Classicality in discrete Wigner functions

Cecilia Cormick,1 Ernesto F. Galvão,2 Daniel Gottesman,2 Juan Pablo Paz,1,3 and Arthur O. Pittenger4
1 Departamento de Física “Juan José Giambiagi”, FCEyN UBA, Pabellón I, Ciudad Universitaria, 1428 Buenos Aires, Argentina
2 Perimeter Institute for Theoretical Physics
3 31 Caroline Street North, Waterloo, Ontario, N2L 2Y5, Canada
4 Theoretical Division, Los Alamos National Laboratory, MS B288, Los Alamos, NM 87545, USA

PACS numbers: 03.67.Lx, 03.67.Hk, 03.65.Ca

I. INTRODUCTION

Continuous-variable Wigner functions $W(q, p)$ have been used for a long time to represent quantum systems in phase space $I J$. The Wigner function $W(q, p)$ is an alternative complete description of quantum states which behaves almost like a phase-space probability density. Not only is it real-valued and normalized but it also yields the correct value of the probability density for the quadrature $aQ + bP$ when integrated along the phase-space line $aq + bp$. However, unlike probability densities, the Wigner function can assume negative values for some quantum states. This negativity of the Wigner function has been considered a defining signature of non-classicality (or quantum coherence and interference) $I J$.

In quantum information science we usually deal with systems with a space of states of a finite dimension $d$. For example, for a system of $n$ qubits the dimension of the (Hilbert) space of states is $d = 2^n$. For such systems, various discrete analogues of the Wigner function have been proposed $I J$ and used to investigate a variety of interesting problems connected with quantum computation such as the phase-space representation of quantum algorithms $I J$, separability $I J$, quantum state tomography $I J$, teleportation $I J$, decoherence in quantum walks $I J$, and error correction $I J$. Here we shall concentrate on a class of discrete Wigner functions $W$ introduced recently by Gibbons et al. $I J$. This elegant approach seems to be a potentially powerful tool to establish connections between phase-space techniques and problems in quantum information and foundations of quantum mechanics.

In this paper we study the set of states with non-negative discrete Wigner functions $W$ for all functions in the class proposed by $I J$, and the group of unitaries that preserve non-negativity of $W$. Each class of Wigner functions $W$ in $I J$ is defined with respect to a fixed, complete set of mutually unbiased bases (MUB), which will be defined in section II. Our first result is a complete characterization of the set of quantum states having non-negative discrete Wigner functions $W$. This is done by proving a conjecture presented by one of us in $I J$ (a related discussion in a somewhat different context, using concepts in high-dimensional geometry appeared in $I J$). Our proof is elementary and constructive, and shows that for any choice of MUB consisting of stabilizer states, the only pure states with non-negative $W$ turn out to be those same stabilizer states, i.e. simultaneous eigenstates of generalized Pauli operators $I J$. We then study the group of unitaries which preserve non-negativity of $W$, and prove that they form a subgroup of the Clifford group. This means such states and unitaries are classical in the sense of allowing for an efficient classical simulation scheme using the stabilizer formalism.

The paper is organized as follows. In Section II we review the discrete Wigner functions $W$ of $I J$. In Section III we characterize the states with non-negative $W$, in Section IV we discuss positivity-preserving unitary dynamics in phase space and in Section V we summarize our results.

II. DISCRETE WIGNER FUNCTIONS

In this section we review the class of discrete Wigner functions proposed in $I J$ and discuss some of their features.

Let us assume that we are describing a quantum state whose Hilbert space dimensionality $d$ is a power of a
prime number \( p \) (\( d = p^n \)). In such cases one can introduce a phase-space grid with \( d \times d \) points and label the position and momentum coordinates \((q, p)\) with elements of the finite Galois field \( GF(p^n) \). At first the use of elements of \( GF(p^n) \) for both phase-space coordinates could be seen as an unnecessary complication, but it turns out to be an essential step. The reason is that by doing this we can endow the phase-space grid with the same geometric properties as the ordinary plane. For example, in the finite \( d \times d \) grid we can define lines as solutions to linear equations of the form \( aq + bp = c \) [where all elements and operations in this equation are in \( GF(p^n) \)]. Each line will then consist of exactly \( d \) points of the grid. The field structure of \( GF(p^n) \) ensures the validity of properties such as: (i) there is only one line joining any given two points, (ii) two lines are either parallel (i.e. with no points in common) or they intersect at a single point. Moreover, it is possible to show that a set of \( d \) parallel lines (which we will call a **striation** \((\underline{1})\)) is obtained by varying the parameter \( c \) in the equation \( aq + bp = c \). Finally, the number of different striations turns out to be \((d + 1)\). The complete set of \((d + 1)\) striations has been studied for a long time in discrete geometry, where it is called a finite affine plane \([24,25]\). We will label the striations with an index \( \kappa = 1, \ldots, d+1 \) and the lines within a striation with an index \( j = 1, \ldots, d \). In this way the \( j \)-th line belonging to the \( \kappa \)-th striation will be denoted as \( \lambda_j^{(\kappa)} \).

A discrete phase space with the above properties was used by Gibbons, Hoffman and Wootters in \([11]\) to define a class of discrete Wigner functions. As mentioned above, the crucial property of the continuous Wigner function is that its integral along any line \( \lambda \) is equal to the expectation value of a projection operator \( \hat{P}_\lambda \), i.e. a probability. This essential feature is generalized to the discrete case in a straightforward way: every line in the \( d \times d \) phase-space grid is associated to a rank one projection operator. As noted in \([11]\), this association cannot be arbitrary and must obey some simple geometric constraints. For example, we can define a set of \( d \times d \) unitary operators \( \hat{T}(q, p) \) acting on the Hilbert space that faithfully represent discrete phase-space translations. For the association between lines and states to respect covariance under translations we must impose that the quantum state associated to a translated line should be identical to the state obtained by acting with the operator \( \hat{T}(q, p) \) on the original state. This covariance constraint can be used to show the validity of some very significant properties: a) the states associated to parallel lines must be orthogonal; b) the overlap between states associated to non-parallel lines must be equal to \( 1/d \). This is important and implies that the \((d+1)\) phase-space striations must be associated to an equal number of mutually unbiased bases (MUB), i.e. bases

\[
MUB^{(\kappa)} = \{|\psi_1^{(\kappa)}\rangle, \ldots, |\psi_d^{(\kappa)}\rangle\}
\]

such that

\[
|\langle \phi_j^{(\kappa')} | \phi_j^{(\kappa)} \rangle|^2 = \frac{1}{d} (1 - \delta_{\kappa,\kappa'}) + \delta_{\kappa,\kappa'} \delta_{j,j'}.
\]

As we see, mutually unbiased bases are orthonormal bases picked in such a way that any state in one basis is an equal–amplitude superposition of all the states of any other basis. A complete set of \((d+1)\) MUB is known to exist if the dimensionality of the space of states is a power of a prime number. In such case, many constructions of MUB have been proposed \([24,25,26,29,30]\). It has been shown that a complete set of \((d+1)\) MUB for \( d \)-dimensional systems can be chosen to consist solely of stabilizer states, i.e. simultaneous eigenstates of sets of (generalized) Pauli operators \([28,29,31]\).

The defining feature of the discrete Wigner functions of \([11]\) is the association between MUB and striations in the discrete phase space. As discussed in \([11]\) this can be done in a variety of ways and each defines a different quantum net \([11]\), which will result in a different definition of the discrete Wigner function \( W \). In this paper we propose a notion of classicality of quantum states which is based on non-negativity of \( W \) for all quantum nets obtainable from a fixed complete set of MUB. It should be noted, however, that there have been proposals of criteria to narrow down the choice of quantum nets: in \([16]\) the criterion is covariance under the so-called discrete squeezing operator; and in \([14]\) the net is chosen so as to enforce a natural relation between a separable state’s \( W \) and the \( W \) of its subsystems.

The quantum net is defined by associating each line \( \lambda_j^{(\kappa)} \) in striation \( \kappa \) to a projector \( \hat{P}_{\lambda_j^{(\kappa)}} = |\phi_j^{(\kappa)}\rangle\langle \phi_j^{(\kappa)}| \) onto a basis state of basis \( \kappa \). Having fixed a quantum net, the discrete Wigner function is uniquely defined by imposing the condition that the sum of its values along any line should be equal to the expectation value of the projector corresponding to that line (see \([11]\) for details). The resulting Wigner function at any phase-space point \( \alpha = (q, p) \) can then be shown to be

\[
W_{\alpha} = \text{Tr} \left( \hat{\rho} \hat{A}(\alpha) \right),
\]

\[
\hat{A}(\alpha) = \frac{1}{d} \left( \sum_{\lambda_j^{(\kappa)} \ni \alpha} \hat{P}_{\lambda_j^{(\kappa)}} - \mathbb{1} \right),
\]

where the sum is over projectors associated with all lines \( \lambda_j^{(\kappa)} \) containing point \( \alpha \). The construction of the striations guarantees that the sum above will contain exactly one projector from each basis.

The operators \( \hat{A}(\alpha) \) are known as **phase-space point operators** and form a complete basis for the space of operators, which is orthogonal in the Schmidt inner product (i.e. \( \text{Tr} \left( \hat{A}(\alpha) \hat{A}(\beta) \right) = \delta_{\alpha,\beta}/d \)). We can rewrite the expression for the Wigner function at phase-space point \( \alpha \) using the probabilities associated with the projectors
\[
p_j^{(\alpha)} = \text{Tr} \left( \hat{p} \hat{P}^{(\alpha)} \right).
\]  
(5)

In terms of these probabilities, the Wigner function at the point \( \alpha \) takes the form
\[
W_\alpha = \frac{1}{d} \left( \sum_{\hat{p}^{(\alpha)} \geq \alpha} p_j^{(\alpha)} - 1 \right).
\]  
(6)

The discrete Wigner function \( W \) can be shown to have many of the features of the continuous Wigner function \( W(q,p) \) [11]: it is real (but can be negative), normalized, and its values are obtained through eq. (6) from measurements onto MUB. Here the MUB projectors play the role that the quadratures \( aQ + bP \) play in \( W(q,p) \), forming a particularly symmetric set of observables whose measurement results completely characterize the state (in a process known as quantum tomography). For a discussion of further properties of \( W \) see [11] [14] [28].

In the discussion that follows we will often be representing quantum states using the probabilities \( p_j^{(\alpha)} \). As the projectors \( \hat{P}^{(\alpha)} \) form an over-complete basis for the space of density matrices, these probabilities completely characterize the state. Since for any stratification \( \sum_j p_j^{(\alpha)} = 1 \), there are only \( (d-1) \) independent probabilities for each basis, resulting in a total of \( (d-1) \cdot (d+1) = (d^2-1) \) independent probabilities, exactly the number of real parameters necessary to describe a general normalized mixed quantum state in \( d \)-dimensional Hilbert space. Each quantum state is represented by a point \( \hat{p} \) in this \( (d^2-1) \)-dimensional probability space.

As mentioned above, for power-of-prime \( d \) it is possible to build a complete set of \( (d+1) \) MUB using only stabilizer states, i.e. joint eigenstates of generalized Pauli operators. Let us discuss more explicitly such constructions for the case \( d = 2^n \), i.e. \( n \) qubits (see [25] for more details). In order to define a complete set of \( (2^n+1) \) MUB we start by partitioning the \( (2^n-1) \) Pauli operators (excluding the identity) into \( (2^n+1) \) sets \( S_i \) of \( (2^n-1) \) Pauli operators each. We will require that the Pauli operators in each set \( S_i \) be mutually commuting, but otherwise the partitioning can be completely arbitrary. If we add the identity and a \( \pm 1 \) phase to the Pauli operators in each set \( S_i \), each will form a maximal Abelian subgroup of the Pauli group. The joint eigenstates of each such set \( S_i \) form a basis for the Hilbert space, and due to properties of the Pauli operators the \( (2^n+1) \) bases thus defined can be shown to be mutually unbiased [25].

The phase-space construction provides a natural procedure for partitioning the Pauli group into disjoint, mutually commuting sets. The idea, which is worth reviewing here, was described in [11] and further elaborated in [13]. Pauli operators represent phase-space translations and can be labelled using binary \( n \)-tuples \( \vec{p} \) and \( \vec{q} \) \((n\text{-tuples})\). \( \vec{q} \) and \( \vec{p} \) contain the coordinates of the field elements \( q \) and \( p \) in a given basis as described below. Each Pauli operator can be written as
\[
\hat{T}(\vec{q},\vec{p}) = \prod_{i=0}^{n-1} \hat{X}_i^{p_i} \hat{Z}_i^{q_i} e^{\frac{\pi}{2} \hat{q}_i \cdot \hat{p}_i},
\]  
(7)

where \( \hat{X}_i \) and \( \hat{Z}_i \) stand for the Pauli operators on qubit \( i \), and the phase is chosen so as to make the operators Hermitian. The definition above will be written in shorthand as
\[
\hat{T}(\vec{q},\vec{p}) = \hat{X}^{\vec{q}} \hat{Z}^{\vec{p}} e^{\frac{\pi}{2} \vec{q} \cdot \vec{p}}.
\]  
(8)

The condition for two Pauli operators to commute turns out to be
\[
[\hat{T}(\vec{q},\vec{p}),\hat{T}(\vec{q}',\vec{p}')] = 0 \text{ iff } \vec{q} \cdot \vec{p}' - \vec{p} \cdot \vec{q}' = 0 \pmod{2}.
\]  
(9)

Let us consider a set of \((d-1)\) Pauli operators
\[
S_{(\vec{a},\vec{b})} = \left\{ \hat{T}(\vec{a}M_j,\vec{b}\widetilde{M}_j) \mid j = 0, 1, \ldots, d-2 \right\},
\]  
(10)

where \( M \) is an arbitrary binary matrix, \( \widetilde{M} \) is its transpose and \( \vec{a}, \vec{b} \) are binary \( n \)-tuples. Any two operators of this set commute. It is interesting to note that \( S_{(\vec{a},\vec{b})} \) forms a maximal Abelian subgroup of the Pauli group if and only if \( M \) is a generating element of the matrix representation of the field \( GF(2^n) \). This can be seen as follows: the product of two elements of \( S_{(\vec{a},\vec{b})} \) is itself an element of this set (up to a sign) iff the matrix \( M \) is such that for every pair of integers \( j, j' \), there is a third integer \( j'' \) such that \( M_j + M_j' = M_j'' \) [where \( 0 \leq j, j', j'' \leq (d-2) \) and \( j \neq j' \)]. Moreover, for the set to have exactly \((d-1)\) different elements, the matrix \( M \) should be such that all powers \( M_j \) for \( j = 0, \ldots, (d-2) \) are nonzero and different from each other. For \( M \) satisfying these conditions, it can be seen that \( M^{d-1} = 1 \). Therefore, the elements of the set \( \{0, 1, M, M^2, \ldots, M^{d-2}\} \) form a finite field, and we see that the matrix \( M \) and its powers form a matrix representation of \( GF(2^n) \). A possible choice for \( M \), used in [13], is the so-called “companion matrix” of the primitive polynomial which defines the product rule in the field. With such a matrix we can build \((d+1)\) disjoint sets of commuting Paulis of the form \( S_{(\vec{a},\vec{b})} \) by choosing the binary \( n \)-tuples \( \vec{a}, \vec{b} \) as explained below.

The association between each phase-space point \((q,p)\) and a Pauli operator \( \hat{T}(\vec{q},\vec{p}) \) must respect the covariance of the construction under phase space translations. This is done as follows: the line formed by all phase-space points satisfying the equation \( bq + ap = c \) is invariant under phase-space translations of the form \( \vec{q}' = \vec{q} + \omega \vec{p} \), \( \vec{p}' = \vec{p} + \omega \vec{q} \) (where \( \omega \) is a generating element of the field). To this phase-space translation we must associate an operator acting in Hilbert space. The natural identification is to associate this with the operator \( \hat{T}(\vec{a}M^j,\vec{b}\widetilde{M}^j) \). Here, the choice of \( n \)-tuples \( \vec{a} \) and \( \vec{b} \) is arbitrary. The important
point is that once this choice is made [i.e., once we arbitrarily assign two \( n \)-tuples to the point \((a, b)\)] we repeatedly apply the matrix \( M(M) \) to the position (momentum) coordinates to obtain the \( n \)-tuples parametrizing the Pauli operators associated to the other phase-space points \( M(i) \). In summary, this construction associates an operator \( T(q, p) \) to every phase space point \((q, p)\) in such a way that the elements of the Abelian subgroup \( S(a, b) \) are associated to points in phase space that belong to the ray defined by the equation \( b q + a p = 0 \) (a ray is defined as a line that contains the origin). In \( S(a, b) \) it was shown that by varying the \( n \)-tuples \((a, b)\) one can construct only \((d + 1)\) different sets \( S(a, b) \). If we define the two \( n \)-tuples \( \vec{1} \equiv (1000 \ldots 0) \) and \( 0 \equiv (0000 \ldots 0) \), these maximal mutually commuting sets of Pauli operators can be conveniently built by choosing \((a, b) = (\vec{1}, 0)\) (which we will associate with the horizontal striation), \((a, b) = (0, \vec{1})\) (the vertical striation) and \((a, b) = (\vec{1}, \vec{1})\) for \( b \neq 0 \) (the other striations).

Thus, the mapping between striations and MUB is naturally determined by the phase-space construction, as lines which are invariant under the transformations \( q' = q + a\omega, \ p' = p + b\omega \) must be associated to states which are invariant under the corresponding transformations in Hilbert space, that is, the translation operators in \( S(a, b) \). Therefore, the lines of the form \( bq + ap = c \) must be associated to common eigenstates of the set \( S(a, b) \). However, there is no criterion telling us how to associate each line in a striation with a projector in the corresponding basis. We can count the number of possible quantum nets as follows: for the ray of a given striation there are \( d \) possible projectors to choose from; once this choice has been made the condition of covariance under translations determines which projector should be associated to each of the other lines in the same striation. As there are \((d + 1)\) rays, the number of possible associations between lines and projectors is \( d(d + 1) \), each of which defines a different quantum net, leading to a different definition of the Wigner function. The projectors associated to each of the lines in the vertical and horizontal striations can be chosen in such a way that the coordinates of each line correspond to the eigenvalues of the Paulis generating the set (the single qubit Paulis \( \hat{Z} \) and \( \hat{X} \), respectively). Then, there are still \( d^d - 1 \) possible quantum nets, each of them given by a particular choice of projectors to be associated to the rays of the remaining oblique striations.

There is a closely related methodology for constructing the Wigner functions for general dimension \( d = p^n \) which emphasizes the link between the exponents of the \( \hat{Z} \) and \( \hat{X} \) operators and the finite geometry of \( V_2[GF(p^n)] \), the two-dimensional vector space over the field \( GF(p^n) \). One defines lines, rays and striations in this two-dimensional space and then, using the properties of the algebraic field extension, defines an isomorphism with \( V_{2n}[GF(p)] \). Vectors in this second space serve as exponents of the \( \hat{Z} \) and \( \hat{X} \) operators, and the commuting classes of generalized Pauli matrices correspond precisely to parallel lines in a striation in the first vector space. Details of this approach and a methodology for projecting to lines are given in \[24\].

## III. STATES WITH NON-NEGATIVE WIGNER FUNCTIONS \( W \)

Following \[24\], let us now characterize the set of states having non-negative discrete Wigner functions \( W \) simultaneously in all definitions proposed by Gibbons et al. \[11\] for power-of-prime dimension \( d \).

**Definition:** The set \( C_d \) is defined as the set of (pure or mixed) density matrices of systems in a \( d \)-dimensional Hilbert space having non-negative discrete Wigner function \( W \) in all phase-space points and for all definitions of \( W \) using a fixed set of mutually unbiased bases.

By definition, the set \( C_d \) is specified as the intersection of a number of half-spaces in the \((d^2 - 1)\)-dimensional \( p \)-space. From \[6\] it can be seen that each half-space inequality is of the form

\[
\sum_{\lambda_j \geq 0} p_j^{(\lambda)} \geq 1, \tag{11}
\]

where the probabilities appearing in the sum are associated with the lines containing phase-space point \( x \), and hence depend on \( x \) and on the quantum net chosen. The intersection of the half-spaces defined by these inequalities is a convex polytope in \( p \)-space, given in an \( H \)-description (\( H \) standing for “Half-space”). Any convex polytope also admits an alternative \( V \)-description (\( V \) for “Vertices”), consisting of the list of vertices whose convex hull defines the polytope.

Galvão showed that for \( d \leq 5 \), the \( H \)-polytope \( C_d \) has a \( V \)-description whose vertices are the MUB projectors \[24\], and conjectured this would also be true for general power-of-prime \( d \). A geometrical argument showing the validity of this conjecture was given in \[21\]. Let us now prove a constructive, analytical proof.

**Theorem 1** For any power-of-prime Hilbert space dimension \( d \), the \( H \)-polytope \( C_d \) is equivalent to the \( V \)-polytope \( C_v \) having the MUB projectors as vertices.

**Proof:** Let us prove the theorem by first showing that the \( V \)-polytope \( C_v \) is contained in \( C_d \), and then the converse. From \[11\] we know that the Wigner function for any MUB projector \( \hat{P}_j^{(\lambda)} = |\phi_j^{(\lambda)}\rangle\langle\phi_j^{(\lambda)}| \) is non-negative. Since the Wigner function depends linearly on the density matrices, \( W \) is non-negative also for any state in the convex hull of the \( \hat{P}_j^{(\lambda)} \). This shows that any state in \( C_v \) is also in \( C_d \), as we wanted to prove.
Let us now prove the converse, i.e. that any polytope $C_d$ is contained in polytope $C_d$. What is required now is to show that any state $\hat{\rho} \in C_d$ can be written as a convex combination of the projectors $\hat{P}_j^{(\kappa)}$:

$$\hat{\rho} = \sum_{\kappa=1}^{d+1} \sum_{j=1}^{d} c_j^{(\kappa)} \hat{P}_j^{(\kappa)},$$  \hspace{1cm} (12)

with all $c_j^{(\kappa)} \geq 0$. Note that the decomposition is not unique.

Let us start by considering the general expression for Wigner function $W$ at phase-space point $\alpha$ [eq. (3)]. Given a state $\hat{\rho}$ there is some Wigner function definition which, at some point $\alpha$, evaluates to a minimum value among all definitions and all points $\alpha$. This happens when the expression for $W_\alpha$ is such that the sum includes only the smallest probability $p_j^{(\kappa)}$ from each MUB $\kappa$. Let us denote these $W$-minimizing probabilities $\hat{p}_j^{(\kappa)} = \min_j \{p_j^{(\kappa)}\}$. States in $C_d$ are those for which all expressions of the form include only the smallest probability $p_j^{(\kappa)}$ and $\hat{\rho}$ involves only the $p_j^{(\kappa)}$ is non-negative. In other words, a state has non-negative $W$ in all definitions if and only if:

$$\sum_{\kappa=1}^{d+1} \hat{p}_j^{(\kappa)} \equiv 1.$$  \hspace{1cm} (13)

This is our hypothesis.

Any density matrix $\hat{\rho}$ can be expanded in terms of the projectors $\hat{P}_j^{(\kappa)} = |\phi_j^{(\kappa)}\rangle \langle \phi_j^{(\kappa)}|$ as in eq. (12), with real (but possibly negative) coefficients $c_j^{(\kappa)}$. A first constraint on the coefficients $c_j^{(\kappa)}$ comes from the requirement that $\text{Tr}(\hat{\rho}) = 1$. Using property (3) of the MUB we can compute the trace, obtaining

$$\text{Tr}(\hat{\rho}) = \sum_{\kappa=1}^{d+1} \sum_{j=1}^{d} c_j^{(\kappa)} = 1.$$  \hspace{1cm} (14)

Now let us use eq. (3) to calculate $p_j^{(\kappa)}$ explicitly from eq. (12), so as to obtain relations between the coefficients $c_j^{(\kappa)}$ and the probabilities $p_j^{(\kappa)}$:

$$p_j^{(\kappa)} = |\langle \phi_j^{(\kappa)} | \hat{\rho} | \phi_j^{(\kappa)} \rangle|^2 = \sum_{\mu=1}^{d+1} \sum_{m=1}^{d} c_m^{(\mu)} |\langle \phi_j^{(\kappa)} | \phi_m^{(\mu)} \rangle|^2 = \sum_{\mu \neq \kappa} \sum_{m=1}^{d} c_m^{(\mu)} + \sum_{m=1}^{d} c_m^{(\kappa)} 1/d = \frac{1}{d} \sum_{m=1}^{d} c_m^{(\kappa)},$$  \hspace{1cm} (15)

where we have used the condition of mutual unbiasedness of the bases [eq. (3)]. Now we can use the trace condition (14) to rewrite this as

$$p_j^{(\kappa)} = c_j^{(\kappa)} + \frac{1}{d} \sum_{m=1}^{d} c_m^{(\kappa)},$$  \hspace{1cm} (16)

or

$$c_j^{(\kappa)} = p_j^{(\kappa)} - \frac{1}{d} \sum_{m=1}^{d} c_m^{(\kappa)}.$$  \hspace{1cm} (17)

Let us add $0 = p_j^{(\kappa)} - p_j^{(\kappa)}$ to the right-hand side of the equation above, to obtain

$$c_j^{(\kappa)} = (p_j^{(\kappa)} - p_j^{(\kappa)}) + x^{(\kappa)}$$  \hspace{1cm} (18)

with

$$x^{(\kappa)} \equiv p_j^{(\kappa)} - \frac{1}{d} \sum_{m=1}^{d} c_m^{(\kappa)}.$$  \hspace{1cm} (19)

Eq. (18) tells us that each coefficient $c_j^{(\kappa)}$ can be written as the sum of a non-negative term $(p_j^{(\kappa)} - p_j^{(\kappa)})$ plus a (possibly negative) constant $x^{(\kappa)}$. We can show, however, that the sum of those constants $x^{(\kappa)}$ has to be non-negative. We do that by using the normalization condition (14) on eq. (18):

$$\sum_{\kappa=1}^{d+1} c_j^{(\kappa)} = 1 \Rightarrow \sum_{\kappa=1}^{d+1} p_j^{(\kappa)} - \sum_{\kappa=1}^{d+1} p_j^{(\kappa)} + \sum_{\kappa=1}^{d+1} x^{(\kappa)} = 1$$

$$\Rightarrow d + 1 - d \sum_{\kappa=1}^{d+1} p_j^{(\kappa)} + \sum_{\kappa=1}^{d+1} x^{(\kappa)} = 1$$

$$\Rightarrow \sum_{\kappa=1}^{d+1} x^{(\kappa)} = \sum_{\kappa=1}^{d+1} p_j^{(\kappa)} - 1.$$  \hspace{1cm} (20)

Now remember that our hypothesis is that $\sum_{\kappa=1}^{d+1} p_j^{(\kappa)} \geq 1$, which implies that $\sum_{\kappa=1}^{d+1} x^{(\kappa)} \geq 0$. Let us now use this fact and expression (18) to obtain an expansion of the density matrix $\hat{\rho}$ in terms of the projection operators $\hat{P}_j^{(\kappa)}$, but now with non-negative coefficients only. Plugging eq. (18) into eq. (12) we obtain:

$$\hat{\rho} = \sum_{\kappa=1}^{d+1} (p_j^{(\kappa)} - p_j^{(\kappa)} + x^{(\kappa)}) \hat{P}_j^{(\kappa)}.$$  \hspace{1cm} (21)

Using the fact that $\sum_{\kappa=1}^{d+1} \hat{P}_j^{(\kappa)} = 1$, we can rewrite this as

$$\hat{\rho} = \sum_{\kappa=1}^{d+1} \left( p_j^{(\kappa)} - p_j^{(\kappa)} + \frac{x}{d+1} \right) \hat{P}_j^{(\kappa)}.$$  \hspace{1cm} (22)

where we defined $x \equiv \sum_{\kappa=1}^{d+1} x^{(\kappa)}$. Note that $\hat{P}_j^{(\kappa)} \geq p_j^{(\kappa)}$ by definition, and our hypothesis guarantees that $x \geq 0$. What we have now is then an expansion of $\hat{\rho}$ in terms of the MUB projectors using only non-negative coefficients. QED

Our proof above is constructive – for any state in $C_d$ we can use equations (3) and (20) to obtain a convex decomposition of the state in terms of the MUB projectors and their associated probabilities, given by eq. (22). As noted in (20), some non-physical states (i.e. described by non-positive Hermitian matrices) can have non-negative
Wigner functions in a single definition of $\mathcal{W}$. Theorem 4 shows that imposing non-negativity of $\mathcal{W}$ for all definitions of $\mathcal{W}$ is sufficient to guarantee that the set $C_d$ contains only physical states.

In the light of Theorem 4 above, let us now discuss in which senses states with non-negative Wigner functions are classical. We have defined the set $C_d$ of states of a $d$-dimensional system with non-negative Wigner functions $\mathcal{W}$ in all definitions of $\mathcal{W}$ using a fixed complete set of MUB. Theorem 5 proves that the only pure states in $C_d$ are the MUB projectors. As mentioned before, it turns out that many constructions of complete sets of MUB use stabilizer states only, i.e., simultaneous eigenstates of Pauli operators $\sigma_x, \sigma_y, \sigma_z$; in the remainder of this paper we will assume that such a MUB construction using stabilizer states was chosen, as was the case in the original work of Gibbons et al. [11]. The stabilizer formalism then provides us with a way to represent pure states in $C_d$ using a number of bits which is polynomial in the number of qubits $[33]$. This contrasts with general quantum states whose classical description requires an exponential number of bits. We thus see that states whose discrete phase-space description avoids Feynman’s “negative probabilities” are classical also in the sense of having a classical-like short description.

The stabilizer formalism provides us with a framework in terms of which pure states in the set $C_d$ have an efficient classical description. Other choices of frameworks are possible, each choice resulting in a different set of quantum states with efficient classical descriptions. One example are the (mixed) separable density matrices of a collection of qubits, each of which has an efficient classical description in terms of single-qubit pure states. An efficient description, however, does not guarantee the existence of an efficient simulation scheme for the dynamics; the dynamics of separable mixed states in NMR quantum computation experiments provides us with an example of this problem [32]. In the next section we discuss the issue of simulability of unitary dynamics within our set $C_d$.

Given these observations, it is not surprising that pure states in $C_d$ can behave non-classically in other ways, that is, with respect to other frameworks. For example, states in $C_d$ can be highly entangled, allowing for proofs of quantum non-locality and contextuality.

IV. UNITARIES PRESERVING NON-NEGATIVITY OF $\mathcal{W}$

In continuous phase-space we can define classical unitaries as the group of unitaries which preserve non-negativity of the Wigner function $\mathcal{W}(q,p)$. It has been shown that this group is formed by all unitaries generated by Hamiltonians which are quadratic forms in phase space $[3]$. In this section we obtain an analogous result for our discrete Wigner functions $\mathcal{W}$: using the ‘classical’ pure states in $C_d$, we define and characterize the group of unitaries $\{U_c\}$ that map pure states in $C_d$ to other pure states in $C_d$. In other words, we characterize the group of ‘classical’ unitaries $\{U_c\}$ that preserve non-negativity of $\mathcal{W}$ for all quantum nets obtained from a fixed complete set of MUB built from stabilizer states.

The structure of the group $\{U_c\}$ may depend on the particular complete set of MUB we choose to define our discrete Wigner functions. We prove that for any MUB construction using Pauli operators, the group $\{U_c\}$ is a subgroup of the Clifford group (the group of unitaries mapping Pauli operators to Pauli operators under conjugation $[31]$). For the particular construction in [11] we present some unitaries in $\{U_c\}$ and discuss their action in phase space.

A. $\{U_c\}$ is a subgroup of the Clifford group

Let us consider the $(4^n - 1)$ Pauli operators acting on the $2^n$-dimensional Hilbert space of $n$ qubits (excluding the identity). We can partition these $(4^n - 1)$ Pauli operators into $(2^n + 1)$ sets $S_i$ of $(2^n - 1)$ commuting Paulis each. The joint eigenstates of the sets $S_i$ form a complete set of $(2^n + 1)$ MUB, as discussed in section 11. We can partition the Paulis in many different ways; each such partitioning defines a different complete set of MUB, which will be denoted as $B_i$. In this section we show that the ‘classical’ unitaries $\{U_c\}$ mapping MUB in a partition $B_i$ to MUB in the same partition form a subgroup of the Clifford group.

The strategy is as follows. We will consider a slightly more general problem, which is to characterize unitaries mapping MUB defined by an arbitrary partition $B_1$ of Pauli operators to MUB defined by a second partition $B_2$. A general unitary $U$ mapping $B_1$ to $B_2$ will in particular map two bases in $B_1$ to two other bases in $B_2$. Let us name them $(S_1 \in B_1) \xmapsto{U} (S_2 \in B_2), (T_1 \in B_1) \xmapsto{U} (T_2 \in B_2)$. The first step is to prove there are two Clifford unitaries $C_j(j = 1, 2)$ that map basis $S_j$ to the computational ($Z$) basis, while mapping basis $T_j$ to the $X$-basis. These standard Clifford unitaries are the key to the proof. This is because $U$ is Clifford if and only if $\hat{U} = C_2 U C_1^\dagger$ is Clifford. So it is enough to show $\hat{U}$ is Clifford (done in Theorem 8), which is easier as by construction $\hat{U}$ are unitaries that preserve both the $Z$ basis and the $X$ basis.

With this more general result in hand, we can consider the case when the two partitions are one and the same ($B_1 = B_2$), and we will have what we wanted to prove, i.e. that our ‘classical’ unitaries $\{U_c\}$ are Clifford group operators.

We wish to show

Theorem 2 Let $U$ be a unitary transformation that maps a complete set of Pauli MUB $B_1$ to a second complete set of Pauli MUB $B_2$. Then $U$ is in the Clifford group, up to a global phase.

The first step involves proving the following Lemma:
Lemma 1 Let $S$ and $T$ be two maximal Abelian subgroups of the Pauli group, with $S \cap T = \{1\}$. Then there exists a Clifford operation which maps $S \mapsto \langle Z_1, \ldots, Z_n \rangle$, $T \mapsto \langle X_1, \ldots, X_n \rangle$.

Proof: Since $S$ and $T$ are maximal Abelian subgroups with trivial intersection, it follows that no (non-identity) element of $T$ commutes with every element of $S$ (or vice-versa). Let $\{M_i \mid i = 1, \ldots, n\}$ be a set of generators of $S$. For any particular element $N \in T$, we can define the syndrome $\sigma(N)$ which is an $n$-tuple whose $i$-th component is given by $\sigma_i = c(N, M_i) \quad i = 1, \ldots, n$. Here $c(N, M) = 0$ if $N$ and $M$ commute and $c(N, M) = 1$ if $N$ and $M$ anticommute. Then it follows that if $N, N' \in T$, $N \neq N'$, then $\sigma(N) \neq \sigma(N')$ (since otherwise $\sigma(NN') = \sigma(N) + \sigma(N') = 0$, and $NN'$ would commute with every element of $S$).

In particular, since there are $2^n$ elements of $T$ and $2^n$ different possible values of $\sigma$, it follows that each value of $\sigma$ is used exactly once. Thus, we can choose $N_i \in T$ such that $\sigma(N_i) = e'_i$ (where $e'_i$ is the vector that is 1 in the $i$-th position and 0 elsewhere). That is, $N_i$ anticommutes with $M_i$ and commutes with $M_j$ ($i \neq j$). The $N_i$’s are independent (because their $\sigma$ vectors are independent) and they commute with each other (because $T$ is Abelian). Therefore, the set of $M_i$’s and $N_i$’s have the same commutation/anticommutation relationships as the $Z_i$’s and the $X_i$’s, so there exists a Clifford group operation that maps $M_i \mapsto Z_i$ and $N_i \mapsto X_i$. This provides the appropriate map on $S$ and $T$.

This Lemma can be adapted so it applies also to $d$-dimensional registers, the main difference being that the syndrome function $\sigma(N)$ assumes values which are vectors modulo $d$ (see [3]). For recent related results on the structure of the Clifford group see [34, 35].

An immediate consequence of this lemma is that for any complete set of Pauli MUB $B$, we can choose any two of its bases, represented by stabilizers $S$ and $T$, and find a Clifford group operation that will map $B$ to another complete set of MUB containing the bases $\langle Z_1, \ldots, Z_n \rangle$ and $\langle X_1, \ldots, X_n \rangle$; and in particular, this Clifford group operation will map $S$ to $\langle Z_1, \ldots, Z_n \rangle$ and $T$ to $\langle X_1, \ldots, X_n \rangle$.

Therefore, if we have a general unitary $U$ that maps Pauli MUB $B_1$ to Pauli MUB $B_2$, we can choose bases $S_1, T_1 \in B_1$ with $S_2 = U(S_1)$, $T_2 = U(T_1)$ (so $S_2, T_2 \in B_2$), and then find Clifford operations $C_1$ and $C_2$ which map $C_1 : S_1 \mapsto \langle Z_1, \ldots, Z_n \rangle$, $C_1 : T_1 \mapsto \langle X_1, \ldots, X_n \rangle$, and then find Clifford operations $C_2$ which map $C_2 : S_2 \mapsto \langle Z_1, \ldots, Z_n \rangle$, $C_2 : T_2 \mapsto \langle X_1, \ldots, X_n \rangle$. Then it follows that $C_2 U C_1^* : \langle Z_1, \ldots, Z_n \rangle \mapsto \langle Z_1, \ldots, Z_n \rangle$, $C_2 U C_1^* : \langle X_1, \ldots, X_n \rangle \mapsto \langle X_1, \ldots, X_n \rangle$. Denoting $\tilde{U} = C_2 U C_1^*$, it is easy to see that $\tilde{U}$ is a Clifford group operator iff $U$ is a Clifford group operation. Thus, to prove Theorem 3 it will be sufficient to prove

Theorem 3 If $\tilde{U}$ is a unitary operation which preserves both the $Z$ basis and the $X$ basis (i.e., maps eigenstates of $\langle Z_1, \ldots, Z_n \rangle$ to other eigenstates of this set of operators, and the same is valid for eigenstates of $\langle X_1, \ldots, X_n \rangle$), then $\tilde{U}$ is a Clifford group operation, up to a global phase.

Proof: Since $\tilde{U}$ preserves the $Z$ basis, it has the form of a classical gate with possibly some phases changed:

$$\tilde{U}\ket{\vec{z}}_Z = e^{i\phi(\vec{z})}\ket{\tilde{g}(\vec{z})}_Z,$$

with $\tilde{g}(\vec{z})$ a permutation of the $2^n$ possible values of $\vec{z}$.

In terms of the $Z$ basis, we can expand elements of the $X$ basis as follows:

$$\ket{\vec{x}}_X = \sum_{\vec{z}} e^{i\pi(\vec{x} \cdot \vec{z})}\ket{\vec{z}}_Z.$$

Therefore, $\tilde{U}\ket{\vec{x}}_X = \sum_{\vec{z}} e^{i\pi(\vec{x} \cdot \vec{z} + \phi(\vec{z}))}\ket{\tilde{g}(\vec{z})}_Z = \sum_{\vec{z}} e^{i\pi\tilde{g}(\vec{z})^{-1}(\vec{x} \cdot \vec{z}) + \phi(\tilde{g}(\vec{z})^{-1}(\vec{x}))}\ket{\vec{z}}_Z.$$

In order to preserve the $X$ basis, we need

$$\tilde{U}\ket{\vec{x}}_X = e^{i\theta(\vec{x})}\ket{\tilde{h}(\vec{x})}_X = \sum_{\vec{z}} e^{i\theta(\vec{x} \cdot \vec{z}) + \pi\tilde{h}(\vec{x} \cdot \vec{z})}\ket{\vec{z}}_Z.$$

where $\tilde{h}(\vec{x})$ is a permutation of the values of $\vec{x}$. Equating (25) and (26), we find

$$\pi\vec{x} \cdot \tilde{g}(\vec{z})^{-1}(\vec{z}) + \phi(\tilde{g}(\vec{z})^{-1}(\vec{z})) = \theta(\vec{x} \cdot \vec{z}) + \pi\tilde{h}(\vec{x} \cdot \vec{z}).$$

This must be true for all $\vec{x}$ and $\vec{z}$. Plugging in $\vec{x} = \vec{0}$, we find

$$\phi(\tilde{g}(\vec{z})^{-1}(\vec{z})) = \theta(\vec{0}) + \pi\tilde{h}(\vec{0} \cdot \vec{z}) = \theta(\vec{0}) + \pi\tilde{h}(\vec{0} \cdot \vec{z}).$$

where $\theta(\vec{0}) = \theta(\vec{0})$ and $\tilde{h}(\vec{0} \cdot \vec{z}) = \tilde{h}(\vec{0})$. Therefore

$$\pi\vec{x} \cdot \tilde{g}(\vec{z})^{-1}(\vec{z}) = \pi\tilde{h}(\vec{0} \cdot \vec{z}) - \theta(\vec{0}).$$

Of course, eqs. 27-29 are understood to modulo 2$\pi$.

It then follows that $\tilde{g}(\vec{z})^{-1}(\vec{z})$ must be affine in $\vec{z}$:

$$\tilde{g}(\vec{z})^{-1}(\vec{z}) = A\vec{z} + \vec{b},$$

(30)

where $A$ is an invertible $n \times n$ binary matrix and thus, by (28), $\phi(\vec{z})$ is also affine in $\vec{z}$:

$$\phi(\vec{z}) = \pi\vec{c} \cdot \vec{z} + d,$$

(31)

with $A\vec{c} = \vec{h}$ and $\pi\vec{b} \cdot \vec{c} + d = \theta(\vec{0})$.$\quad$ Thus we find

$$\tilde{U}\ket{\vec{z}}_Z = e^{i\pi(\vec{c} \cdot \vec{z} + d)}|A^{-1}\vec{z} - A^{-1}\vec{b}|.$$ 

(32)

We can easily identify this as a Clifford group operation, up to the global phase $e^{id}$. $|\vec{z}| \mapsto |A^{-1}\vec{z}|$ can be performed with CNOT gates, $|\vec{z}| \mapsto |\vec{z} - A^{-1}\vec{b}|$ can be
performed with $X$ operations, and $|\vec{z}\rangle \mapsto e^{i\pi(\vec{c} \cdot \vec{z})}|\vec{z}\rangle$ can be performed with $Z$ operations.

The proof for $d$-dimensional registers is almost identical, except that we must replace $\pi$ everywhere with $2\pi/d$, and we need scalar multiplication gates as well as SUM gates to perform $|\vec{z}\rangle \mapsto |A^{-1}\vec{z}\rangle$.

For prime Hilbert space dimensions there is a unique Pauli MUB construction that uses all stabilizer states $\{U_c\}$. In that case our set of pure classical states coincides with the set of stabilizer states, and our group of classical unitaries $\{U_c\}$ coincides with the Clifford group. This is not the case for power-of-prime dimensions, where a complete set of Pauli MUB contains only a proper subset of the stabilizer states, resulting in classical unitaries $\{U_c\}$ which form a proper subgroup of the Clifford group.

The Gottesman-Knill theorem states that Clifford group operations on stabilizer states can be simulated efficiently on a classical computer. For discrete Wigner functions defined with respect to stabilizer MUB, our Theorems $\text{1}$ and $\text{2}$ guarantee that the group of classical unitaries $\{U_c\}$ applied on the set of classical pure states in $C_d$ can be efficiently simulated, i.e. with a number of time-steps that increases only polynomially in the number of qubits. This is to be contrasted with general quantum computation, which uses states and operations outside of our classical sets, and which is thought to provide exponential speedup for some problems. The necessity of negativity of $W$ for achieving universal quantum computation had been noted in $\text{24}$ for a particular computational model proposed recently by Bravyi and Kitaev.

Our results point to an interesting convergence between two different notions of classicality. The first defines classical states as those whose description can be made in terms of non-negative quasi-probability distributions, in this case the discrete Wigner functions of $\text{11}$. The second is motivated by quantum computation: classical states and operations are those which can be efficiently simulated on a classical computer. For a related discussion of simulability in the context of continuous variables see $\text{55}$.

In this section and the previous one we only made claims of classicality for pure states in $C_d$ and their associated unitary dynamics. The problem seems to become much more involved when we consider mixed states in $C_d$ and their associated dynamics, which in this case will be (in general non-unitary) completely positive maps. It is not clear whether an efficient simulation scheme for this more general definition of classical dynamics can be devised.

**B. Unitaries in $\{U_c\}$ and their action in phase space**

We have just shown that when we build a complete set of MUB using Pauli operators, the ‘classical’ unitaries in $\{U_c\}$ turn out to form a subgroup of the Clifford group. The exact characterization of this subgroup will depend on which Pauli MUB construction we pick. In this section we restrict ourselves to the stabilizer MUB construction for $N$ qubits sketched in section $\text{11}$ and present some ‘classical’ unitaries in $\{U_c\}$ together with their associated action in phase space.

1. **Discrete phase-space translation operators**

In $\text{19}$ it was shown that the Pauli operators act as discrete phase-space translations mapping phase-space lines into other lines. This means that the Pauli operators themselves are in our group $\{U_c\}$. Translation operators $\hat{T}(\vec{q},\vec{p})$ operate on quantum states in such a way that their Wigner functions are transformed as flows: each phase-space point operator is mapped into another one because of the covariance condition imposed on the quantum net. Thus, the effect of a translation operator on a state’s Wigner function is a translation in phase space, and in this sense its action is “classical-like”. Since translation operators act as flows in phase space, they preserve positivity of $W$ for any single association between lines and MUB projectors.

2. **Discrete squeezing operator**

The discrete squeezing operator $\text{13}$ maps the horizontal and the vertical striations into themselves, while cycling through all oblique striations. An explicit Clifford circuit for $U_s$ is given in $\text{14}$. When acting on translation operators, $U_s$ maps them into other translations in a way that resembles a squeezing flow in phase space (see Fig. 1.1b):

$$U_s\hat{T}(\vec{q},\vec{p})U_s^\dagger = \pm\hat{T}(\vec{q}M,\vec{p}M^{-1}).$$  \hspace{1cm} (33)

Besides covariance with respect to the discrete phase-space translations, we can impose on the quantum net also the constraint of covariance under $U_s$. In doing so, the freedom in picking the quantum net will be limited to the choice of which MUB projector to associate to a fixed oblique line, since the covariance requirement determines...
all other associations. In this way, the number of possible choices is greatly reduced from $d^{d-1}$ to $d$.

If we choose a quantum net which is covariant under the squeezing operator, it can be shown that $U_s$ will map phase-space point operators into other point operators, i.e. $U_s$ acts like a phase-space flow. This may not be the case if the quantum net is not chosen to be covariant under $U_s$.

By definition, the group $\{U_s\}$ consists of unitaries that preserve non-negativity of the Wigner function $W$ for all possible quantum nets. This does not imply that operators in $\{U_s\}$ will preserve positivity for any single definition of $W$. This is because some states may have positive $W$ for a single definition, but negative $W$ for other definitions (and hence lie outside the set $C_d$). In the case of $U_s$, preservation of positivity for each single definition of $W$ is only guaranteed when $U_s$ acts as a flow in phase space, and this only happens when the quantum net is chosen to be covariant with respect to $U_s$.

3. Finite Fourier transform

The finite Fourier transform $F$ maps the horizontal and the vertical striations into one another; oblique striations are interchanged in pairs, and one of them [the “main diagonal”], which corresponds to the eigenstates of the set of translations obtained by setting $\vec{a} = \vec{b}$ in eq. (10) is mapped into itself. For the particular case in which the canonical basis of the Galois field is self-dual, $F$ is just the Hadamard transform. The effect of $F$ on the translation operators is $-\lambda$ to a sign $-\lambda$ a reflection with respect to the main diagonal of phase space (see Fig. 1c):

$$F(T(\vec{1M})^j, M^k)F^\dagger = \pm T(\vec{1M})^j, M^k$$  

The fact that $F$ interchanges translation operators by a reflection might suggest that its action on the states could be analogous, that is, that $F$ could reflect a state’s Wigner function with respect to the main diagonal, perhaps for some particular quantum nets (as is the case with $U_s$).

For $F$ to act on lines as a reflection with respect to the main diagonal, there should be one MUB projector associated to the axis of reflection, and $F$ should map this projector into itself. This projector should, then, be a common eigenstate of $F$ and all the Paulis that define the basis to which the state belongs. Using the fact that $F$ anti-commutes with some of them, and that the eigenvalues of $F$ and the Paulis are different from zero, it can be seen that no state can fulfill this requirement. Thus, there is no association between lines and MUB projectors that makes the action of $F$ on the Wigner function be a reflection flow.

Moreover, it can be seen that there is no quantum net for which $F$ acts as a flow in any way, because for $F$ to be a flow it should map phase-space point operators into other point operators. For this to happen, the $(d + 1)$ lines that intersect in any given point must be mapped by $F$ into other lines that intersect in only one point. The vertical and the horizontal rays (i.e. lines containing the origin) are interchanged by $F$, so the other rays must be mapped into rays too (so that all the resulting lines intersect at the origin). This requires the ray in the main diagonal to be mapped into itself, and, as pointed out in the previous paragraph, this cannot be achieved.

Therefore, $F$ provides an example of an operator in $\{U_s\}$ which cannot be interpreted in terms of a flow for any choice of associations between lines and states, and so has no obvious continuous phase-space analogue.

V. CONCLUSION

We have characterized the set $C_d$ of states whose discrete Wigner functions $W$ (as defined in [1]) are non-negative. We showed that the only pure states in $C_d$ are the mutually unbiased bases projectors used to define $W$, as conjectured in [2]. Constructions of complete sets of MUB using stabilizer states are known for any power-of-prime dimension $d$, and we restricted our attention to Wigner functions defined using such stabilizer MUB. Our characterization of $C_d$ then ensures that pure states with non-negative Wigner functions admit an efficient classical description using the stabilizer formalism. Moreover, we proved that the unitaries which preserve non-negativity of $W$ for all such functions $W$ form a subgroup of the Clifford group. It is known that Clifford operations on stabilizer states can be simulated efficiently on a classical computer. We have thus identified a relation between two different notions of classicality: states which are classical in the sense of having non-negative quasi-probability distributions (the discrete Wigner functions of [1]) can also be simulated efficiently on classical computers. Since general quantum computation is thought to be hard to simulate classically, our results mean that negativity of $W$ is necessary for exponential computational speedup with pure states.

There are many open problems worth investigating. The complete characterization of non-negativity preserving unitaries for different constructions of complete sets of MUB is still unsolved. It would also be interesting if one could relate non-classicality to negativity of $W$ in a quantitative way. Another research direction is to investigate the relationship between $W$ and a notable open problem, that of the existence of complete sets of mutually unbiased bases for general Hilbert space dimensions (see [21, 22, 24]). The original idea behind continuous-variable Wigner functions was to help visualize quantum dynamics in the familiar framework of classical phase space. Some research has been done on the visualization of quantum information protocols in discrete phase-space [12, 13, 14, 14, 15, 18, 13]; further work might bring insights into existing applications, or suggest new ones.
Acknowledgments. EFG was partly supported by Canada’s NSERC. JPP was partially funded by Fundación Antorchas and a grant from ARDA. AOP was partially supported by NSF grants EIA-0113137 and DMS-0309042.