Formalism of thermal waves applied to the characterization of materials thermal effusivity

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Thermal characterization of materials, especially civil engineering materials, in the way of non-destructive methods, are more and more widespread. In this article, we show an original point of view to describe the used method, the thermal waves, to obtain the thermal impedance of the studied system, using a specific sensor – a fluxmeter. The identification technique, based on a frequential approach, is optimized by applying a random input to the system. This kind of random heating is shown to provide a frequency range where the thermal effusivity is able to be identified and not correlated to another parameter. The strength of the method is also the determination of the contact resistance of the system, that allows to validate the identification process. Experimental results obtained from a sample with well-known thermal properties (polyvinyl chloride) are used to validate the proposed method. © 2011 American Institute of Physics. [doi:10.1063/1.3600899]

I. INTRODUCTION

The measurement of thermophysical properties is more and more developed, using different methods.1 In recent years, thermal wave physics has played an important role in the area of non-destructive evaluation of subsurface defects2,3 in metallic, composite materials, and the characterization of civil engineering materials. It is achieved by observing, recording, and analyzing the thermal response at the material surface to a heat stimulus. Especially in the domain of thermography, significant works have been done and understanding achieved in the propagation of thermal waves in materials.4

Non-destructive testing methods are used to detect defects in materials and evaluation of materials properties without causing changes to their usable properties.

Usually, an available form of answer on thermal stimulation is modification of temperature and speed of temperature changes it contains information about values of capacity and thermal conductivity, characterizing internal structure of a testing object. The basic foundation of non-destructive testing methods is that every class of testing objects reacts on stimulation in a specific manner. There can be different reactions of using stimulation of a particular type to cause the beginning of thermal wave and reaction of an object against its spreading. Examples of that stimulation can be used are as follows: sound waves, infrared lamps, microwaves, flash lamps, lasers, others.

The stimulation has most often the character of an impulse or sinusoidal wave. The result of an object against stimulation is registered by the thermal system. Their study describes the thermal materials properties. But in most of the cases, thermal characterization methods involve experiments requiring special precautions that can only be taken in the laboratory, as the information is readily accessible but obtained at the cost of very strict experimental conditions that call for often complex and expensive experimental setups. For several years, our research team has been developing thermophysical characterization methods based on the study of thermal impedance.5 It is a function that represents the relation between the frequency components of temperature and the flux density in a plane for each frequency. From an experimental point of view, it is determined simply by measuring the flux density and temperature simultaneously in a measurement plane. In practice, a fluxmeter6 in which a thermocouple has been placed is put in contact with the sample. The changes in flux density and temperature measured in this way are different from those in the material access plane. The reasons for this perturbation are the presence of the sensor and the sensor/material contact resistance. This original approach is based on a frequent study of changes at the surface of materials.

The purpose of this article is to show that it is possible to obtain the thermal impedance from the formalism of the thermal waves, and use it as a characterization method to less effusive materials such as polyvinyl chloride (PVC).

II. EXPERIMENTAL SETUP AND MODELLING OF THERMAL IMPEDANCE

A. Constitutive equations for the temperature field in an isotropic homogeneous medium

The thermal state of a homogeneous medium is described by its temperature field \( \theta (x, y, z, t) \), which is a function of space and time. The variable \( \theta \) is the solution of a partial differential equation known as the heat equation. In the case of
unidirectional heat transfer, this equation is limited to

\[
\frac{\partial \theta(x, t)}{\partial t} = a \frac{\partial^2 \theta(x, t)}{\partial x^2}. \tag{1}
\]

The thermophysical parameter \(a \, [m^2/s]\) is the thermal diffusivity, which represents the speed at which a material is able to propagate a thermal disturbance.

The temperature gradient causes an exchange of heat represented by the flux \(\phi(x, t)\). The two parameters \(\theta\) and \(\phi\) are linked by Fourier’s law, which is given here in the case of an isotropic material,

\[
\phi(x, t) = -\lambda \frac{\partial \theta(x, t)}{\partial x}. \tag{2}
\]

It can easily be demonstrated that it is possible to write a similar relation to the heat equation with the flux density,

\[
\frac{\partial \phi(x, t)}{\partial t} = a \frac{\partial^2 \phi(x, t)}{\partial x^2}. \tag{3}
\]

### B. Resolution of the thermal wave propagation equations

#### 1. General case

In this section, allowance will be made only for the case of solicitations and sinusoidal time functions. Sinusoidal functions are a basis for decomposition enabling all the other functions to be considered:

\[
\tilde{\theta}(x, t) = \theta_{\text{Max}} \cdot \cos(\omega t - \varphi(x)), \tag{4}
\]

\[
\tilde{\phi}(x, t) = \phi_{\text{Max}} \cdot \cos(\omega t - \varphi'(x)). \tag{5}
\]

Here, \(\theta_{\text{Max}}\) and \(\phi_{\text{Max}}\) are respectively the temperature and flux ranges at the abscissa \(x\) while \(\varphi\) and \(\varphi'\) are the phase shifts.

In the particular case of harmonic parameters, the following complex formulation may be adopted:

\[
\tilde{\theta}(x, t) = \theta_{\text{Max}} \cdot e^{j(\omega t - \varphi(x))} = \tilde{\theta}_x e^{j\omega t}, \tag{6}
\]

\[
\tilde{\phi}(x, t) = \phi_{\text{Max}} \cdot e^{j(\omega t - \varphi'(x))} = \tilde{\phi}_x e^{j\omega t}, \tag{7}
\]

in which the functions of \(x\): \(\tilde{\theta}_x = \theta_{\text{Max}} \cdot e^{j\varphi(x)}\) and \(\tilde{\phi}_x = \phi_{\text{Max}} \cdot e^{j\varphi'(x)}\) are independent of time \(t\), hence the formulation

\[
\frac{\partial^2 \tilde{\theta}}{\partial x^2} = \gamma^2 \tilde{\theta}, \tag{8}
\]

\[
\frac{\partial^2 \tilde{\phi}}{\partial x^2} = \gamma^2 \tilde{\phi}, \tag{9}
\]

with \(\gamma^2 = \frac{j \omega}{a}\).

Solving this coupled equation system by using Fourier’s law gives

\[
\tilde{\theta}(x, \omega) = A e^{-\gamma x} + B e^{+\gamma x}, \tag{10}
\]

\[
\tilde{\phi}(x, \omega) = \frac{1}{\tilde{Z}_c} (A e^{-\gamma x} - B e^{+\gamma x}), \tag{11}
\]

where \(\tilde{Z}_c\) is the characteristic impedance of the medium,

\[
\tilde{Z}_c = \frac{\sqrt{a}}{\lambda} = \frac{1}{b \sqrt{j \omega}}. \tag{12}
\]

In this expression \(b = \lambda / \sqrt{a} = \sqrt{\kappa \rho c}\) represents the thermal effusivity of the material.

The thermal effusivity is the essential parameter of a transient regime. It represents the ability of the material to absorb heat energy.

At any moment, the temperature and flux density may be considered as the superimposition of two thermal waves:

– one propagating in the direction of increasing \(x\) values, denoted as propagation term \(e^{-\gamma x}\). This “thermal wave” has a temperature amplitude \(A\) and flux density amplitude \(A / \tilde{Z}_c\) at the origin \((x = 0)\);

– the other, denoted as propagation term \(e^{+\gamma x}\), propagating in the opposite direction. This return wave has amplitudes \(B\) and \(-B / \tilde{Z}_c\) at the origin for the temperature and flux density respectively.

#### 2. Particular case of a semi-infinite medium

When \(x\) tends towards infinity \((\infty)\), the amplitudes of the temperature and flux density variations must tend towards 0 (Fig. 1). This requires \(B = 0\) in Eqs. (10) and (11), which thus become

\[
\tilde{\theta}(x, \omega) = A e^{-\gamma x}, \tag{13}
\]

\[
\tilde{\phi}(x, \omega) = \frac{1}{\tilde{Z}_c} A e^{-\gamma x}. \tag{14}
\]

In particular, for \(x = 0\) (access face)

\[
\tilde{\theta}(0, \omega) = A, \tag{15}
\]

\[
\tilde{\phi}(0, \omega) = \frac{1}{\tilde{Z}_c} A. \tag{16}
\]

The amplitudes at any point can be written in relation to the amplitudes at the origin,

\[
\tilde{\theta}(x, \omega) = A e^{-\gamma x} = \tilde{\theta}(0, \omega) \cdot e^{-\gamma x}, \tag{17}
\]

\[
\tilde{\phi}(x, \omega) = \frac{A}{\tilde{Z}_c} e^{-\gamma x} = \tilde{\phi}(0, \omega) \cdot e^{-\gamma x}. \tag{18}
\]
It can be seen that the amplitudes at all points in a semi-infinite medium are linked by the characteristic impedance,

$$\frac{\theta(x, \omega)}{\phi(x, \omega)} = \frac{\theta(0, \omega)}{\phi(0, \omega)} = Z_e = \frac{1}{b\sqrt{j\omega}}.$$  \hspace{1cm} (19)

All that remains in Eqs. (17) and (18) are the propagation terms $e^{-\gamma x}$. Heat diffusion in the case of a semi-infinite medium may be assimilated to the propagation of a progressive wave.

C. Notion of thermal impedance

1. Definition

Thermal impedance is a complex parameter, defined in the frequency domain as the ratio of the temperature and flux spectra. It is used for thermophysical characterization purposes or for non-destructive testing.

In the frequency domain, the impedance $Z$ is defined by

$$Z(f) = \frac{\theta(f)}{\phi(f)} = \frac{\theta(\omega)}{\phi(\omega)},$$ \hspace{1cm} (20)

where $Z$ is a function of the frequency $f$ or of the pulse $\omega$.

2. Case of a homogeneous wall

In the case of a homogeneous medium of finite dimensions, a homogeneous wall is considered (Fig. 2). Its thickness $\ell$ is small in comparison with the transverse dimensions. In these conditions, if it is subject to uniform stress on one of the access faces, the changes are independent of the coordinates $y$ and $z$. The conduction regime is one-dimensional.

$Z_e$ will be used to denote the input impedance of such a system, with $\theta_e, \phi_e$ as input parameters, and $Z_t$ the output impedance of such a system, with $\theta_t, \phi_t$ as output parameters.

If the solutions considered are of the complex exponential type, the thermal impedance of the output plane ($x = \ell$) is expressed by

$$Z_s(\omega) = \frac{\theta_s(\omega)}{\phi_s(\omega)} = \frac{\theta_s(\ell, \omega)}{\phi_s(\ell, \omega)},$$ \hspace{1cm} (21)

with $\theta_s = \theta(\ell, \omega)$ and $\phi_s = \phi(\ell, \omega)$.

3. Resolution

The following hyperbolic lines are introduced into the general expressions:

$$e^{y x} = ch(y x) + sh(y x),$$ \hspace{1cm} (22)

$$e^{-y x} = ch(y x) - sh(y x),$$ \hspace{1cm} (23)

where $ch$ and $sh$ are hyperbolic cosines and sines, respectively.

Equations (17) and (18) become

$$\theta(x, \omega) = (A + B)ch(y x) - (A - B)sh(y x),$$ \hspace{1cm} (24)

$$\phi(x, \omega) = \frac{1}{Z_e}((A - B)ch(y x) - (A + B)sh(y x)).$$ \hspace{1cm} (25)

It may be noted that

$$A + B = \theta_s ch(\gamma \ell) + Z_e \phi_s sh(\gamma \ell),$$ \hspace{1cm} (26)

$$A - B = \theta_s sh(\gamma \ell) + Z_e \phi_s ch(\gamma \ell).$$ \hspace{1cm} (27)

Hence the relations

$$\theta(x, \omega) = \theta_s ch(\gamma (\ell - x)) + Z_e \phi_s sh(\gamma (\ell - x)),$$ \hspace{1cm} (28)

$$\phi(x, \omega) = \phi_s ch(\gamma (\ell - x)) + \frac{\theta_s}{Z_e} sh(\gamma (\ell - x)).$$ \hspace{1cm} (29)

The following are thus obtained at $x = 0$:

$$\theta(0, \omega) = \theta_s ch(\gamma \ell) + Z_e \phi_s sh(\gamma \ell),$$ \hspace{1cm} (30)

$$\phi(0, \omega) = \phi_s ch(\gamma \ell) + \frac{\theta_s}{Z_e} sh(\gamma \ell).$$ \hspace{1cm} (31)

These two expressions can be used to determine the thermal impedance of the input plane ($x = 0$):

$$Z_e(\omega) = \frac{\theta_e(\omega)}{\phi_e(\omega)} = \frac{\theta(0, \omega)}{\phi(0, \omega)}.$$ \hspace{1cm} (32)

This gives

$$Z_e(\omega) = Z_e \frac{\theta_s ch(\gamma \ell) + Z_e \phi_s sh(\gamma \ell)}{Z_e \phi_s ch(\gamma \ell) + \theta_s sh(\gamma \ell)},$$ \hspace{1cm} (33)

or again

$$Z_e(\omega) = \frac{Z_s + Z_e th(\gamma \ell)}{Z_e + Z_s th(\gamma \ell)},$$ \hspace{1cm} (34)

where $th$ indicates the hyperbolic tangent.

Relation (34) expresses the input thermal impedance of a homogeneous wall as a function of its output impedance.
4. Case of a semi-infinite medium

Considering a semi-infinite medium ($\ell \to \infty$, $th(\gamma \ell) \approx 1$), the impedance of the homogeneous medium only depends on the thermophysical parameter – the thermal diffusivity $b$,

$$Z_{c, \infty}(\omega) = Z_c(\omega) = \frac{1}{b \sqrt{j \omega}}. \quad (35)$$

This is the characteristic impedance $Z_c$ introduced previously.

This characteristic impedance corresponds to the reference behaviour of a homogeneous material, and can be compared with an experimental impedance. This approach can be used to detect a possible anomaly.

5. Case of contact resistance

In this work, a flux and temperature sensor is placed on the system (Fig. 3).

Contact conditions lead to a temperature drop across the interface (sensor and medium). It is modeled by means of a contact resistance $R_c$. If it is neglected in the model of behavior, it can cause significant errors in estimating the parameters of the material. The value of $R_c$ depends on surface roughness and conditions of positioning. It is interesting to determine it for each test. On the basis of relations (30) and (31), the problem is represented by the following relations:

$$\theta(0, \omega) = \theta_s + R_c \phi_s, \quad (36)$$

$$\phi(0, \omega) = \phi_s. \quad (37)$$

The thermal impedance of the input plane has thus been obtained:

$$Z_e(\omega) = \frac{\theta_s(\omega)}{\phi_s(\omega)} = \frac{\theta(0, \omega)}{\phi(0, \omega)} = R_c + Z_s. \quad (38)$$

6. Case of sensor (fluxmeter)

The sensor (heat fluxmeter), is a very thin complex system made of copper, constantan and kapton. At low frequencies (less than 0.1 Hz) the sensor may be considered to be an homogeneous material and the following approximations may be made,

$$ch(\gamma \ell) = c h \left(\frac{j \omega}{a} \ell \right) \approx 1, \quad (39)$$

$$sh(\gamma \ell) = s h \left(\frac{j \omega}{a} \ell \right) \approx \sqrt{\frac{j \omega}{a} \ell}. \quad (40)$$

Therefore,

$$Z_e \cdot sh(\gamma \ell) \approx Z_c \gamma \ell = \frac{1}{b \sqrt{j \omega}} \cdot \sqrt{\frac{j \omega}{a} \ell} = \frac{\ell}{b \sqrt{a}} = \frac{\ell}{\lambda} = R_f, \quad (41)$$

$$\frac{1}{Z_e} \cdot sh(\gamma \ell) \approx \frac{1}{Z_c} \gamma \ell = b \sqrt{j \omega} \cdot \sqrt{\frac{j \omega}{a} \ell} = j \omega b \ell \sqrt{a} = j \omega \rho c \ell = j \omega C_f, \quad (42)$$

where $R_f$ corresponds to the resistance of the fluxmeter in $\text{K m}^2/\text{W}$, while $C_f$ is the calorific capacity of the fluxmeter in $\text{J/(m}^2 \text{K)}$.

Relations (30) and (31) may thus be written as follows:

$$\theta(0, \omega) = \theta_s + R_f \phi_s, \quad (43)$$

$$\phi(0, \omega) = \phi_s + j \omega C_f \theta_s, \quad (44)$$

which leads to thermal impedance in the input plane

$$Z_e(\omega) = \frac{\theta_s(\omega)}{\phi_s(\omega)} = \frac{\theta(0, \omega)}{\phi(0, \omega)} = \frac{R_f + Z_s}{1 + j \omega C_f Z_s}. \quad (45)$$

At the low frequencies studied, the sensor may be modelled by a cell including local thermal resistance and thermal capacity.

Considering a semi-infinite medium, the thermal impedance is equal to the characteristic impedance of the medium.

D. Case of a three-layer medium

1. Case of material A – resistance – material A (semi-infinite)

The presence of a pure resistance in a conduction system is an important case. Delamination in a structure or insulant in a building wall are two examples of this configuration.

The existence of delamination or a pure resistance in a homogeneous material means that a three-level system needs to be studied: material – resistance – material (Fig. 4).
E. Case of the experimental study of a semi-infinite medium

One of the applications in this work aims to show that it is possible to avoid sensor and contact disturbance when a semi-infinite medium is studied by using relatively high frequencies.

At these frequencies (of the order of $10^2$ Hz), the system to be taken into account has three layers: sensor – contact – semi-infinite material. A similar approach to that used earlier means that the impedances are determined in the following order:

- at the entrance to the semi-infinite medium,
- at the entrance to the resistance $R_c$,
- in the sensor measuring plane.

This gives respectively

$$\tilde{Z}_1(\omega) = \tilde{Z}_1(\omega) = \frac{1}{b\sqrt{j\omega}}.$$  

$$\tilde{Z}_2(\omega) = R_c + \tilde{Z}_3,$$  

$$\tilde{Z}_3(\omega) = \frac{R_f + \tilde{Z}_2}{1 + j\omega C_f \tilde{Z}_2},$$  

i.e.,

$$\tilde{Z}_4(\omega) = \tilde{Z}_4(\omega) = \frac{R_c + R_f + \tilde{Z}_4(\omega)}{1 + j\omega C_f (R_c + \tilde{Z}_4(\omega))}.$$  

III. ANALYSIS OF IMPEDANCE SENSITIVITY TO THERMOPHYSICAL PARAMETERS

The objective of the application is to determine the thermal effusivity of the material from an estimation of the impedance on a frequencies band. A sensitivity analysis can determine which parameters are identifiable and what is the optimal frequency band to achieve such identification.

Usually, a sensitivity analysis is the study of how the variation (uncertainty) in the output of a mathematical model can be apportioned, qualitatively or quantitatively, to different sources of variation in the input of the model. A put another way, it is a technique for systematically changing parameters in a model to determine the effects of such changes.

The understanding of how the model behaves in response to changes in its inputs, is of fundamental importance to ensure a correct use of the models. Sensitivity Analysis has the role of ordering by importance the strength and relevance of the inputs in determining the variation in the output.

Here, the objective of such a sensitivity study is to define the influence of each system parameter and to optimize the choice of the frequency range to be used for identification purposes. In an inverse technique procedure, thermophysical parameters are estimated by seeking the grouping in which the experimental impedance is best approximated by the theoretical impedance. The possibility of simultaneously identifying these parameters can be discussed on the basis of a sensitivity study. This is performed by observing variations in the given function when subjected to a change in one of the parameters. Thus, analysis takes into account the range of variation.
and meets the conditions for de-correlating the quantities. In this way, the frequency range under study can be optimized as a function of the required objectives. It may even enable the model to be reduced, as certain parameters would prove to have negligible influence on the observation range.

The impedance is a complex function of frequency. The sensitivity of the moduli and phases to various parameters is studied in parallel. The sensitivity functions $S_{ip}$ of both the moduli and the phase of $Z$ to the parameter $p_i$ will be defined by the relation

$$s_{Z,p_i}(f) = \frac{\Delta Z}{\Delta p_i} \bigg/ \frac{Z}{p_i}.$$  (57)

In this expression, $Z$ represents alternatively the modulus or argument of the impedance. In order to make interpretation easier over a wide frequency range, the ratio of the relative variations in the parameter to the response function is calculated and expressed as a percentage of the tested function. Since $S_{ip}$ is defined as a ratio of two non-dimensional functions, it follows that it is also non-dimensional. The calculation, numerically obtained, involves introducing nominal values of the various parameters. Such a constraint is not in contradiction with the aim of determining these parameters. The sensitivity functions are only used qualitatively. They highlight the predominance of given parameters in the behavior of the response function and their possible correlation. They also make it possible to choose optimum frequency ranges for identifying the main parameters. A rough estimate of the values is enough for this use of the sensitivity functions. In this case, the following values were chosen: $C_f = 650 \text{ J m}^{-2} \text{ K}^{-1}$ for the sensor capacity, $R_c = 5 \times 10^{-2} \text{ K m}^2 \text{ W}^{-1}$ for the contact resistance, $R_f = 5 \times 10^{-4} \text{ K m}^2 \text{ W}$ for the resistance of the fluxmeter and $b = 500 \text{ J m}^{-2} \text{ s}^{-1/2} \text{ K}^{-1}$ for the thermal effusivity of the material.

Figure 5 shows a global view (for better appreciation) of the modulus sensitivity for each parameter. This example makes clear the choice of the frequency range between $\sim 10^{-3}$ Hz to some $10^{-2}$ Hz, because, in this range, the impedance is highly sensitive to the thermal effusivity of the material; it should also be noted that there is high sensitivity to resistance $R_c$, which increases with frequency, and both sensitivity curves $b$ and $R_c$ are not proportional, indicating that the sensitivities are not correlated in the studied frequency range. The two parameters can be determined simultaneously. It should be noted that the sensitivity to the sensor capacity is much lower. Capacity values obtained by identification will be assigned a high level of uncertainty, because of a low sensitivity in the impedance estimation. The result could be fixed at a nominal value without hampering any identification of the others. The study shows that disturbance from the sensor ($C_f$, $R_f$) can be overlooked.

Assuming a lower frequency range, we could neglect the $R_c$ parameter; however, in such a case, an experiment of sufficiently long time should be required that questions the unidirectionality of the exchanges.

Finally, to check if parameters $b$ and $R_c$ are correlated, the sensitivity to $R_c$ is plotted as a function of sensitivity to $b$. If those parameters are correlated, the graph will be a line going through the origin. Both sensitivities are non-dimensional and expressed in a percent form in Fig. 6. Hence, with regard to this study, parameters $b$ and $R_c$ can be simultaneously determined in the studied area, from the same test.

**IV. DATA PROCESSING**

**A. Calculation of impedance**

The principle involves considering characterization of the material as the study of a linear system that does not vary in time.  It may be assumed that, when the system is excited by a flux stress, its response is a change in surface temperature. The system is represented in the following form.

The analogue signal of temperature $T(t)$ observed at a constant rate may be represented by the series $\{T(1); T(2); T(3); \ldots; T(p)\}$.

It is assumed that this signal is the response to an excitation of flux $F(t)$. $F(t)$ is also observed at discrete times $\{F(1); F(2); F(3); \ldots; F(p)\}$.

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*FIG. 5. Modulus sensitivity to parameters as a function of frequency.*

*FIG. 6. Evolution of the sensitivity to the resistance $R_c$ depending on the sensitivity to the thermal effusivity $b$.***

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The samples of the two signals may be linked by the following linear relation:

\[
a_0 T(k) + a_1 T(k-1) + \cdots + a_p T(k-p) = b_0 F(k) + b_1 F(k-1) + \cdots + b_q F(k-q). \tag{58}
\]

This equation constitutes a discrete linear model of order \((p, q)\). It expresses the fact that the value of function \(T\) at a given instant depends on the past and present excitation values \(F\) and on the previous values of \(T\). Normalizing with \(a_0\) gives the commonly used expression,

\[
T(k) = -\sum_{i=1}^{p} a_i T(k-i) + \sum_{i=0}^{q} b_i F(k-i), \tag{59}
\]

hence \(\theta(z)\) is the \(z\) transform of the sequence \(T(k)\) and \(\phi(z)\) that of \(F(k)\):

\[
\theta(z) = \sum_{k=-\infty}^{+\infty} T(k)z^{-k}, \tag{60}
\]

\[
\phi(z) = \sum_{k=-\infty}^{+\infty} F(k)z^{-k}. \tag{61}
\]

From the time-dependent equation linking the input and the output signals of the linear system, an equivalent equation may be written connecting the various \(z\) transforms,

\[
\theta(z) + a_1 z^{-1} \theta(z) + \cdots + a_p z^{-p} \theta(z) = b_0 \phi(z) + b_1 z^{-1} \phi(z) + \cdots + b_q z^{-q} \phi(z), \tag{62}
\]

or

\[
\frac{\theta(z)}{\phi(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_q z^{-q}}{1 + a_1 z^{-1} + \cdots + a_p z^{-p}} = Z(z), \tag{63}
\]

where \(Z(z)\) is referred to as the \(z\) transfer function of the discrete-time linear system.

The \(z\) transform is obtained from the relation proposed above. If this transformation is calculated for \(z = e^{j\omega T_e}\), the expression for the discrete Fourier transform is obtained,

\[
\theta(\omega) = \sum_{k=-\infty}^{+\infty} T(kT_e) e^{-j\omega kT_e}, \tag{64}
\]

where \(T_e\) represents the sampling rate.

In this case, the transfer function gives the frequency response \(Z(f)\) of the system. In the present case, \(Z(f)\) is the input impedance of the system.

Parameters \(a_i\) and \(b_i\) are determined by a least-error squares estimation procedure that involves adopting the group that minimizes the difference between each value of the output signal and the value predicted by relation (59).

**B. Determination of thermophysical parameters of system**

The thermophysical parameters of the system are estimated by fitting the theoretical model to the experimental impedance. The theoretical impedance is a non-linear function of the thermophysical parameters and frequency. It is adjusted using an iterative procedure based on minimizing a least-error squares criterion. The Nelder-Mead simplex (direct search) method was used for this purpose.

**C. Excitation signal**

This method has the advantage of not requiring any control of the boundary conditions and being able to exploit random signals. It is perfectly suited to in situ transposition under any disturbance conditions whatsoever. The experiments for the present study were performed in the laboratory and stress was imposed on the system by dissipating heat through a flat resistance. Pseudo-random binary signals were used. This type of signal has the advantage of selecting and exciting a wide spectral band while limiting the amount of energy introduced into the system. This is an important point if one wishes to study wet materials without upsetting the water contents.

**D. Experimental aspects**

The theory developed requires the ability to measure heat flow and temperature in the same plane. For this we used plan fluxmeters with tangential gradients. Their operating principle consists to establish temperature differences across flat thermoelectric junctions, this causing micro-constrictions lines of heat flow through the sensor plane. Different studies have been conducted with this kind of sensors. They produce a positive or negative voltage depending on the orientation of the heat flow, the corresponding measurement is a thermal balance proportional to the sensitive surface of the sensor. Its design is based on the printed circuit technology which reduces its thickness (about 500 \(\mu\)m), the low inertia of the sensor allows response times lower than the second. The type plate thermocouple is integrated into the printed circuit for measuring temperature in the same plane as that of the heat flow. A guard ring having the same structure as the sensing element has been fulfilled so as to extend the surface of the measuring instrument, which eliminates the edge effects at the sensitive surface (heat flux lines distortion). These provisions are necessary to ensure the unidirectionality of heat flow through the fluxmeter and the independence of its sensitivity towards the thermophysical characteristics of the material on which it is installed. The sensors used have a sensitive area of 0.13 m \(\times\) 0.13 m and are equipped with a guard ring leading to the sensor dimensions 0.25 m \(\times\) 0.25 m. The measurements are made using a multimeter analog scanner with high resolution multichannel voltage (0.1 \(\mu\)V-VDC). The integrated cards are equipped with two lines of commands and they ensure the stresses on the system from the signal generated by the computer.

Figure 3 shows the experimental procedure of thermal characterization of a material sample. The objective is to apply a thermal stress on the surface by means of electrical resistance. To ensure that the majority of the energy dissipated through which the fluxmeter and reaches the material, a thermal insulation is placed above the resistance. The quantities of flow and temperature are recorded by the fluxmeter, under the resistance and the sample surface. The unidirectionality of transfer is ensured by an insulating belt surrounding the
material studied, the sensor is equipped with a guard ring and its surface is less sensitive than that of the sample. Edge effects are avoided. The weight placed on the device keeps all in a stable position and reduces contact resistance. The thermal stress imposed by the heater is based on a pseudo-random binary signal generated by the computer, clock rate of 30 s. Data acquisition is done every 2 s for a total of 750 points.

E. Model and processing validation stage

1. Preliminary characterization of the sample

Validation was carried out on a PVC sample. It is a well-known composite material, homogeneous and stable at ambient temperature (about 20 °C).

A study of the heat transfer through the sample made it possible to determine a reference thermal conductivity value. This study was carried out in the usual way with a conduction test bench. Analysis of a storage process in the same setup allowed us to measure the thermal capacity. On the basis of these preliminary measurements (determination of both thermal conductivity and specific heat), it was possible to deduce a reference thermal effusivity value with the relation

\[ b = \sqrt{\lambda \rho c}, \]  

where \( \lambda \) is the thermal conductivity, and \( \rho c \) is the volumetric specific heat. Results are illustrated in Table I.

2. Experimental results

From the experimental device (Fig. 3), we obtain simultaneously the temperature and the flux of the PVC sample. Figures 7 and 8 represent the flux density passing through the

\[ \text{FIG. 7. Flux density as a function of time.} \]

\[ \text{FIG. 8. Temperature as a function of time.} \]

We can see that the difference in temperature not exceed ∼3 °C. Then, as explained previously, the \( z \) transfer function is determined from these dependent changes. The impedance obtained in this case is represented in Figure 9 in the form of a graph showing the phases plotted as a function of the moduli.

Thermophysical parameters \( (b, R_c, C_f) \) of the theoretical model are identified by minimizing a deviation function between the experimental points and those derived from the theoretical curve considering solving the quadratic minimization problem (least-squares method). As the impedance is a non-linear function of the parameters, the approximation is obtained with an iterative algorithm.

The good agreement between the measured impedance (from the temperature and flux using \( z \) transfer function – (63)) and the optimized impedance (from the theoretical impedance – (56)), by the Nelder-Mead method (simplex) applied to the model led to the following results (Table II).

It is obvious that the result is strongly dependant of the thermal contact resistance between the sensor and the sample, and it influences on the heat exchanges. From different tests, we noticed that this contact resistance is not constant for each

\[ \text{TABLE II. Comparison of the results obtained on the PVC sample.} \]

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Laboratory</th>
<th>Experimental</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thermal conductivity (W m(^{-1})K(^{-1}))</td>
<td>0.214</td>
<td>0.214</td>
</tr>
<tr>
<td>Volumetric heat capacity × 10(^6) (J m(^{-3})K(^{-1}))</td>
<td>1.32</td>
<td>1.32</td>
</tr>
<tr>
<td>Contact resistance (K m(^2)W(^{-1}))</td>
<td>0.0058</td>
<td>0.0058</td>
</tr>
<tr>
<td>Fluxmeter capacity (J m(^{-2})K(^{-1}))</td>
<td>600</td>
<td>600</td>
</tr>
<tr>
<td>Thermal effusivity (J m(^{-2})s(^{-1/2})K(^{-1}))</td>
<td>531</td>
<td>547</td>
</tr>
<tr>
<td>Deviation 100 × ( \frac{b_{\text{exp}} - b_{\text{labo}}}{b_{\text{labo}}} ) (%)</td>
<td>3.1</td>
<td>3.1</td>
</tr>
</tbody>
</table>
In the described method, the huge advantage is that we can identify this resistance and discuss the probity of the obtained parameters from the optimization sequence. Usually, in different methods, the contact resistance is neglected, or its value is fixed at a nominal value in the model.

V. DISCUSSION AND CONCLUSIONS

The study shows that it is possible to use the formalism of thermal waves to thermally characterize a material. In the experimental configuration considered in this work, a flux and temperature sensor is placed on the surface of a homogeneous material. This method enables in a simple way to write the thermal impedance in the frequency domain. We show that it is sensitive to the thermal effusivity of the material and to the contact resistance sensor / material and these two parameters can be identified simultaneously. This estimation, based on an inverse method, is performed using a very common spreadsheet: a simplex algorithm.

The method was validated on a PVC sample, a well-known thermal properties material with low effusivity, which is a typical value for building materials.

In further works, the method could be applied on different materials, with a very high contact resistance (for example: concrete), or on wet materials because of the advantage to limit the amount of energy introduced into the system.

The formalism of thermal waves could also be extended to materials of finite sizes by introducing a factor of reflection.

17E. Antczak, D. Defer, M. Elaomi, A. Chauchois, and B. Duthoit, NDT & E Int. 40(6), 428 (2007).