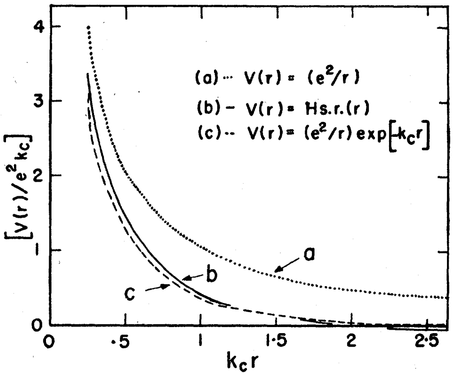

FIG. 1. $H_{\text{s.r.}}(r)$ compared with (e^2/r) and $(e^2/r)\exp(-k_cr)$.

Let us now consider method (b), in which we seek to eliminate n' of the particle variables in terms of the field variables p_k. As is clear from the form of (12), this is a much more formidable task, one which we are not able to carry out explicitly. However, as we shall see throughout this paper, we can still draw a number of useful conclusions concerning the effect of such an elimination without actually solving for the $\mathbf{x}_i$ in terms of the p_k. In particular, we shall see in Sec. III how one may use a canonical transformation to replace (to lowest order in the field-particle coupling constant) n' of the individual particle degrees of freedom by as many collective degrees of freedom.

We now wish to justify our neglect of U and to investigate to what extent corrections arising from the inclusion of H_{I} will be of importance. In the remainder of this section we confine our attention to the lowest state of the system. We first show that the exact lowest state eigenfunction ψ_0 of our Hamiltonian (11) automatically satisfies the subsidiary conditions (12). For as we have noted the subsidiary condition operators Ω_k commute with the Hamiltonian H, so that the wave function ψ_0 can, in general, be expressed in terms of a series of simultaneous eigenfunctions of H and the Ω_k. The lowest state of the system is nondegenerate and hence corresponds to a single eigenvalue of the operator Ω_k, which we may call α_k. To determine the value of α_k, we consider a space displacement of the entire system (field plus electrons) through a distance $\Delta\mathbf{x}$, so that

$$\mathbf{x}\rightarrow\mathbf{x}'+\Delta\mathbf{x},\quad \mathbf{p}_i\rightarrow\mathbf{p}_i',$$
$$\mathbf{x}_i\rightarrow\mathbf{x}_i'+\Delta\mathbf{x}.$$

From Eqs. (2) and (3) we see that the effect of this displacement on the field coordinates is given by

$$p_k\rightarrow p_k'e^{-i\mathbf{k}\cdot\Delta\mathbf{x}},$$
$$q_k\rightarrow q_k'e^{i\mathbf{k}\cdot\Delta\mathbf{x}}.$$

The Hamiltonian is thus invariant under this displacement, while the subsidiary condition applied to our lowest state wave function becomes

$$e^{-i\mathbf{k}\cdot\Delta\mathbf{x}}\Omega_k'\psi_0=\alpha_k\psi_0.$$

However, since the lowest state is nondegenerate, it is not changed by this displacement, and we must have $\Omega_k'\psi_0=\alpha_k\psi_0$. Thus we find $\alpha_k=\alpha_k e^{i\mathbf{k}\cdot\Delta\mathbf{x}}$, which can only be satisfied if $\alpha_k=0$.

Thus, if we could obtain an exact solution for the lowest state eigenfunction ψ_0, we would automatically satisfy the subsidiary condition $\Omega_k\psi_0=0$. We may, in general, expect that if we obtain an approximate solution for ψ_0, we will not be able to satisfy the subsidiary condition, but that any error we make in determining the energy of the lowest state will not be increased by our failure to satisfy this subsidiary condition, since an exact solution satisfies the subsidiary condition and leads to the lowest possible energy state. The situation with regard to the excited states of the system will be somewhat different, and we will return to this question later.

Let us take as our approximate ψ_0, the wave function (22). In this approximation the energy of the lowest state is given by

$$\begin{aligned}E&=\tfrac{3}{5}E_0+\sum_{k<k_c}\frac{\hbar\omega_p}{2}-\frac{2\pi ne^2}{k^2}+\langle H_{\text{s.r.}}\rangle_{\text{Av}}\\&=\tfrac{3}{5}E_0+n'\frac{\hbar\omega_p}{2}-\frac{ne^2}{\pi}k_c+\langle H_{\text{s.r.}}\rangle_{\text{Av}},\end{aligned}\tag{25}$$

where E_0 is the energy of an electron at the top of the Fermi distribution, and $\langle H_{\text{s.r.}}\rangle_{\text{Av}}$ is the exchange energy arising from the screened Coulomb interaction term, $H_{\text{s.r.}}$, Eq. (18). We will not be concerned with evaluating $\langle H_{\text{s.r.}}\rangle_{\text{Av}}$ at present (reserving this for Paper IV), as we are here primarily interested in evaluating the corrections arising from U and H_{I}. We estimate these terms using perturbation theory. With the wave function (22) the average values of U and H_{I} vanish. U, in fact, has non-vanishing matrix elements only between the lowest state (with zero quanta) and a two-quantum state, while H_{I} connects the lowest state and a one-quantum state. From second-order perturbation theory, we have

$$\Delta U=-\sum_n\frac{|U_{0n}|^2}{E_n-E_0},\tag{26}$$

where

$$U_{0n}=\frac{2\pi e^2}{m}\frac{\hbar}{2\omega_p}\boldsymbol{\varepsilon}_k\cdot\boldsymbol{\varepsilon}_l,\tag{27}$$

if the state n has two quanta of momentum $\mathbf{k}$ and $\mathbf{l}$, respectively; and

$$E_n-E_0=2\hbar\omega_p+\frac{\hbar(\mathbf{k}+\mathbf{l})^2}{2m}-\frac{\hbar(\mathbf{k}+\mathbf{l})}{m}\cdot\mathbf{p}_i,\tag{28}$$

if the electron in the initial state has momentum $\mathbf{p}_i$. In (28) we may, for the purpose of this rough estimate, approximate $E_n-E_0=2\hbar\omega_p$ since, as we shall see in Paper IV, $\hbar(\mathbf{k}+\mathbf{l})^2/2m-[\hbar(\mathbf{k}+\mathbf{l})/m]\cdot\mathbf{p}_i$ is always appreciably less than $2\hbar\omega_p$ as long as $k, l<k_c$. We then find

$$\begin{aligned}\Delta U&=-\left(\frac{\pi e^2\hbar}{m\omega_p}\right)^2 n\sum_{\substack{k<k_c\\ l<k_c}}\frac{(\boldsymbol{\varepsilon}_k\cdot\boldsymbol{\varepsilon}_l)^2}{2\hbar\omega_p}\\&=-\left(\frac{\hbar\omega_p}{4}\right)^2\frac{1}{n}\frac{(n')^2}{6\hbar\omega_p}=-\frac{1}{48}\left(\frac{n'}{n}\right)\frac{n'\hbar\omega_p}{2}.\end{aligned}\tag{29}$$

Thus ΔU introduces a fractional change in the zero point energy, per oscillator, of $(1/48)(n'/n)$, and since n' is never greater than n (and is, in fact, often quite a bit smaller), this change is negligible. Thus, we are justified in neglecting completely the term U.

We may estimate the corrections arising from H_{I} in similar fashion. We have

$$\Delta H_{\text{I}}=-\sum_n\frac{|H_{\text{I}0n}|^2}{E_n-E_0},\tag{30}$$

where

$$(H_{\text{I}})_{n0}=(2\pi\hbar/\omega_p)^{\frac{1}{2}}(e/m)\boldsymbol{\varepsilon}_k\cdot(\mathbf{p}_i-\hbar\mathbf{k}/2),\tag{31}$$

if the state n has one quantum of momentum $\mathbf{k}$ present, and

$$E_n-E_0=\hbar\omega_p+\frac{\hbar k^2}{2m}-\frac{\hbar\mathbf{k}\cdot\mathbf{p}_i}{m}\cong\hbar\omega_p.\tag{32}$$

We then find

$$\begin{aligned}\Delta H_{\text{I}}&=-\frac{2\pi\hbar}{\omega_p}\frac{e^2}{m^2}\sum_{i,k<k_c}\frac{[\boldsymbol{\varepsilon}_k\cdot(\mathbf{p}_i-\hbar\mathbf{k}/2)]^2}{\hbar\omega_p}\\&=-\frac{n'}{3n}\left\{\sum_i\frac{p_i^2}{2m}-\frac{9n}{40}\frac{\hbar^2k_c^2}{m}\right\}.\end{aligned}\tag{33}$$

Since, as we shall see, $k_c \lesssim k_0$, the wave vector of an electron at the top of the Fermi distribution, we see that the second term in the parenthesis in (33) is generally somewhat smaller than the first, and the first term corresponds to a fractional correction in the kinetic energy per electron (and thus in its effective mass) of $\sim n'/3n$. This may be appreciable if $n' \sim n$ but otherwise is small. This term implies a similar order of magnitude correction for the frequency of the collective oscillations, since $\sum_i p_i^2/2m$ and $n'\hbar\omega_p/2$ are roughly of the same order of magnitude. Thus we find that we are justified in neglecting H_{I} in order to obtain a qualitative and rough quantitative understanding of the behavior of our system, but that the effects arising from H_{I} should definitely be taken into account in a careful quantitative treatment. This we shall give in Sec. III.

Thus far we have not specified the value of k_c, and hence the number of collective degrees of freedom we find it desirable to introduce in our treatment. We may obtain a rough qualitative estimate of n' by minimizing our approximate expression for the lowest state energy (25) with respect to k_c (or n'). For the purpose of this rough estimate, let us neglect the dependence of $\langle H_{\mathrm{s.r.}}\rangle_{\mathrm{Av}}$ on k_c. We then note that the second term in (25) will be negative for those k for which $(2\pi ne^2/k^2) > \hbar\omega_p/2$. Hence we obtain the minimum value for (25) if we include in this summation, only those k for which this inequality is satisfied. This criterion yields

$$k_c^2 = \frac{4\pi ne^2}{\hbar\omega_p} = \frac{k_0^2}{2.14}\left(\frac{r_0}{a_0}\right)^{\frac{1}{2}}, \tag{34}$$

where r_0 is the interelectronic spacing, defined by

$$n = (4\pi r_0^3/3)^{-1}, \tag{35}$$

and a_0 is the Bohr radius. For a typical metal like Na, we have $(r_s/a_0) \sim 4$ and hence $k_c \sim k_0$. From (10) we see that in this case $n' \sim n/2$. In Paper IV where we give a more detailed treatment of that choice of k_c which minimizes the energy, including the effects of H_{I}, and $\langle H_{\mathrm{s.r.}}\rangle_{\mathrm{Av}}$, we find for Na, $k_c \sim 0.68k_0$, and $n' \sim n/8$ in fair agreement with this rough estimate.

Finally we may remark that with the choice of k_c(34), the energy of the lowest state is

$$\begin{aligned} E &= \tfrac{3}{5}E_0 + \frac{n'\hbar\omega_p}{2} - \frac{ne^2k_c}{\pi} + \langle H_{\mathrm{s.r.}}\rangle_{\mathrm{Av}} \\ &= \tfrac{3}{5}E_0 - \frac{2}{3}\frac{ne^2}{\pi}k_c + \langle H_{\mathrm{s.r.}}\rangle_{\mathrm{Av}}. \end{aligned} \tag{36}$$

The energy $-\frac{2}{3}(ne^2/\pi)k_c$ represents a long-range correlation energy, i.e., that energy associated with the long-range correlations in electronic positions described by the wave function (23). In contrast to the exchange energy, this term represents Coulomb correlations between electrons of both kinds of spin. For Na it is, per electron,

$$-\frac{2}{3\pi}e^2k_c \cong -\frac{2}{3\pi}e^2k_0 \cong -0.4\frac{e^2}{r_0}, \tag{37}$$

a not inconsiderable energy. In Paper IV we return to a more careful estimate of the long-range correlation energy.

III.

In this section we wish to consider the effect of the field particle interaction term H_{I} on the motion of the electrons and the collective oscillations. We do this with the aid of a canonical transformation which is chosen to eliminate H_{I} in first approximation. Thus we seek a canonical transformation to a new representation in which the coupling between the fields and the electrons is described by a term H_{II}, which is appreciably smaller than H_{I}, and may consequently be neglected to a good degree of approximation (comparable, say, with our neglect of U). We shall then see that the effects of the coupling between the electrons and the collective field variables, as described by H_{I}, are threefold: there is an increase in the electronic effective mass, the frequency of the collective oscillations is increased and becomes k dependent, and the effective electron-electron interaction is modified. As we anticipated on the basis of our perturbation-theoretic estimate of H_{I} in the preceding section, none of these effects is so large as to destroy the qualitative conclusions we reached there, although the quantitative estimates of the energy and wave functions of our system are somewhat altered.

The measure of the smallness of H_{II}, and hence the extent to which we are successful in carrying out our canonical transformation, is the expansion parameter

$$\alpha = \left\langle \left(\frac{\mathbf{k}\cdot\mathbf{p}_i}{m\omega}\right)^2 \right\rangle_{\mathrm{Av}}, \tag{38}$$

where we average over the particle momenta and the collective field wave vectors, and ω is the frequency of the collective oscillations. We find

$$\alpha \cong \frac{9}{25} \times \frac{1}{3}\frac{k_c^2 p_0^2}{m^2\omega_p^2} \cong \tfrac{1}{2}\beta^2\frac{a_0}{r_s}, \tag{39}$$

where we have replaced ω by its approximate value ω_p and

$$\beta = k_c/k_0. \tag{40}$$

It is clear that by choosing β or k_c small enough, our expansion parameter α may be made as small as we like. We shall assume throughout the remainder of this paper that such a choice has been made, i.e., that $\alpha \ll 1$. In Paper IV we show that this criterion is satisfied in that $\alpha \sim 1/16$ for the electronic densities encountered in metals, if we take for β that value which minimizes the total energy. Another parameter of whose smallness we shall have occasion to make use is the ratio of the number of collective degrees of freedom, n', to the total number of degrees of freedom, $3n$. For most metals, with the above choice of β, we find $(n'/3n)\sim 1/25$.

We shall make the further approximation of neglecting the effects of our canonical transformation on $H_{\text{s.r.}}$, the short-range Coulomb interaction between the electrons. From Eq. (11), we see that if we neglect H_{I}, the collective oscillations are not affected at all by $H_{\text{s.r.}}$. Thus $H_{\text{s.r.}}$ can influence the q_k only indirectly through H_{I}. But, as we shall see, the *direct* effects of H_{I} on the collective oscillations are small. Thus, it may be expected that the *indirect* effects of $H_{\text{s.r.}}$ on the q_k through H_{I} are an order of magnitude smaller and may be neglected in our treatment which is aimed at approximating the effects of H_{I}. We will justify this procedure in greater detail in the following section.

With regard to the subsidiary conditions (11), we shall find that to order α, the subsidiary conditions in our new representation involve only the new particle coordinates $(\mathbf{X}_i, \mathbf{P}_i)$. Thus we may write our new wave function in terms of products like

$$\Phi_{\text{field}}\chi(\mathbf{X}_1\cdots\mathbf{X}_n),$$

and the subsidiary conditions will only act on the $\chi(X_i)$. The n' subsidiary conditions may thus be viewed as consisting of n' relationships among the particle variables, which effectively reduce the number of individual electronic degrees of freedom from $3n$ to $3n-n'$. This reduction is necessary, since in this new representation the n' collective degrees of freedom must be regarded as independent. For the field coordinates no longer appear in the subsidiary conditions, and hence describe real collective oscillation, which is independent of the electronic motion in this new representation.

There is a close resemblance between our Hamiltonian (11), which describes a collection of electrons interacting via longitudinal fields, and the Hamiltonian we considered in I, which described a collection of electrons interacting via the transverse electromagnetic fields. In fact, we shall see that our desired canonical transformation is just the longitudinal analog of that used in Paper I to treat the organized aspects of the transverse magnetic interactions in an electron gas. In order to point up this similarity and to simplify the commutator calculus, we introduce the creation and destruction operators for our longitudinal photon field, a_k and a_k^*, which are defined by[9]

$$\begin{aligned} q_k &= (\hbar/2\omega)^{\frac{1}{2}}(a_k - a_{-k}^*), \\ p_k &= i(\hbar\omega/2)^{\frac{1}{2}}(a_k^* + a_{-k}), \end{aligned} \tag{41}$$

and which possess the commutation properties

$$\begin{aligned} [a_k, a_{k'}] &= [a_k^*, a_{k'}^*] = 0, \\ [a_k, a_{k'}^*] &= \delta_{kk'}, \end{aligned} \tag{42}$$

in virtue of (41). In terms of these variables, we then write our Hamiltonian and supplementary conditions schematically as

$$\begin{gathered} H = H_{\text{part}} + H_{\text{I}} + H_{\text{field}} + H_{\text{s.r.}}, \\ \Omega_k\Phi = 0 \quad (k<k_c), \end{gathered} \tag{43}$$

where, using (11), (12), and (41) and neglecting U (Eq. 15),

$$H_{\text{part}} = \sum_i \frac{p_i^2}{2m} - \sum_{k<k_c} \frac{2\pi e^2 n}{k^2}, \tag{44a}$$

$$\begin{aligned} H_{\text{I}} = (e/m) \sum_{i,k<k_c} (2\pi\hbar/\omega)^{\frac{1}{2}} \{ &\boldsymbol{\varepsilon}_k\cdot(\mathbf{p}_i - \hbar\mathbf{k}/2) a_k e^{i\mathbf{k}\cdot\mathbf{x}_i} \\ &+ e^{-i\mathbf{k}\cdot\mathbf{x}_i} a_k^* \boldsymbol{\varepsilon}_k\cdot(\mathbf{p}_i - \hbar\mathbf{k}/2)\}, \end{aligned} \tag{44b}$$

$$\begin{aligned} H_{\text{field}} = \sum_{k<k_c} &\frac{\hbar\omega}{2}(a_k^* a_k + a_k a_k^*) + \frac{\hbar\omega}{4}(\omega_p^2 - \omega^2) \\ &\times (a_k^* a_k + a_k a_k^* - a_k a_{-k} - a_{-k}^* a_k^*), \end{aligned} \tag{44c}$$

$$H_{\text{s.r.}} = 2\pi e^2 \sum_{\substack{k>k_c \\ i\neq j}} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}_i - \mathbf{x}_j)}}{k^2}, \tag{44d}$$

$$\Omega_k = a_k^* + a_{-k} - \left(\frac{8\pi e^2}{k^2\hbar}\right)^{\frac{1}{2}} \sum_i e^{+i\mathbf{k}\cdot\mathbf{x}_i}. \tag{44e}$$

We note that H_{field} takes the form (44c), because we have expanded in terms of creation operators of frequency ω rather than ω_p.

We now consider a transformation from our operators $(\mathbf{x}_i, \mathbf{p}_i, a_k, a_k^*)$ to a new set of operators $(\mathbf{X}_i, \mathbf{P}_i, A_k, A_k^*)$, which possess the same eigenvalues and satisfy the same commutation rules as our original set.[10] The relation between these two sets may be written as

$$\mathbf{x}_i = e^{-iS/\hbar}\mathbf{X}_i e^{+iS/\hbar} \tag{45}$$

(with similar equations for $\mathbf{p}_i$, a_k, and a_k^*); (45) may be viewed as an operator equation, and we may take S the generating function of our canonical transformation to be a function of the new operators $(\mathbf{X}_i, \mathbf{P}_i, A_k, A_k^*)$ only. The operator relationship between the old and new Hamiltonians is

$$H = e^{-iS/\hbar}\mathcal{H}e^{iS/\hbar} = H_{\text{new}}, \tag{46}$$

where $\mathcal{H}$ represents that Hamiltonian which is the same function of the new coordinates as H is of the old, and H_{new} denotes the Hamiltonian expressed in terms of the new coordinates.

---

[9] ω is here unspecified, but will later be chosen to be the frequency of the collective oscillations.

[10] Quantum mechanical transformation theory is developed in, for instance, P. A. M. Dirac, *Principles of Quantum Mechanics* (Oxford University Press, London, 1935), second edition.

The problem of finding the proper form of S to realize our program was solved by a systematic study of the equations of motion. We do not have space to go into the details of this study here but confine ourselves to giving the correct transformation below. We shall then demonstrate that it leads to the desired results. Our canonical transformation is generated by

$$\begin{aligned} S=-(ei/m)\sum_{ik<k_c}(2\pi\hbar/\omega)^{\frac{1}{2}}\Bigg\{&\frac{\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_i-\hbar\mathbf{k}/2)A_k}{\omega-\mathbf{k}\cdot\mathbf{P}_i/m+\hbar k^2/2m} \\ &\times\exp(i\mathbf{k}\cdot\mathbf{X}_i)-\exp(-i\mathbf{k}\cdot\mathbf{X}_i)A_k^* \\ &\times\frac{\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_i-\hbar\mathbf{k}/2)}{\omega-\mathbf{k}\cdot\mathbf{P}_i/m+\hbar k^2/2m}\Bigg\}. \end{aligned} \tag{47}$$

On comparison with Eq. (45) of I, this generating function may be seen to be just the longitudinal analog of the "transverse" generating function given there. [The additional term in $\hbar k/2$ arises because $\mathbf{k}\cdot\mathbf{P}_i$ does not commute with $\exp(i\mathbf{k}\cdot\mathbf{X}_i)$.] Since H_{inter} and H_{field} are also analogous to the transverse terms encountered in I, we may expect that many of the results obtained there may be directly transposed to this longitudinal case. The differences in the treatments will arise from a consideration of $H_{\text{short-range}}$ and the subsidiary conditions.

We find it convenient to write the relationship between any old operator, O_{old} and the corresponding new operator O_{new} as

$$\begin{aligned} O_{\text{old}}&=\exp(-iS/\hbar)O_{\text{new}}\exp(iS/\hbar) \\ &=O_{\text{new}}+(i/\hbar)[O_{\text{new}},S] \\ &\quad-(1/2\hbar^2)[[O_{\text{new}},S],S]+\cdots, \end{aligned} \tag{48}$$

and we will classify terms in this series according to the power of S they contain; i.e., $[O,S]$ is the first-order commutator of O and S. We then find, keeping only first-order commutators, that

$$\begin{aligned} \mathbf{p}_i=\mathbf{P}_i+(e/m)\sum_{k<k_c}\left(\frac{2\pi\hbar}{\omega}\right)^{\frac{1}{2}}\mathbf{k}\Bigg\{&\frac{\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_i-\hbar\mathbf{k}/2)}{\omega-\mathbf{k}\cdot\mathbf{P}_i/m+\hbar k^2/2m} \\ &\times A_k\exp(i\mathbf{k}\cdot\mathbf{X}_i)+\exp(-i\mathbf{k}\cdot\mathbf{X}_i)A_k^* \\ &\times\frac{\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_i-\hbar\mathbf{k}/2)}{\omega-\mathbf{k}\cdot\mathbf{P}_i/m+\hbar k^2/2m}\Bigg\}+\cdots, \end{aligned} \tag{49}$$

$$\begin{aligned} a_k=A_k-(e/m)\sum_i&\left(\frac{2\pi}{\hbar\omega}\right)^{\frac{1}{2}} \\ &\times\exp(-i\mathbf{k}\cdot\mathbf{X}_i)\frac{\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_i-(\hbar\mathbf{k})/2)}{\omega-\mathbf{k}\cdot\mathbf{P}_i/m+\hbar k^2/2m}+\cdots, \end{aligned} \tag{50}$$

$$\begin{aligned} e^{i\mathbf{k}\cdot\mathbf{x}_i}=\exp(i\mathbf{k}\cdot\mathbf{X}_i)&+\sum_{l<l_c}\left(\frac{2\pi e^2\omega}{\hbar l^2}\right)^{\frac{1}{2}} \\ &\times\Bigg\{\frac{1}{\omega-\mathbf{l}\cdot\mathbf{P}_i/m+\hbar l^2/2m+\hbar\mathbf{l}\cdot\mathbf{k}/m} \\ &-\frac{1}{\omega-\mathbf{l}\cdot\mathbf{P}_i/m+\hbar l^2/2m}\Bigg\}A_l\exp[i(\mathbf{l}+\mathbf{k})\cdot\mathbf{X}_i] \\ &-\sum_{l<l_c}\left(\frac{2\pi e^2\omega}{\hbar l^2}\right)^{\frac{1}{2}}\exp(-i\mathbf{l}\cdot\mathbf{X}_i)A_l^* \\ &\times\Bigg\{\frac{1}{\omega-\mathbf{l}\cdot\mathbf{P}_i/m+\hbar l^2/2m+\hbar\mathbf{l}\cdot\mathbf{k}/m} \\ &-\frac{1}{\omega-\mathbf{l}\cdot\mathbf{P}_i/m+\hbar l^2/2m}\Bigg\}\exp(i\mathbf{k}\cdot\mathbf{X}_i)+\cdots, \end{aligned} \tag{51}$$

and we shall use these relationships in determining H_{new}.

We now proceed in a manner directly analogous to that of Paper I. We classify terms in H_{new} by considering the corresponding schematic terms in H [Eq. (44a)–(44d)] from which they may be considered to arise. Every term, $\mathcal{T}$, in H, leads to a zero-order (commutator) term, $\mathcal{T}$, which is the same function of the new variables as it was of the old variables, and in addition, a first order commutator, $+(i/\hbar)[\mathcal{T},S]$, a second-order term, $-(1/2\hbar^2)[\mathcal{T},[\mathcal{T},S]]$, etc. A convenient grouping of the terms in H exists which considerably simplifies the calculation of H_{new}. To demonstrate this grouping, we consider

$$H_a=\sum_i P_i^2/2m+\sum_{k<k_c}(\hbar\omega/2)(a_k^*a_k+a_ka_k^*).$$

The first-order commutator arising from H_a is

$$\begin{aligned} +(i/h)[\mathcal{H}_a,S]=-(e/m)&\sum_{i,k<k_c}(2\pi\hbar/\omega)^{\frac{1}{2}} \\ &\times\{\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_i-\hbar\mathbf{k}/2)A_k\exp(i\mathbf{k}\cdot\mathbf{X}_i) \\ &+\exp(-i\mathbf{k}\cdot\mathbf{X}_i)A_k^*\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_i-\hbar\mathbf{k}/2)\}. \end{aligned} \tag{52}$$

By Eq. (44c) we see that the above term is just the negative of H_{I}, expressed in terms of the new variables. Thus, the first-order commutator of $\mathcal{H}_a$ with S cancels the term arising from the zero-order commutator of $\mathcal{H}_{\text{I}}$. $\mathcal{H}_{\text{I}}$ and $\mathcal{H}_a$ are thus "connected" in that a simple relationship exists between the various order commutators arising from these terms; in fact, the nth order commutator of $\mathcal{H}_a$ with S is equal to the negative of the $(n-1)$th-order commutator of $\mathcal{H}_{\text{I}}$ with S. The terms in H_{new} arising from the connected terms H_a+H_{I}, may consequently be written in the following series:

$$H'=\mathcal{H}_a+\sum_{n=1}^{\infty}[\mathcal{H}_{\text{I}},S]_n\left\{\frac{1}{n!}-\frac{1}{(n+1)!}\right\}(i/\hbar)^n, \tag{53}$$

where $[\mathcal{H}_I, S]_n$ is the nth-order commutator of $\mathcal{H}_I$ with S.

We shall see that the effects of the field-particle interaction (up to order α) are contained in the first correction term to $\mathcal{H}_a$, $[(i/2\hbar)S, \mathcal{H}_I]$. The higher-order commutators will be shown to lead to effects of order α^2 or $\alpha(n'/3n)$ and may hence be neglected. The evaluation of our lowest-order term, $(i/2h)[S, \mathcal{H}_I]$ is lengthy, but straightforward. We find, after some rearrangement of terms, that

$$
\begin{aligned}
(i/2h)[\mathcal{H}_I, S] &= \frac{4\pi e^2}{m}\sum_{ki}\left(\frac{\hbar}{4\omega}\right) \\
&\times\left\{\frac{2\omega(\mathbf{k}\cdot\mathbf{P}_i/m)-(\mathbf{k}\cdot\mathbf{P}_i/m)^2+(\hbar^2k^4/4m^2)}{(\omega-\mathbf{k}\cdot\mathbf{P}_i/m)^2-(\hbar^2k^4/4m^2)}\right\} \\
&\times(A_kA_k^*+A_k^*A_k)+\frac{4\pi e^2}{m}\sum_{ki}\left(\frac{\hbar}{4\omega}\right) \\
&\times\left\{\frac{2\omega(\mathbf{k}\cdot\mathbf{P}_i/m)+(\mathbf{k}\cdot\mathbf{P}_i/m)^2-(\hbar^2k^4/4m^2)}{(\omega+\mathbf{k}\cdot\mathbf{P}_i/m)^2-(\hbar^2k^4/4m^2)}\right\} \\
&\times(A_kA_{-k}+A_k^*A_{-k}^*)-(\pi e^2/m^2) \\
&\times\sum_{\substack{k<k_c\\ i,j;i\neq j}}\frac{[\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_i-\hbar\mathbf{k}/2)][\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_j+\hbar\mathbf{k}/2)]}{\omega[\omega-\mathbf{k}\cdot\mathbf{P}_j/m-\hbar k^2/2m]} \\
&\times\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)]+\exp[-i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)] \\
&\times\frac{\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_i-\hbar\mathbf{k}/2)\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_j+\hbar\mathbf{k}/2)}{\omega(\omega-\mathbf{k}\cdot\mathbf{P}_j/m-\hbar k^2/2m)} \\
&-\frac{2\pi e^2}{m^2}\sum_{ik<k_c}\frac{(\boldsymbol{\varepsilon}_k\cdot\mathbf{P}_i)^2}{(\omega-\hbar k^2/2m)^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2}. \qquad (54)
\end{aligned}
$$

In obtaining (54) we have neglected a number of terms which are quadratic in the field variables and are multiplied by a phase factor with a nonvanishing argument, $\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{X}_i]$. These are terms like

$$
\frac{4\pi e^2}{m^2}\sum_{\substack{k<k_c,i\\ l<k_c\\ l\neq -k}}\left(\frac{\hbar}{4\omega}\right)\frac{\boldsymbol{\varepsilon}_l\cdot(\mathbf{P}_i-\hbar\mathbf{l}/2)A_lA_k}{\omega-\mathbf{l}\cdot\mathbf{P}_i/m+\hbar l^2/2m} \times\exp[i(\mathbf{k}+\mathbf{l})\cdot\mathbf{X}_i]. \qquad (55)
$$

Such terms are of exactly the same character as those we considered earlier in U [Eq. (15)], except that they are smaller by a factor of $\sim(\mathbf{l}\cdot\mathbf{P}_i/m\omega)$. Exactly the same arguments that we applied in showing that U could be neglected may be applied to terms like (55), with the result that we find that these terms are also completely negligible, leading in fact, to an energy correction which is smaller than that arising from U by a factor of α [Eq. (39)].

The remaining lowest-order term in H_{new} is just the zero-order term from $H_{\text{field}}-\sum_{k<k_c}(\hbar\omega/2)(a_k^*a_k+a_ka_k^*)$,

$$
\mathcal{H}_{\text{field}}-\tfrac{1}{2}\sum_{k<k_c}(\hbar\omega/2)(A_k^*A_k+A_kA_k^*)=\sum_{k<k_c}(\hbar/4\omega) \times(\omega_p^2-\omega^2)[A_k^*A_k+A_kA_k^*-A_kA_{-k}-A_{-k}^*A_k^*]. \qquad (56)
$$

We will now show that if we define ω by the dispersion relation,

$$
1=\frac{4\pi e^2}{m}\sum_i\frac{1}{(\omega-(\mathbf{k}\cdot\mathbf{P}_i/m))^2-\hbar^2k^4/4m^2}, \qquad (57)
$$

then the sum of (56) and the first two terms of (54) vanishes. To see this we note that multiplying (57) by $\omega^2-\omega_p^2$ on both sides, and rearranging terms on the right-hand side, yields

$$
\begin{aligned}
\omega^2-\omega_p^2&=\frac{4\pi e^2}{m}\sum_i\frac{\omega^2-[\omega-(\mathbf{k}\cdot\mathbf{P}_i/m)]^2+\hbar^2k^4/4m^2}{[\omega-(\mathbf{k}\cdot\mathbf{P}_i/m)]^2-\hbar^2k^4/4m^2} \\
&=\frac{4\pi e^2}{m}\sum_i\frac{2\omega(\mathbf{k}\cdot\mathbf{P}_i/m)+(\hbar^2k^4/4m^2)-(\mathbf{k}\cdot\mathbf{P}_i/m)^2}{[\omega-(\mathbf{k}\cdot\mathbf{P}_i/m)]^2-\hbar^2k^4/4m^2}, \qquad (58)
\end{aligned}
$$

from which the above statement follows for the $(A_k^*A_k+A_kA_k^*)$ terms in (54). The (A_kA_{-k} and $A_k^*A_{-k}^*$) terms likewise go out when we replace $\mathbf{k}$ by $-\mathbf{k}$ in (57) and (58) and use the resulting relations to compare (56) and (54).

The results in lowest order of our canonical transformation on the Hamiltonian may thus be expressed schematically as follows:

$$
H_{\text{new}}^{(0)}=H_{\text{electron}}+H_{\text{coll.}}+H_{\text{res part}}, \qquad (59)
$$

where

$$
\begin{aligned}
H_{\text{electron}}&=\sum_i\frac{P_i^2}{2m}-\frac{2\pi e^2}{m^2}\sum_{ik<k_c}\frac{(\boldsymbol{\varepsilon}_k\cdot\mathbf{P}_i)^2}{(\omega-\hbar k^2/2m)^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2} \\
&+2\pi e^2\sum_{\substack{i,jk>k_c\\ i\neq j}}\frac{\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)]}{k^2}-2\pi ne^2\sum_{k<k_c}\frac{1}{k^2}, \qquad (60)
\end{aligned}
$$

$$
H_{\text{coll.}}=\tfrac{1}{2}\sum_{k<k_c}(\hbar\omega)(A_k^*A_k+A_kA_k^*), \qquad (61)
$$

and

$$
\begin{aligned}
H_{\text{res part}}&=-\frac{\pi e^2}{m^2}\sum_{\substack{k<k_c\\ ij:i\neq j}}\frac{[\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_i-\hbar\mathbf{k}/2)][\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_j+\hbar\mathbf{k}/2)]}{\omega[\omega-\mathbf{k}\cdot\mathbf{P}_j/m-\hbar k^2/2m]} \\
&\times\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)]+\exp[-i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)] \\
&\times\frac{[\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_i-\hbar\mathbf{k}/2)][\boldsymbol{\varepsilon}_k\cdot(\mathbf{P}_j+\hbar\mathbf{k}/2)]}{\omega[\omega-(\mathbf{k}\cdot\mathbf{P}_j/m)-(\hbar k^2/2m)]}. \qquad (62)
\end{aligned}
$$

The effect of our transformation on the subsidiary conditions may be obtained in similar fashion. Our new subsidiary conditions are given by

$$(\Omega_k)_{\text{new}}\psi = \exp(-iS/\hbar)\Omega_k' \exp(iS/\hbar)\psi$$

$$= \left\{\Omega_k' - \frac{i}{\hbar}[S, \Omega_k'] - \frac{1}{2\hbar^2}[S, [S, \Omega_k']] + \cdots\right\}$$

$$\times \psi = 0, \quad (k < k_c), \quad (63)$$

where ψ is our new system wave function, and Ω_k' is the same function of the new variables that Ω_k was of the old variables. We find

$$(\Omega_k)_{\text{new}} = A_k^* \left(1 - \frac{4\pi e^2}{m} \sum_i \frac{1}{[\omega - (\mathbf{k}\cdot\mathbf{P}_i/m)]^2 - \hbar^2 k^4/4m^2}\right)$$

$$+ A_{-k}\left(1 - \frac{4\pi e^2}{m} \sum_i \frac{1}{[\omega + (\mathbf{k}\cdot\mathbf{P}_i/m)]^2 - \hbar^2 k^4/4m^2}\right)$$

$$- \left(\frac{8\pi e^2}{k^2\hbar\omega}\right)^{\frac{1}{2}} \sum_i \frac{\omega^2}{\omega^2 - (\mathbf{k}\cdot\mathbf{P}_i/m - \hbar k^2/2m)^2}$$

$$\times \exp(ik\cdot\mathbf{X}_i) - \left(\frac{8\pi e^2}{k^2\hbar\omega}\right)^{\frac{1}{2}} \sum_{\substack{l<l_c,i \\ l\neq k}} \left(\frac{2\pi e^2\omega}{\hbar l^2 m^2}\right)^{\frac{1}{2}}$$

$$\times \left\{\left[\frac{1}{\omega + (\mathbf{l}\cdot\mathbf{P}_i/m) + \hbar l^2/2m - \hbar\mathbf{l}\cdot\mathbf{k}/m}\right.\right.$$

$$\left. - \frac{1}{\omega + \mathbf{l}\cdot\mathbf{P}_i/m + \hbar l^2/2m}\right] A_{-l} \exp[+i(\mathbf{k}-\mathbf{l})\cdot\mathbf{X}_i]$$

$$- \exp(-i\mathbf{l}\cdot\mathbf{X}_i) A_1^* \left[\frac{1}{\omega - \mathbf{l}\cdot\mathbf{P}_i/m + \hbar l^2/2m + \hbar\mathbf{l}\cdot\mathbf{k}/m}\right.$$

$$\left.\left. - \frac{1}{\omega - \mathbf{l}\cdot\mathbf{P}_i + hl^2/2m}\right] \exp(i\mathbf{k}\cdot\mathbf{X}_i)\right\}$$

$$- \frac{1}{2\hbar^2}[S, [S, \Omega_k']] + \cdots. \quad (64)$$

$(\Omega_k)_{\text{new}}$ is considerably simplified when we note that the first two terms vanish when we apply the dispersion relation [Eq. (57)] for both plus and minus k. The fourth term consists of a linear term in the field coordinates multiplied by a nonvanishing phase factor, and the effect of such a term in the subsidiary condition is the same as that of a term like (55) in the Hamiltonian. Since there is no point in obtaining the subsidiary condition to a higher order of accuracy than is maintained in our Hamiltonian, we may neglect this term. With this approximation, our subsidiary condition reduces to one which does not involve (in lowest orders) the field variables, and is given by

$$(\Omega_k)_{\text{new}}\psi = \sum_i \frac{\omega^2}{\omega^2 - [(\mathbf{k}\cdot\mathbf{P}_i/m) - \hbar k^2/2m]^2}$$

$$\times \exp(i\mathbf{k}\cdot\mathbf{X}_i)\psi = 0, \quad (k < k_c). \quad (65)$$

IV.

The physical consequences of our canonical transformation follow from the lowest-order Hamiltonian, $H_{\text{new}}^{(0)}$ [Eq. (59)] and the associated set of subsidiary conditions on our system wave function [Eq. (65)]. We discuss these briefly and then show that the higher-order terms in H_{new} and $(\Omega_k)_{\text{new}}$ are actually negligible. We first note that our field coordinates occur only in $H_{\text{coll.}}$, and thus describe a set of uncoupled fields which carry out real independent longitudinal oscillations, since the subsidiary conditions no longer relate field and particle variables, and since there are no field-particle interaction terms in H_{new}. The frequency of these collective oscillations is given by the dispersion relation [Eq. (57)], which is the appropriate quantum-mechanical generalization of the classical dispersion relation derived in II, as well as being the longitudinal analog of the quantum-dispersion relation for organized transverse oscillation, which we obtained in I.

This dispersion relation plays a key role in our collective description, since it is only for $\omega(k)$ which satisfy it that we can eliminate the unwanted terms in the Hamiltonian [Eq. (54)] and the unwanted field terms in the subsidiary condition. For sufficiently small k, we may expand (57) in powers of $(\mathbf{k}\cdot\mathbf{P}_i/m\omega)$ and $(\hbar k^2/m\omega)$ and so always obtain a solution for $\omega(k)$. If we do this, and assume an isotropic distribution of $\mathbf{P}_i$, we find

$$\omega^2 = \omega_p^2 + \frac{k^2}{nm^2} \sum_i P_i^2 + \frac{\hbar^2 k^4}{4m^2}, \quad (66)$$

and hence

$$\omega = \omega_p \left(1 + \frac{k^2}{2nm^2} \sum_i \frac{P_i^2}{\omega_p^2} + \frac{\hbar^2 k^4}{8m^2\omega_p^2}\right). \quad (67)$$

These appropriate dispersion relations are, in fact, quite sufficient for our purpose, since the expansions involved in obtaining them are the same that we have used in obtaining H_{new}.

We have treated ω as a pure number thus far, although we see from (57) or (67) that ω is, in fact, an operator, since it contains $\mathbf{P}_i$. We have ignored this fact, for instance, in working out our commutation relations and obtaining H_{new}. This treatment of ω as a pure number is only strictly justified if our system wave function is an eigenfunction of $\mathbf{P}_i$, which is not the case. Thus ω contains and, in turn, can contribute to the Hamiltonian, off-diagonal terms which cause transitions between states of different energy. These terms could then, in principle, be eliminated from the

Hamiltonian by a further canonical transformation. However because the dependence of ω on P_i is already or order α, this elimination would produce terms of order α^2 which are truly negligible. We are justified in neglecting the off-diagonal elements of the operator ω.

According to (67), in consequence of the electron-field interaction the frequency of the collective oscillations has become k dependent. We may obtain an order-of-magnitude estimate of the fractional change in this frequency by averaging the dispersion relation (67) over all $k<k_c$ and carrying out the indicated sum over particle momenta. In obtaining this mean value of $\sum_i P_i^2$, we should use the appropriate eigenfunctions of our new Hamiltonian (59). However, as we shall see later, the correct particle eigenfunctions can be replaced for many applications by plane waves, so that $\sum_i P_i^2$ may be approximately evaluated by assuming a Fermi distribution of electrons at absolute zero. We then find

$$\langle\omega\rangle_{\mathrm{Av}}=\omega_p\left(1+\frac{3}{10}\frac{\langle k^2\rangle_{\mathrm{Av}}P_0^2}{m^2\omega_p^2}+\frac{\hbar^2\langle k^4\rangle_{\mathrm{Av}}}{8m^2\omega_p^2}\right)$$

$$=\omega_p(1+3\alpha[1+(3/10)\beta^2]), \quad (68)$$

where α is given by (39) and β by (40). Since $\beta\lesssim 1$, we see that the effect of the k^4 term is small compared to the k^2 term. This result holds true quite generally, in that where an expansion in powers of $\alpha=\langle(\mathbf{k}\cdot\mathbf{P}_i/m\omega)^2\rangle_{\mathrm{Av}}$ is justified, the terms of order $(h^2k^4/4m^2\omega^2)$ are negligible. The average fractional increase in the frequency is thus of order 3α. As we have remarked, for the electronic densities encountered in metals, α turns out to be $\sim 1/16$, so that this constitutes at most a 20 percent correction in the collective oscillation frequency.

The effect on the electrons of the elimination (in lowest order) of the electron-field interaction may be seen by considering the second term in H_{electron} and $H_{\text{res part}}$. We first note that in the approximation of small α, the second term in H_{electron} becomes

$$E_{\text{red}}=-\frac{k_c^3}{18\pi^2 n}\sum_i\frac{P_i^2}{2m}=-\frac{n'}{3n}\sum_i\frac{P_i^2}{2m}.$$

If we combine this with the first term, $\sum_i P_i^2/2m$, we obtain

$$E_{\text{red}}+\sum_i\frac{P_i^2}{2m}\left(\frac{3n-n'}{3n}\right)=\sum_i\frac{P_i^2}{2m^*},$$

where

$$m^*=m\times 3n/(3n-n'). \quad (69)$$

Thus the "new" electrons behave as if they had an effective mass m^*, which is given by (69), and which is slightly greater than the "bare" electron mass m. This increase in the effective electronic mass has a simple physical interpretation. For we note that according to Eqs. (47) and (48),

$$\mathbf{X}_i=\mathbf{x}_i-\frac{ei}{m}\sum_{k<k_c}\left(\frac{2\pi\hbar}{\omega}\right)^{\frac{1}{2}}\left\{\frac{\boldsymbol{\varepsilon}_k\omega A_k}{\omega-\mathbf{k}\cdot\mathbf{P}_i/m+\hbar k^2/2m}\right.$$
$$\times\exp(i\mathbf{k}\cdot\mathbf{x}_i)-\exp(-i\mathbf{k}\cdot\mathbf{x}_i)$$
$$\left.\times\frac{\boldsymbol{\varepsilon}_k\omega A_k{}^*}{\omega-\mathbf{k}\cdot\mathbf{P}_i/m+\hbar k^2/2m}\right\}. \quad (70)$$

The $\mathbf{X}_i$ thus represents the "bare" electron plus an associated cloud of collective oscillation; the increased effective mass may be regarded as an inertial effect resulting from the fact that these electrons carry such a cloud along with them.

$H_{\text{res part}}$, in the approximation of small α, may be written as

$$H_{\text{r.p.}}=\frac{-2\pi e^2}{m^2}\sum_{\substack{k<k_c\\ i,j;i\neq j}}\frac{(\boldsymbol{\varepsilon}_k\cdot P_i)(\boldsymbol{\varepsilon}_k\cdot P_j)}{\omega_p{}^2+k^2\langle V^2\rangle_{\mathrm{Av}}}\exp[ik\cdot(\mathbf{X}_i-\mathbf{X}_j)]$$
$$=-2\pi e^2\sum_{\substack{k<k_c\\ i,j;i\neq j}}\frac{(\boldsymbol{\varepsilon}_k\cdot\mathbf{P}_i)(\boldsymbol{\varepsilon}_k\cdot\mathbf{P}_j)/(m^2\langle V^2\rangle_{\mathrm{Av}})}{k^2+K^2}$$
$$\times\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)], \quad (71)$$

where $\langle V^2\rangle_{\mathrm{Av}}=\sum_i P_i^2/m^2n$ and $K^2=\omega_p{}^2/\langle V^2\rangle_{\mathrm{Av}}$. If we assume that the electrons form a completely degenerate gas, then for most metals,

$$K^2=(5/3)(\omega_p{}^2/v_0{}^2)\cong k_0{}^2. \quad (72)$$

Thus

$$H_{\text{r.p.}}\cong-2\pi e^2\sum_{\substack{i\neq j\\ k<k_c}}\frac{(\boldsymbol{\varepsilon}_k\cdot\mathbf{P}_i)(\boldsymbol{\varepsilon}_k\cdot\mathbf{P}_j)/(m^2\langle V^2\rangle_{\mathrm{Av}})}{k^2+k_0{}^2}$$
$$\times\exp[i\mathbf{k}\cdot(\mathbf{X}_i-\mathbf{X}_j)]. \quad (73)$$

$H_{\text{res part}}$ thus describes an extremely weak attractive velocity dependent electron-electron interaction. For if the summation in (73) were over all k, it would correspond to a screened interaction of range $\sim(1/k_0)$; however, the summation is only for $k<k_c$, where $k_c<k_0$, so that we are describing here that part of a screened interaction beyond the screening length. A more detailed analysis confirms that this qualitative estimate, and justifies our neglecting $H_{\text{r.p.}}$ in comparison with $H_{\text{s.r.}}$ in considering the effects of electron-electron interaction.

Let us now consider the effect of the higher-order terms, such as $[S,[S,\mathcal{H}_{\mathrm{I}}]]$. {The higher-order commutators arising from $\mathcal{H}_{\text{field}}-\sum_{k<k_c}(\hbar\omega/2)(A_k{}^*A_k+A_kA_k{}^*)$ will be of this same type, since the zero-order commutator from this term cancelled part of $[S,\mathcal{H}_{\mathrm{I}}]$.} The calculation of $[S,[S,\mathcal{H}_{\mathrm{I}}]]$ is quite straightforward, but scarcely worth going into here, since by comparison of Eqs. (49), (50), (51), and (44), it may easily be seen that the lowest order non-negligible terms terms will resemble H_{I} but will be at least of order $(\mathbf{k}\cdot\mathbf{P}_i/m\omega)$ smaller. These terms which we earlier de-

noted by H_{II} could be eliminated by a further transformation. However, since as we have seen, the elimination of H_{I} led to effects of order α (or $n'/3n$), the effects so obtained would then be of order α^2, and we may neglect them entirely in our approximation of small α. Exactly the same conclusions apply with respect to the higher-order commutators of the subsidiary condition operator, $(\Omega_k)_{\text{new}}$, since it is not fruitful for us to evaluate $(\Omega_k)_{\text{new}}$ to any greater accuracy than that obtaining for H_{new}.

It is interesting to note that included in these higher-order terms is the influence of our effective mass correction, Eq. (69), on the frequency of the collective oscillations. Thus, on evaluating these terms, one finds

$$\omega^2=\omega_p{}^2+\frac{k^2}{n(m^*)^2}\sum_i P_i{}^2+\frac{h^2k^4}{4(m^*)^2},$$

instead of the dispersion relation (66). This is, of course, just what might be expected, since the successive elimination of the field-particle interaction terms leads to a mass renormalization, familiar from quantum electrodynamics, in that everywhere m appears, it should properly be replaced by m^*.[11] This correction is here quite negligible, usually leading to a fractional change in the collective oscillation frequency of less than 1 percent. For this change is $\sim(\alpha n'/n)$, and for the electronic densities encountered in metals,

$$\alpha(n'/n)\cong(1/16)(3/25)=(3/400).$$

Our only other approximation has been to neglect the effect of the canonical transformation on $H_{\text{s.r.}}$, which will lead, indirectly, to the effect of $H_{\text{s.r.}}$ on the collective oscillations. Suppose we consider a typical first-order term arising from $[S, H_{\text{s.r.}}]$. This will be like

$$2\pi e^2\sum_{\substack{k>k_c\\ l<k_c\\ i,j}}\left(\frac{2\pi e^2}{\hbar l^2}\right)^{\frac{1}{2}}\frac{\exp(-i\mathbf{l}\cdot\mathbf{X}_j)}{k^2}\frac{(\hbar\mathbf{l}\cdot\mathbf{k}/m)A_l{}^*\exp(i\mathbf{k}\cdot\mathbf{X}_i)\exp(-i\mathbf{k}\cdot\mathbf{X}_j)}{(\omega-\mathbf{l}\cdot\mathbf{P}_i/m+\hbar l^2/2m)(\omega-\mathbf{l}\cdot\mathbf{P}_i/m+\hbar l^2/2m+\hbar\mathbf{l}\cdot\mathbf{k}/m)}. \tag{74}$$

These terms thus consist of a nonvanishing phase factor multiplying a field variable and a short-wavelength density fluctuation. The structure of (74) is quite similar to that of U [Eq. (15)], the difference being that the short-wavelength density fluctuation $\sum_j\exp(-i\mathbf{k}\cdot\mathbf{X}_j)$ here plays the same role as the collective field variable (which is essentially a long-wavelength density fluctuation) did in U. If we had a term for which $\mathbf{k}=\mathbf{l}$, (74) would reduce to a term like H_{I}, just as the third term in (11) reduced to $\omega_p{}^2\sum_k\times(q_kq_{-k}/2)$. Thus we might expect that (74) bears about the same relationship to H_{I}, as U does to $(\omega_p{}^2/2)\times\sum_k q_kq_{-k}$. However, it is quite a bit more difficult to establish the smallness of (74) mathematically than it was for U, since a perturbation theoretic estimate involves the consideration of intermediate states in which two electrons are excited. We note that the main effect of $H_{\text{s.r.}}$ is to produce short-range correlations in particle positions, analogous to the long-range correlations produced by the long-range part of the Coulomb potential, in the sense that the particles tend to keep apart and thus tend to reduce the effectiveness of $H_{\text{s.r.}}$. Because of the analytical difficulties involved in a justification along these lines we prefer to justify our neglect of (74) in a more qualitative and physical fashion.

We see that (74) describes the effect of the collective oscillations on the short range collisions between the electrons, and conversely, the effect of the short-range collisions on the collective oscillations. We may expect that these effects will be quite small, since $H_{\text{s.r.}}$ is itself a comparatively weak interaction. The short-range electron-electron collisions arising from $H_{\text{s.r.}}$ will act to damp the collective oscillations, a phenomenon which has been treated in some detail classically by Bohm and Gross.[12] A test for the validity of our approximation in neglecting terms like (74) is that the damping time from the collisions be small compared with the period of a collective oscillation. In this connection we may make the following remarks:

(1) Electron-electron collisions are comparatively ineffective in damping the oscillations, since momentum is conserved in such collisions, so that to a first approximation such collisions produce no damping. [Such collisions produce damping only in powers of $(\mathbf{k}\cdot\mathbf{P}_i/m\omega)$ higher than the first.]

(2) The exclusion principle will further reduce the cross section for electron-electron collision.

(3) If H_{I} is neglected, collisions have no effect on the collective oscillations. This means that the major part of the collective energy is unaffected by these short-range collisions, since only that part coming from H_{I}, (which is of order α relative to $\hbar\omega_p$) can possibly be influenced. Thus at most 20 percent of the collective energy can be damped in a collision process.

All of these factors combine to reduce the rate of damping, so that we believe this rate is not more than 1 percent per period of an oscillation and probably is quite a bit less. A correspondingly small broadening of the levels of collective oscillation is to be expected. It is for these reasons that we feel justified in neglecting the effects of our canonical transformation on $H_{\text{s.r.}}$.

---

[11] Note, however, that m is not replaced by m^* in our expression for ω_p, Eq. (14), since the collective oscillations are not affected by the field-electron interaction in this order.

[12] D. Bohm and E. P. Gross, Phys. Rev. **75**, 1864 (1949).

V.

The motion of the electrons in our new representation is considerably more complicated than that of the collective fields. The major reason for this complication is our set of subsidiary conditions (65), which essentially act to reduce the number of individual electron degrees of freedom from $3n$ to $3n-n'$, where n' is the number of collective degrees of freedom and is given by

$$n'=\frac{k_c^3}{6\pi^2}=\frac{\beta^3k_0^3}{6\pi^2}=\frac{\beta^3n}{2}. \tag{75}$$

We may obtain a better understanding of the role of these subsidiary conditions by making use of the density fluctuation concept which we developed in Paper II. There we saw that classically the collective component of the density fluctuation ρ_k was proportional to

$$R_k{}^c=\sum_i\frac{1}{\omega^2-(\mathbf{k}\cdot\mathbf{P}_i/m)^2}e^{i\mathbf{k}\cdot\mathbf{x}_i}.$$

In a quantum-theoretical treatment of the density fluctuations, the collective component is found to be proportional to

$$R_k{}^q=\sum_i\frac{1}{\omega^2-[(\mathbf{k}\cdot\mathbf{P}_i/m)-\hbar k^2/2m]^2}e^{i\mathbf{k}\cdot\mathbf{x}_i}. \tag{76}$$

This result may be seen to follow directly from the quantum generalization of the methods of II given in Appendix I. In the preceding expressions, $\mathbf{x}_i$ and $\mathbf{p}_i$ of course refer to the "original" position and momentum of the electron, i.e., the Hamiltonian in terms of these variables is given by Eq. (1). On the other hand, our "new" electron variables (X_i, P_i) describe electron motion in the absence of any collective oscillation, since there are no terms in our Hamiltonian (59) which couple the electrons and the collective oscillation. Consequently we should expect that the collective component of the density fluctuation when expressed in terms of these "new" variables should vanish, since these variables are chosen to describe "pure" individual electron motion and are incapable of describing, or taking part in, collective oscillation. But this is just what our subsidiary conditions assert, as may be seen by comparing (76) and (65). Thus, if we carry out a transformation to "individual" electron variables, we must expect a set of subsidiary conditions given by (65), since these guarantee that we have developed a consistent description of the state of the electron gas in the absence of collective oscillation.

The physical content of the subsidiary condition also follows from the density fluctuation concept. For we may rewrite the subsidiary condition, Eq. (65) as

$$\sum_i\exp(+i\mathbf{k}\cdot\mathbf{X}_i)\psi=\sum_i\frac{(\mathbf{k}\cdot\mathbf{P}_i/m-\hbar k^2/2m)^2\exp(i\mathbf{k}\cdot\mathbf{X}_i)}{\omega^2-[(\mathbf{k}\cdot\mathbf{P}_i/m)-\hbar k^2/2m]^2}. \tag{77}$$

Since we are dealing with $k<k_c$, for which $(\mathbf{k}\cdot\mathbf{P}_i/m\omega)^2\ll1$, we see that the subsidiary condition asserts that in terms of our new coordinates and momenta, the density fluctuations of long wavelength are greatly reduced. This reduction is due to the fact that the major portion of the long-wavelength density fluctuations is associated with the collective oscillations, and described in terms of these in our collective description.

In our new representation, the subsidiary conditions (65) continue to commute with the Hamiltonian (59). This follows since the commutation relations are unchanged by a canonical transformation; it may easily be directly verified from (65) and (59) that these commute within the approximations we have made. Consequently, just as was the case with (11) and (12), if we correctly solved for the exact lowest state eigenfunction of our Hamiltonian H_{new}, we would automatically satisfy the subsidiary conditions (65), since the ground state of our system is nondegenerate. For this reason, the energy of the lowest state of our system is relatively insensitive to whether we satisfy the subsidiary conditions or not. For since the lowest state wave function does satisfy the subsidiary condition, moderate changes in this wave function, involving corresponding failures to satisfy the subsidiary conditions, will provide quite small changes in the energy. Conversely, because of this insensitivity of the ground state energy to the degree of satisfaction of the subsidiary condition, it will take a quite good approximation to the lowest eigenfunction of H_{new} to satisfy the subsidiary conditions to a fair degree of approximation.

It should be noted that the lowest state wave function satisfies the subsidiary condition because of the effects of the term $H_{\text{r.p.}}$ in the Hamiltonian. For as we have seen, the subsidiary condition describes a long range correlation in the particle positions, which is independent of the amplitude of collective oscillation. In the approximation that we are using, this correlation has to be due to the residual interaction between the particles, since the subsidiary conditions will automatically be satisfied if we solve for the lowest state wave function. At first sight, it might be thought that the short-range potential $H_{\text{s.r.}}$ might also play an important role in establishing these correlations, since it corresponds to a fairly strong interaction potential when the particles are close to each other. However, from the definition of $H_{\text{s.r.}}$ in Eq. (18), we see that it has no Fourier components corresponding to $k<k_c$. As a simple perturbation theoretical calculation shows, the only effect of $H_{\text{s.r.}}$ in the first approximation is to turn a plane wave function.

$$\psi_0=\exp(i\sum_n\mathbf{k}_n\cdot\mathbf{x}_n),$$

into the function

$$\psi=\psi_0(1+\sum_{\substack{m,n\\k>k_c}}C_{mn}\exp[i\mathbf{k}\cdot(\mathbf{x}_m-\mathbf{x}_n)]),$$

where C_{mn} is a suitable expansion coefficient, which can be obtained by a detailed calculation.[13] But since the sum is restricted to $k > k_c$, $H_{\text{s.r.}}$ introduces only *short-range* correlations, which have nothing to do with the subsidiary conditions. On the other hand, $H_{\text{r.p.}}$ which has only *long-range* fourier components (i.e., $k < k_c$) introduces only corresponding long-range correlations. Thus, in the present approximation, it is $H_{\text{r.p.}}$ that is responsible for the long-range correlations implied by the subsidiary condition.

On the basis of the above conclusions, we may deduce the following physical picture. The long range Coulomb forces produce a tendency for electrons to keep apart, as a result of which the Coulomb force itself tends to be neutralized. But this neutralization could not be perfect; for if it were, then there would be no force left to produce the necessary correlations in particle positions. Our calculations show that $H_{\text{r.p.}}$ is the small residual part of the Coulomb force which must remain unneutralized in order to produce the long-range correlations needed for agreeing. Because this force is so small, it will produce only correspondingly small changes in the particle momenta, so that in most applications a set of plane waves will provide a good approximation to the particle wave function (in the new representation, of course).

All of the above applies rigorously only in the ground state. In the excited states, similar conclusions apply; but the application of the subsidiary conditions is more difficult, because the wave functions of the excited states are no longer now degenerate. Here, we could in general expand an arbitrary eigenfunction of $H_{\text{new}}^{(0)}$ as a series of eigenfunctions of $(\Omega_k)_{\text{new}}$. To satisfy the subsidiary conditions, we then retain only those terms in this series for which $(\Omega_k)_{\text{new}}=0$. This reduction in the number of possible eigenfunctions corresponds to the reduction in the number of individual electron degrees of freedom implied by (65). The exact treatment of the problem of the excited states is quite complex and will be reserved for a later paper by one of us. However, we may expect that if the reduction in the number of individual electron degrees of freedom is comparatively small [i.e., $(n'/n) \ll 1$], then their effect on the energy spectrum of the electron gas will be correspondingly reduced.

We conclude this section by summing up the results of our canonical transformation to the collective description. We have obtained a Hamiltonian describing collective oscillation plus a system of individual electrons interacting via a screened Coulomb force, with a screening radius of the order of the inner-electronic distance. Although the individual electron wave functions are restricted by a set of n' subsidiary conditions, which act to reduce the number of individual electron degrees of freedom and to inhibit the long-range density fluctuations associated with the individual electron motion, for many purposes the effect of these subsidiary conditions may be neglected. In Paper IV we examine the physical conclusions we are led to by the use of the collective description for the motion of electrons in metals. We shall see that these are in good agreement with experiment and enable us to resolve a number of hitherto puzzling features of the usual one-electron theory.

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APPENDIX I

In this appendix we treat the collective fluctuations in charge density by finding the equations of motion of the associated operators, thus developing a direct quantum-mechanical extension of the methods used in Paper II. We use the electron field second-quantization formalism, in order to facilitate comparison with the work of Tomonaga and to take into account explicitly the fact that the electrons obey Fermi statistics.

Following the usual treatments,[14] we describe the electrons by the field quantities $\psi_\sigma(\mathbf{x})$ which satisfy the anti-commutation relations $[\psi_\sigma(\mathbf{x}), \psi_\sigma(\mathbf{x}')]_+ = [\psi_\sigma{}^*(\mathbf{x}), \psi_{\sigma'}{}^*(\mathbf{x}')]_+ = 0$ and $[\psi_\sigma(\mathbf{x}), \psi_{\sigma'}{}^*(\mathbf{x}')]_+ = \delta(\mathbf{x}-\mathbf{x}')\delta_{\sigma\sigma'}$. σ refers to the electron spin and takes on two values corresponding to the two orientations of the electron spin. We work in the Heisenberg representation. The Hamiltonian which determines the equation of motion of the ψ's is

$$H = -\int \psi^*(\mathbf{x}) \frac{\hbar^2}{2m} \Delta \psi(\mathbf{x}) d\mathbf{x} + \frac{e^2}{2} \int \int \frac{\rho(\mathbf{x})\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d\mathbf{x} d\mathbf{x}', \tag{A1}$$

where

$$\rho(\mathbf{x}) = \textstyle\sum_\sigma \psi_\sigma{}^*(\mathbf{x})\psi_\sigma(\mathbf{x}). \tag{A2}$$

It is convenient to Fourier-analyze $\psi_\sigma(\mathbf{x})$ and $\psi_\sigma{}^*(\mathbf{x})$ by

$$\begin{aligned} \psi_\sigma(\mathbf{x}) &= \sum_k c_{k\sigma} e^{i\mathbf{k}\cdot\mathbf{x}}, \\ \psi_\sigma{}^*(\mathbf{x}) &= \sum_k c_{k\sigma}{}^* e^{-i\mathbf{k}\cdot\mathbf{x}}, \end{aligned} \tag{A3}$$

where the $c_{k\sigma}$ and $c_{k\sigma}{}^*$ obey the anticommutation relations

$$\begin{aligned} [c_{k\sigma}, c_{k'\sigma'}]_+ &= [c_{k\sigma}{}^*, c_{k'\sigma'}{}^*]_+ = 0, \\ [c_{k\sigma}, c_{k'\sigma'}{}^*]_+ &= \delta_{kk'}\delta_{\sigma\sigma'}, \end{aligned} \tag{A4}$$

in virtue of the anticommutation relations satisfied by the $\psi_\sigma(x)$. We also find

$$\rho(x) = \textstyle\sum_k \rho_k e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{A5}$$

where, using (A2) and (A3),

$$\rho_K = \textstyle\sum_k c_{k\sigma}{}^* c_{k+K\sigma}. \tag{A6}$$

In terms of $c_{k\sigma}$ and ρ_K, our Hamiltonian (A1) becomes

$$H = \sum_k c_{k\sigma}{}^* c_{k\sigma} \frac{\hbar^2 k^2}{2m} + 2\pi e^2 \sum_k \frac{\rho_k \rho_{-k}}{k^2}. \tag{A7}$$

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[13] The additional terms describe correlations in particle positions.

[14] See, for instance, G. Wentzel, *Quantum Theory of Wave Fields* (Interscience Publishers, New York, 1949).

The second quantization formalism we are using here is of course equivalent to the use of an antisymmetrized many-electron wave function in the usual configuration-space representation (which we use elsewhere in this paper). For instance, the density fluctuation operator ρ_k is equivalent to the configuration space operator $\sum_i \exp(-i\mathbf{k}\cdot\mathbf{X}_i)$ we introduce earlier. Thus the results obtained in this appendix may be directly compared to those obtained in the previous sections of this paper, and in Paper II.

In Paper II, we saw that classically ρ_k could be split into an oscillatory part q_k, and an additional part which represented the charge density of a set of screened electrons moving at random. We shall now show that a similar q_k can be introduced quantum mechanically, and is proportional to

$$q_K=\sum_{k\sigma}\frac{c_{k\sigma}{}^{*}c_{k+K\sigma}}{\omega^2-(\hbar\mathbf{k}\cdot\mathbf{K}-\hbar^2K^2/2m)^2}. \tag{A8}$$

In the usual coordinate representation, this operator is

$$q_K=\sum_i\frac{1}{\omega^2-(\mathbf{K}\cdot\mathbf{P}_i/m-\hbar K^2/2m)^2}\exp(-i\mathbf{K}\cdot\mathbf{X}_i). \tag{A8a}$$

In the limit of $\hbar\rightarrow 0$, this reduces to the q_k of Paper II (Eq. 16).

As in Paper II, Eq. (17) we find it convenient to introduce the quantities $\xi_{K,\omega}$, which are, quantum mechanically

$$\xi_{K,\omega}=\sum_k\frac{c_{k\sigma}{}^{*}c_{k+K\sigma}}{\omega-(\hbar\mathbf{k}\cdot\mathbf{K}-\hbar K^2/2m)}, \tag{A9}$$

and are related to q_K by

$$q_K=(1/2)[(\xi_{K,\omega}-\xi_{K,-\omega})/\omega]. \tag{A10}$$

If the $\xi_{K,\omega}$ satisfy

$$\dot{\xi}_{K,\omega}+i\omega\xi_{K,\omega}=0, \tag{A11}$$

then it immediately follows on differentiation of (A10) that

$$\ddot{q}_K+\omega^2q_K=0.$$

We have $\dot{\xi}_{K,\omega}=(1/i\hbar)[\xi_{K,\omega},H]$. If we use the commutation properties [Eq. (A4)], we find that

$$\begin{aligned}\dot{\xi}_{K,\omega}+i\omega\xi_{K,\omega}=&\sum_{k\sigma}c_{k\sigma}{}^{*}c_{k+K\sigma}\\&+2\pi e^2\sum_{\alpha k\sigma}c_{k\sigma}{}^{*}\frac{c_{k+K-\alpha,\sigma}\rho_{-\alpha}}{\alpha^2}\left\{\frac{1}{\omega-\hbar\mathbf{k}\cdot\mathbf{K}+\hbar K^2/2m}\right.\\&\left.-\frac{1}{\omega-\hbar(\mathbf{k}-\boldsymbol{\alpha})\cdot\mathbf{K}+\hbar K^2/2m}\right\}+2\pi e^2\sum_{\alpha k\sigma}\\&\times\frac{\rho_\alpha c_{k\sigma}{}^{*}c_{k+K+\alpha,\sigma}}{\alpha^2}\frac{1}{\omega-\hbar\mathbf{k}\cdot\mathbf{K}+\hbar K^2/2m}\\&-\frac{1}{\omega-\hbar(\mathbf{k}+\boldsymbol{\alpha})\cdot\mathbf{K}+\hbar K^2/2m}.\end{aligned} \tag{A12}$$

We now split the sums over $\boldsymbol{\alpha}$ and $\mathbf{K}$ into two parts. In the second term on the right hand side of (A12), we see that those terms for which $\boldsymbol{\alpha}=\mathbf{K}$ give us a factor of n, the total number of particles, while the remaining terms, with $\alpha\neq K$ lead to nonlinear contributions, since there appear here effectively two factors, each of order ρ_K. It can be shown that the neglect of those terms for which $\alpha\neq K$ is equivalent to the "random phase approximation," as applied for instance in the neglect of U Eq. (15). Similarly, in the third term on the right-hand side of (A12) we find the terms for which $\boldsymbol{\alpha}=-\mathbf{K}$ give us a factor of n, while those with $\boldsymbol{\alpha}\neq-\mathbf{K}$ may be neglected in the random phase approximation.

With these approximations, we then obtain

$$\begin{aligned}&\xi_{K,\omega}+i\omega\xi_{K,\omega}\\&\quad=\rho_K\left\{1-\sum_{k\sigma}\frac{4\pi e^2}{m}\frac{c_{k\sigma}{}^{*}c_{k\sigma}}{(\omega-\mathbf{k}\cdot\mathbf{K}\hbar)^2-\hbar^2K^4/4m}\right\}.\end{aligned} \tag{A13}$$

Thus we see that $\xi_{K,\omega}$ and hence q_K, oscillates harmonically provided ω satisfies the dispersion relation

$$1=\frac{4\pi e^2}{m}\sum_k{}'\frac{1}{(\omega-\hbar\mathbf{k}\cdot\mathbf{K})^2-\hbar^2K^4/4m^2}, \tag{A14}$$

where $\sum_k{}'$ here denotes the sum over all occupied electronic states. This dispersion relation is, however, identical with that we found in Sec. II Eq. (57). Thus we see that the same results can be obtained by solving for the operator equations of motion as can be obtained by the canonical transformation method.

However, a word of caution should be injected at this point. For if one naively diagonalizes the terms on the right-hand side on (A12), assuming the electrons occupy a Fermi distribution at $T=0$, one obtains additional "exchange" terms which apparently contribute to order k^2 in the dispersion relation (A14). This in turn introduces an apparent contradiction between the results herein obtained and the dispersion relation (57). The resolution of this contradiction lies in the fact that the electrons in consequence of the Coulomb interactions do not behave like a gas of free particles (as is tacitly assumed in diagonalizing A12), but rather exhibit long-range correlations in positions which act to reduce the long-wavelength density fluctuations. This reduction in the long-wavelength density fluctuations has the result that no "exchange" contributions to the dispersion relation appear up to order k^4. Physically this result follows from the fact that the long-range correlations act to keep the particles far apart, so that they have less chance to feel the effects of the exclusion principle. This result follows quite simply in our treatment in the body of this paper where we take into account the exclusion principle by antisymmetrizing the individual electronic wave functions. However, it is rather difficult to establish the equivalent result in the above second-quantization formalism, so we do not enter on this question farther here.

APPENDIX II

Tomonaga[5] has developed a very interesting one-dimensional treatment of the degenerate gas of Fermi particles in which the excitations are described in terms of a Bose field, and in which he obtains plasma oscillations for the degenerate electron gas. His method, however, appears to be intrinsically restricted to this one-dimensional case. It also involves the approximation that the wave function of the electron gas is not very different from that of a collection of free electrons with a Fermi distribution at absolute zero. In this appendix we shall exhibit the relationship between Tomonaga's methods and ours.

To do this let us first find the equation of motion of the operator ρ_K. We find it convenient to make the simple transformation[15]

$$\rho_K=\sum_k c_k^* c_{k+K}=\sum_k c_{k-K/2}^* c_{k+K/2}. \tag{A15}$$

The equations of motion of ρ_K may be obtained by commuting it with the Hamiltonian (A7). We find

$$\dot{\rho}_K=-i\sum_k (\hbar k\cdot K/m)c_{k-K/2}^* c_{k+K/2}, \tag{A16}$$

$$d^2\rho_K/dt^2=-\sum_k (\hbar k\cdot K/m)^2 c_{k-K/2}^* c_{k+K/2}-\omega_p^2\rho_K - \sum_{K'\neq K}\frac{(\mathbf{K}\cdot\mathbf{K}')}{(K')^2}\rho_{K'}\rho_{K-K'}. \tag{A17}$$

If we neglect the nonlinear terms on the right-hand side of the above equation, we see that ρ_K still does not quite oscillate harmonically. This is because of the term $-\sum_k(h\mathbf{k}\cdot\mathbf{K}/m)^2 c_{k-K/2}^* C_{k+K/2}$, which is the quantum-analog of the term $\sum_i(\mathbf{k}\cdot\mathbf{V}_i)^2 e^{-i\mathbf{k}\cdot\mathbf{x}_i}$ appearing in Paper II, Eq. (9). As in the classical treatment, this term arises from the fact that we have a collection of different electrons, each moving with a different velocity and each therefore contributing differently to ρ_K. Hence, for the same reasons given in Paper II, it is necessary to seek the function q_K [given in (A8)] which oscillates harmonically in spite of the random motions of the individual electrons.

However in the *one-dimensional* case a considerable simplification is possible when the wave function is approximately that associated with a Fermi distribution of electrons at absolute zero. For in this case, either the operator $c_{k-K/2}^*$ or the operator $c_{k+K/2}$ will be zero except in a small region of width K at the top of the distribution. If K is small, then the term $(K\cdot k)^2 = K^2k^2$ can be approximated as $K^2k_0^2$, where k_0 is the wave vector of an electron at the top of the distribution. We then get

$$d^2\rho_K/dt^2=-(h^2K^2k_0^2/m^2+\omega_p^2)\rho_K, \tag{A18}$$

and we see that ρ_K oscillates harmonically, which is the result of Tomonaga.

In the three-dimensional case, such a simplification is not possible. For the Fermi distribution is now spherical, and the factor $\mathbf{k}\cdot\mathbf{K}\cong k_0K\cos\vartheta$, where ϑ is the angle between $\mathbf{k}$ and $\mathbf{K}$ for the electrons at the top of the Fermi distribution. Thus the various terms $c_{k-K/2}^* \times c_{k+K/2}$ can no longer be given a common factor, and the simple result (A18) can no longer be obtained. The reason for this change may be given a simple physical interpretation. In a one-dimensional problem, the electrons at the top of the Fermi distribution have only one velocity, and therefore all electrons contribute approximately in unison to ρ_K. In the three-dimensional case, each electron contributes differently, so that the function ρ_K is altered in time, and a new function is introduced which cannot be expressed as a simple function of ρ_K.

It should also be noted that our criterion for the validity of the collective approximation is different from that of Tomonaga. For we require the smallness of $\alpha=\langle(\mathbf{k}\cdot\mathbf{P}_i/m\omega)^2\rangle_{\mathrm{Av}}$, while Tomonaga requires the smallness of $(\Delta W/\hbar\omega)$ where ΔW is the mean excitation energy of the electron distribution over the ground state Fermi energy.

Finally, we shall demonstrate explicitly the relationship between Tomonaga's variables and ours. From (A8), (A9), and (A10) we may write for our collective variables in the one-dimensional case:

$$q_K=\sum_k\frac{c_{k-K/2}^* c_{k+K/2}\omega_p^2}{\omega^2-(\hbar kK/m)^2}, \tag{A19}$$

$$p_K=\frac{i}{\omega(K)}\sum_k\frac{c_{k-K/2}^* c_{k+K/2}}{\omega^2-(\hbar kK/m)^2}(\hbar kK/2m\omega)\omega_p^2. \tag{A20}$$

Now Tomonaga breaks his sums over k into two parts, corresponding to positive and negative values of k. We shall do the same, noting that the only nonzero contributions are in a small region of width K near the top of the distribution. We get

$$q_K=q_K^{+}+q_K^{-}=\left\{\frac{\omega_p^2}{2[\omega^2-(\hbar k_0K/m)^2]}\right\} \times\sum_{+k} c_{k-K/2}^* c_{k+K/2}+\sum_{-k} c_{k-K/2}^* c_{k+K/2},$$

$$p_K=p_K^{+}+p_K^{-}=\left\{\frac{\omega_p^2 i}{2[\omega^2-(\hbar k_0K/m)^2]}\right\} \times\sum_{+k} c_{k-K/2}^* c_{k+K/2}+\sum_{-k} c_{k-K/2}^* c_{k+K/2}.$$

If we note from (A18) that $\omega^2-(hk_0K/m)^2=\omega_p^2$, we then obtain for the operators

$$a_k=q_K-i\omega P_K=\sum_{+k} c_{k-K/2}^* c_{k+K/2},$$

$$a_k^*=q_K+i\omega P_K=\sum_{-k} c_{k-K/2}^* c_{k+K/2}.$$

These are just the Eqs. (2.5) of Tomonaga.

Thus, in the one-dimensional case, with Tomonaga's assumption of an approximate Fermi distribution of free electron momenta, we obtain the same results as Tomonaga.

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[15] We here suppress the spin index, since this will play no role in what follows.