Vortex velocity field in inhomogeneous media: A numerical study in Bose-Einstein condensates

D. M. Jezek and H. M. Cataldo
Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, RA-1428 Buenos Aires, Argentina and Consejo Nacional de Investigaciones Científicas y Técnicas, Buenos Aires, Argentina

(Received 23 January 2008; published 1 April 2008)

We present a numerical calculation of the velocity field of an off-axis vortex within a harmonic trapping potential. We consider a condensate with a large number of particles which, in spite of being in the Thomas-Fermi regime, possesses a non-totally-negligible vortex core size. Via a simple, yet realistic, modeling of the vortex-state density, we derive, by treating the equation of continuity outside and inside the vortex core, respective equations for the velocity field and the vortex precession velocity. We find that the vortex precession velocity is given by two contributions: the background velocity field evaluated around the border of the core and another term which depends on the core shape. Our findings concerning the velocity field outside the vortex core are in good agreement with previous theoretical predictions in a narrow region around it, while far away from the vortex we observe a field with a large asymmetry with respect to the vortex position. We show that a better approximation may be obtained by adding the velocity field produced by an antivortex located outside the condensate.

DOI: 10.1103/PhysRevA.77.043602

PACS number(s): 03.75.Lm, 03.75.Hh, 03.75.Kk

I. INTRODUCTION

Quantized vortices have been studied for a long time in homogeneous superfluid systems, where it is well known that in the absence of dissipation, a vortex must move with the velocity of the background superflow [1]. An inhomogeneous superfluid, on the other hand, may be regarded as a potentially richer scenario for observing a vortex dynamics, since it may give rise to several new remarkable features. As Bose-Einstein condensates (BECs) are experimentally obtained using trapping potentials that yield inhomogeneous particle densities, the study of the induced velocity field due to the presence of a vortex placed in a density-varying region has acquired great interest [2].

In an axisymmetric trap a single off-axis vortex exhibits, in addition to the ordinary circulating velocity field of a vortex, which constitutes the whole field for a vortex located at the axis of the trap [3,4], an induced velocity field [2]. Moreover, if the condensate is confined by a static harmonic trap, the off-axis vortex is subjected to a precession movement [5–9] related to the above induced velocity field, which, evaluated near the vortex position, defines the vortex background velocity.

An explicit approximate solution for such an induced background field in a two-dimensional system, within the Thomas-Fermi (TF) approximation, was recently published [2]. This field was derived by assuming that the divergence of the particle current density vanishes, which is valid outside the vortex core, while near the vortex the authors state that such a field is approximately spatially uniform.

To the best of our knowledge, no numerical calculations on this issue has been reported so far. So we are presenting in this work such a numerical study, showing that there is a rather good agreement with the theoretical prediction near the vortex position, although it seems to be partially affected by some kind of boundary effect, which was disregarded in the aforementioned theoretical study. This is more evident for the field far from the vortex, since a better agreement than that arising from Ref. [2] is found by considering the field corresponding to an antivortex located outside the condensate, resembling an image vortex.

We want to mention that theoretical research involving stationary configurations of vortices has been recently developed from two-dimensional clusters of vortex-antivortex pairs [10,11] to more general three-dimensional systems [12]. In stationary conditions, the velocity field at the position of each vortex, due to the remaining ones, should balance the induced field generated by the inhomogeneity. Thus, further investigations on these fields should be useful in determining such configurations.

There is another source, in addition to the vortex background velocity, which may contribute to the vortex precession. This stems from the vortex core structure and was recently analyzed in Ref. [13]. In the present work, we further investigate this issue and propose a more general model of the vortex core. This procedure led us to find a better agreement with the numerical result of the precession velocity.

This work is organized as follows. In Sec. II we analyze the equation of continuity in two disjointed regions. On the one hand, we consider the region outside the vortex core, where the density may be regarded as that of the ground state and an approximate calculation of the induced background velocity field may be performed [2]. On the other hand, we focus on the vortex core, where the density evolves in time owing to the precession movement and a formulae for the vortex velocity can be derived by setting an adequate model for the core density. In Sec. III we present and discuss the results arising from the numerical simulation, while in Sec. IV we summarize the main conclusions of our study.

II. EQUATION OF CONTINUITY: VELOCITY FIELD AND VORTEX PRECESSION

It is useful for describing the velocity field \( \mathbf{v} \) to introduce the relative coordinate \( \mathbf{r}_\perp(t) = \mathbf{r} - \mathbf{r}_0(t) \) measured from the vortex axis:
from Fig. 1. In addition, it will be a good approximation outside the core, while inside the vortex core, i.e., $j_1$, assume that $1/H_{11633} = -1/H_{11036}$. 

The particle current density is then given as $\rho_0 \nabla \cdot \mathbf{v} = 0$. 

If $\rho_0$ denotes the smooth ground state density, we may introduce the form factor $F = \rho_0 / \rho_0$ for describing the shape of the core. In fact, in the TF approximation one has $F = 1$ outside the core, while inside the vortex core $F$ exhibits a rapid decrease, and vanishes at its center, as can be inferred from Fig. 1. In addition, it will be a good approximation to assume that $F$ is almost isotropic around the core center——i.e., $F(r_\perp)$. The particle current density $j = \rho_0 \mathbf{v}$ is then given by

$$j = F(r_\perp) \rho_0 \left[ \frac{\hbar}{m} \mathbf{r}_\perp \times \mathbf{v}_\perp + \mathbf{v}_s \right].$$

Then taking into account the equation of continuity $\partial \rho_0 / \partial t = \nabla \cdot j$, we have, on the one hand,

$$\nabla \cdot j = \rho_0 \nabla \cdot \mathbf{v}_f + F \nabla \rho_0 \cdot \mathbf{v}_f + F \rho_0 \nabla \cdot \mathbf{v}_f + \rho_0 \nabla \cdot \mathbf{v}_i + F \rho_0 \nabla \cdot \mathbf{v}_i.$$

On the other hand, calculating the partial derivative of the density with respect to time we obtain

$$\frac{\partial \rho_0}{\partial t} = -\rho_0 \nabla \cdot \mathbf{v}_0,$$

and from the continuity equation we get

$$\rho_0 \nabla \cdot \mathbf{v}_0 = F \nabla \rho_0 \cdot \mathbf{v}_f + \rho_0 \nabla \cdot \mathbf{v}_i + F \rho_0 \nabla \cdot \mathbf{v}_i,$$

which should hold at every point of the condensate. We shall first consider the region outside the vortex core, where we have $F = 1$ and $\nabla F = 0$, which yields

$$0 = \nabla \rho_0 \cdot \mathbf{v}_f + \nabla \nabla_0 \cdot \mathbf{v}_i + \rho_0 \nabla \cdot \mathbf{v}_i.$$

In a recent work, Sheehy and Radzihovsky [2] have obtained an approximate solution of the above equation for the induced background velocity $\mathbf{v}_i$, yielding the expression

$$\mathbf{v}_i(r) = \frac{\hbar}{2m} \left[ \mathbf{r}_0 \cdot \nabla \rho_0 (r_0) \right] \ln \left( \frac{r - r_0}{\rho_0 (r_0)} / \rho_0 (r_0) \right),$$

which is valid in a narrow region around the vortex core.

On the other hand, in order to obtain the vortex precession velocity $\mathbf{v}_0$, one should solve Eq. (7) inside the core, where $\nabla F \neq 0$. Then, the term containing $\nabla \rho_0 \cdot v_i$ in (7) vanishes according to Eq. (9), while the term $F \rho_0 \nabla \cdot v_i$ can be neglected, as we shall show in Sec. III. Finally, we are led to the following equation which is valid inside the core:

$$\rho_0 \nabla \cdot \mathbf{v}_0 = F \nabla \rho_0 \cdot \mathbf{v}_f + \rho_0 \nabla \cdot \mathbf{v}_i + F \rho_0 \nabla \cdot \mathbf{v}_i.$$

Now, defining $F'(r_\perp) = \partial F / \partial r_\perp^2$, we have

$$\nabla F(r_\perp) = 2F'(r_\perp) \mathbf{r}_\perp,$$

and Eq. (10) becomes

$$2F' \rho_0 \mathbf{r}_\perp \cdot \mathbf{r}_0 = \frac{\hbar}{2m} F \nabla \rho_0 \cdot \mathbf{r}_\perp + 2F' \rho_0 \mathbf{r}_\perp \cdot \mathbf{v}_i,$$

where we have replaced $v_i$ according to Eq. (1). The triple product on the right-hand side of (12) may be rearranged to yield

$$2F' \rho_0 \mathbf{r}_\perp \cdot \mathbf{r}_0 + \frac{\hbar}{mr_\perp} F' \mathbf{r}_\perp \cdot \mathbf{r}_\perp + 2F' \rho_0 \mathbf{r}_\perp \cdot \mathbf{v}_i = 0.$$

In a previous work [13], the authors have modeled the vortex core through a linear dependence of the wave function with respect to $r_\perp$, which corresponds to a quadratic dependence of the density. However, we have observed from our numerical calculations, that in the TF limit such a density shape is better described by a linear dependence on $r_\perp$ (see next section). This led us to assume a generic power law $F \sim r_\perp^2$, which replaced in Eq. (13) yields
valid in the core region, excluding the point \( \mathbf{r} = \mathbf{r}_0 \), where the superfluid density vanishes. Nevertheless, since \( \rho_0 \) is a smooth function, \( \rho_0 \) and \( \nabla \rho_0 \) may be evaluated at \( \mathbf{r}_0 \) because they do not change appreciably inside the core. Then, given that the vector inside the square brackets in (14) does not depend on \( \mathbf{r} \), it must equal zero, yielding

\[
\mathbf{r}_0 = -\frac{\hbar}{2m\beta_0} \hat{\mathbf{z}} \times \nabla \rho_0 + \mathbf{v}_i
\]

for the precession velocity of the vortex. The first term on the right-hand side of Eq. (15) corresponds to the contribution arising from the vortex core, while the second term corresponds to the local background velocity. A similar expression for \( B = 1 \) has been recently reported by Nilsen et al. [13] using a different approach.

### III. NUMERICAL CALCULATION

The time evolution of the condensate wave function confined by a time-independent trapping potential is described by the Gross-Pitaevskii equation (GPE) [14,15]. For our numerical study we have chosen a harmonic trapping potential with \( \omega_y = 2\pi \times 100 \text{ Hz} \) and \( \omega_z = 200\omega_y \). We have set \( \omega_y \gg \omega_z \) to suppress excitation in the \( z \) direction in order to be able to utilize the two-dimensional form of the GPE [16]. We have taken, as usual, the harmonic oscillator length \( l = \sqrt{\hbar/(m\omega_y)} \) as the unit of length, while \( \omega_y^{-\frac{1}{2}} \) was chosen as the unit of time. The boson scattering length corresponding to \( ^{87}\text{Rb} \), \( a = 98.983 \text{aBohr} \) (\( a_0 = \text{Bohr radius} \)), was utilized, while a particle number of \( N = 10^3 \) ensured a good TF limit. We performed numerical simulations of the time-dependent GPE within a fifth-order Runge-Kutta algorithm. We considered a system of size \( 17 \times 17 \), analyzing the results arising from two different grids of meshes 0.1 and 0.05. Although we found that both grids produced a very similar performance, we preferred to present the results arising from the smaller mesh, since it provided a better definition for the magnitudes inside the core. For numerical reasons we have worked with a vortex slightly displaced from the grid point (8,0); specifically, it was located at \( x_0 = 8.0245, y_0 = 0.0 \).

To numerically analyze our treatment of the continuity equation inside the vortex core, we plotted in Fig. 2 the terms on the right-hand side of Eq. (10), which were assumed to be the only non-negligible contributions to \( \mathbf{v} \cdot \mathbf{j} \) in that region. We also plotted for comparison the neglected term \( F\rho_0 \nabla \cdot \mathbf{v}_i \). It may be seen that the most important contribution corresponds to \( \rho_0 \mathbf{F} \cdot \mathbf{v}_i \), while \( F\nabla \rho_0 \cdot \mathbf{v}_i \) represents about one-third of such a contribution.

We show in Fig. 3 the renormalized form factor \( F/4 \) as a function of \( x \) and \( y \) in (a) and (b), respectively. Recall that in Sec. II we assumed \( F \sim \mathcal{F}^B \), which means \( F \sim |x - 8.0|^2 \mathcal{F}^B \) and \( F \sim |y - 0.0|^2 \mathcal{F}^B \) in Figs. 3(a) and 3(b), respectively. Thus, it is clear that a value \( B = 1 \) [13]—i.e., a positive curvature of the form factor—could only be representative of a very small portion around the center of the core, not visible in our graph, while the greatest part seems to be well approximated by straight lines—i.e., \( B = 1/2 \) (zero curvature)—except near the core border, where an even smaller value of \( B \) (negative curvature) seems to be appropriate. Here it is important to notice that the value of the local healing length

\[
\xi = \frac{\hbar}{m\omega_y R \sqrt{1 - r^2/R^2}} \approx 0.08,
\]

where \( R \approx 14.5 \) corresponds to the TF condensate radius, clearly underestimates the core radius \( \sim 0.4 \) (see Fig. 3). We also depict in Fig. 3 the \( y \) component of the induced background velocity field around the vortex. The solid squares correspond to the values arising from the numerical simulation, while the theoretical prediction of Eq. (9) is represented by solid lines. We recall that such an expression was extracted by assuming \( |\mathbf{r} - \mathbf{r}_0|/\rho_0(\mathbf{r}_0) \approx 1 \), which sets up an upper limit for their radius of validity around the core center of about 2. On the other hand, since that expression assumes that the divergence of the current density \( \mathbf{j} \) vanishes, it could only be accurate outside the vortex core. In summary, we have that the theoretical estimate is expected to be accurate for \( 0.4 \leq |\mathbf{r} - \mathbf{r}_0| \leq 2 \), as is in fact observed in Fig. 3(b). In contrast, Fig. 3(a) shows an appreciably higher asymmetrical set of numerical values, which clearly differ from the theoretical prediction (note the symmetry of the expression (9) about \( x = x_0 \)). Particularly, the higher values found at the right of the vortex seem to suggest that an adequate far-field approximation could be represented by an antivortex located at the right of the condensate, resembling an image vortex. Using the TF radius of the condensate \( R \approx 14.5 \), the theoretical value of the image position [17] yields \( x \approx 26 \). However, we have found that a much better antivor-
the extra velocity field can be obtained if we place the antivortex closer to the condensate around $x/H_{11229}^21$. In Fig. 4 we plot an enlarged view of Fig. 3, extending along the whole condensate, together with the field of this antivortex. It may be seen that the above mentioned symmetry around the position of the vortex is partially restored when subtracting the velocity field of this antivortex. Moreover, we notice that the difference between the numerical values and the antivortex field becomes smaller for points far from the core. In addition, such a far-field approximation turns out to be much better than that proposed in Ref. [2], which consists in a vortex dipole, with a positive vortex at the trap center and a negative vortex at the location of the true vortex.

To understand in a simple way the appearance of this imagelike field, we may think on particles that move around the vortex position without deforming the condensate density shape outside the core. Given that the density has its maximum at the center of the trap, the number of particles at the right of $x_0$ in Fig. 4 is much smaller that the number at the left. Thus, in order to maintain the density shape, the velocity field at $x=x_0$ should be clearly greater than the field at $x<x_0$. More specifically, the number of particles that pass through the $x$ axis per unit time at the right of the vortex must approximate the corresponding number at the left. Now, since such numbers are given by the integral of the $y$ component of the current density along the corresponding interval of the $x$ axis, it becomes clear that this effect must depend on global properties of the condensate, rather than on local ones, as assumed, e.g., in Eq. (9).

Finally, we may estimate from Fig. 3 that the background superfluid velocity around the vortex core turns out to be $v_i=0.23 \pm 0.02$. On the other hand, from our numerical simulation of the time evolution we have found that the vortex precession velocity is $v_p=0.34 \pm 0.01$. Thus, we may get the following estimate for the core contribution $v_c$ to the precession velocity:

$$v_c = v_p - v_i = 0.11 \pm 0.02,$$

a value which may be compared with that arising from the modeling of the core of Ref. [13], $v_c \approx 0.055$. In summary,
given that our numerical estimate of $v_\phi$ turned out to be around twice the one predicted by assuming $B=1$ [13], we may conclude that a vortex core model with a smaller value of $B$ of about 1/2 in Eq. (15) leads to a much more accurate value of the vortex velocity, while agreeing with our previous discussion regarding the core shape.

IV. SUMMARY AND CONCLUSIONS

Our numerical study of the induced background velocity field yielded values that show a rather good agreement with the theoretical estimates of Ref. [2], near the vortex core. The agreement was less satisfactory along the $x$ direction, since we obtained a higher-asymmetrical set of numerical values with respect to the vortex center. We found that a possible explanation for this discrepancy may be related to the fact that in the theoretical estimate (9), only the local shape of the condensate is taken into account. This suggests that the difference between both results could be accounted for by considering the field produced by an antivortex located outside the condensate, resembling an image vortex, like those utilized for systems confined by sharp boundaries. Thus, we have found that the field produced by this antivortex seems to be the possible source of the asymmetry arising near the position of the vortex and also it yields an acceptable far-field approximation, which improves that proposed in Ref. [2].

On the other hand, we have shown that the difference between the vortex precession velocity and the induced background velocity evaluated in the vicinity of the core border strongly depends on the density inside the core region. Thus, after performing a numerical study of this density profile, we proposed a more realistic model than previously [13], which led to values of the core contribution to the precession velocity in better agreement with the numerical estimate.

Finally, we believe that the present study has shed light on some features of vortex velocity fields in inhomogeneous media, which may be helpful for studying several stationary configurations of vortices.