

Eigensolution of the Coulomb Hamiltonian via supersymmetry

A. Valance, T. J. Morgan, and H. Bergeron

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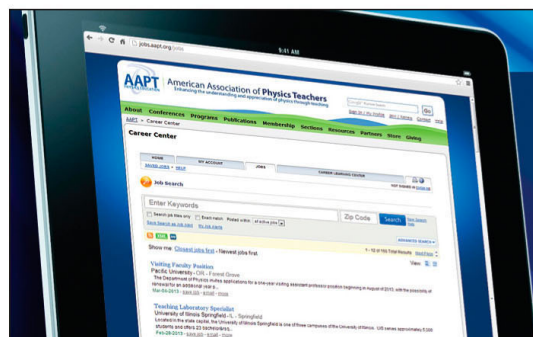
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length, and time on each side of Eq. (B2), it is found that $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$, and $\gamma = -2$; thus we can write

$$f_m = F(m)(EI/\mu l^4)^{1/2}. \quad (\text{B3})$$

Equation (B3) is valid for any boundary conditions of the beam, the different conditions influencing the function $F(m)$. One of the possible set of boundary conditions is that appropriate to the cantilever. It is seen that Eq. (B3) is consistent with the result obtained by the more detailed treatment in Appendix A.

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Eigensolution of the Coulomb Hamiltonian via supersymmetry

A. Valance^{a)} and T. J. Morgan

Department of Physics, Wesleyan University, Middletown, Connecticut 06457

H. Bergeron

LURE (CNRS, CEA, MEN), Bât. 209D, U. P. S., 91405 Orsay Cedex, France

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The eigenvalues and eigenfunctions of the nonrelativistic hydrogen atom are obtained using supersymmetric quantum mechanics arguments and basic properties of the Schrödinger equation.

I. INTRODUCTION

The particular case of the hydrogen atom is treated in most quantum mechanics textbooks by solving the Schrödinger equation. The purpose of this article is to present a different approach to obtain the properties of the nonrelativistic hydrogen atom spectrum, using some supersymmetry quantum mechanics (SSQM) arguments and two simple mathematical properties of the radial Schrödinger equation. The method is appealing, since it avoids the algebraic details of the usual mathematical polynomial solution, while demonstrating in an elegant manner the main results of the hydrogen atom.

An exhaustive and pedagogical presentation of supersymmetry in quantum mechanics has been given by Haymaker and Rau.¹ Also, a pedagogical and focusing study of the exactly solvable potentials using SSQM has recently been given by Dutt *et al.*² Many other studies on SSQM in different fields of physics have been published and are cited in Refs. 1 and 2.

The goal of this article is specific: to obtain the characteristics of the hydrogen atom via an alternative view that

utilizes ideas involved in SSQM. The presentation of this article is also intended to emphasize the pedagogical aspects of the procedure. The article is organized as follows. In the first part of this paper, we demonstrate some important results of SSQM via a method that involves elementary quantum mechanics only. In the second part, we recall some useful results about the radial Schrödinger equation for the hydrogen atom. In the third part, we derive, using SSQM arguments, that there is a supersymmetric pattern for the hydrogen spectrum when it is viewed as several spectra, each one corresponding to the different values of the angular momentum quantum number l . Finally, we give a method to determine all the eigenfunctions via SSQM, starting from the ground state in each l subspace, which is easily obtained using simple mathematical arguments.

II. SUPERSYMMETRY IN QUANTUM MECHANICS

First, we establish the well-known relations for supersymmetric Hamiltonians³ in terms of four propositions.

Consider an operator A and its Hermitian adjoint A^+ . We build the following H operator

$$H = AA^+, \quad (1)$$

which is Hermitian.

Let $|\Psi\rangle$ be a normalized eigenket of H associated with the eigenvalue E , then

$$H|\Psi\rangle = AA^+|\Psi\rangle = E|\Psi\rangle. \quad (2)$$

Also, we observe that the eigenspectrum is positive,

$$\langle\Psi|AA^+|\Psi\rangle = \|A|\Psi\rangle\|^2 = E. \quad (3)$$

Equation (2) leads, on multiplication from the left by A^+ , to

$$A^+A(A^+|\Psi\rangle) = E(A^+|\Psi\rangle). \quad (4)$$

Therefore, any eigenvalue E associated with the eigenket $|\Psi\rangle$ of the operator $H = AA^+$ is also an eigenvalue of the operator $H_S = A^+A$, except when

$$A^+|\Psi\rangle = 0 \text{ (proposition 1)}. \quad (5)$$

Now we assume that the spectrum of H is not degenerate, and denote the normalized eigenkets of H_S by $|\Psi_S\rangle$.

Starting from

$$H_S|\Psi_S\rangle = E|\Psi_S\rangle, \quad (6)$$

and

$$\langle\Psi_S|\Psi_S\rangle = 1, \quad (7)$$

we obtain

$$|\Psi_S\rangle = \alpha A^+|\Psi\rangle, \quad (8)$$

where α is a constant, the value of which is deduced from (2) and (7): $\alpha^2 E = 1$. Therefore, we find the following relations linking the eigenfunctions of $|\Psi_S\rangle$ and $|\Psi\rangle$,

$$\begin{aligned} |\Psi_S\rangle &= E^{-1/2} A^+|\Psi\rangle, \\ |\Psi\rangle &= E^{-1/2} A|\Psi_S\rangle \text{ (proposition 2)}. \end{aligned} \quad (9)$$

Now we will specify the operator A . Let $v(x)$ be a Hermitian operator of a one-dimensional system, and p the momentum operator in atomic units $-id/dx$ in the $|x\rangle$ representation. If

$$A = (ip + v)/\sqrt{2}, \quad (10)$$

then

$$A^+ = (-ip + v)/\sqrt{2}, \quad (11)$$

and

$$AA^+ = (p^2 + v^2 + v')/2, \quad (12)$$

$$A^+A = (p^2 + v^2 - v')/2, \quad (13)$$

where we have used the commutator result

$$[v(x), p] = iv', \quad (14)$$

with the prime denoting derivative with respect to x . Equations (11) and (12) can be rewritten to display the well-known supersymmetric Hamiltonians H and H_S as,³

$$H = AA^+ = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} v^2 + \frac{1}{2} v', \quad (15)$$

$$H_S = A^+A = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} v^2 - \frac{1}{2} v'. \quad (16)$$

Also, it will be useful to establish the following relation obtained by subtracting (13) from (12)

$$[A, A^+] = v'. \quad (17)$$

Now we want to relate any one-dimensional Hamiltonian H_1 to H . This Hamiltonian, associated with the potential $V_1(x)$, is written in a. u. as

$$H_1 = p^2/2 + V_1(x). \quad (18)$$

As H is positive, we must renormalize to $H_1 - E_{1,1}$ ($E_{1,1}$ being the ground-state eigenvalue of H_1). Then using (12) and (18), H_1 can be factorized as

$$H_1 = AA^+ + E_{1,1}, \quad (19)$$

when $v(x)$ is the solution of the differential equation

$$v^2 + v' = 2(V_1 - E_{1,1}). \quad (20)$$

The right side of (20) can be deduced from the Schrödinger equation for the ground state:

$$2(V_1 - E_{1,1}) = \Psi''_{1,1}/\Psi_{1,1}, \quad (21)$$

with $\Psi_{1,1}(x)$ being the ground-state eigenfunction of H_1 . Therefore, we easily verify that

$$v = \Psi'_{1,1}/\Psi_{1,1} = (\ln \Psi_{1,1})' \quad (22)$$

corresponds to a particular solution of (20). Then A may be written as

$$A = \left(\frac{d}{dx} + \frac{\Psi'_{1,1}}{\Psi_{1,1}} \right) / \sqrt{2} \text{ (proposition 3)}. \quad (23)$$

Now we consider another Hamiltonian H_2 characterized by the potential $V_2(x)$,

$$H_2 = p^2/2 + V_2. \quad (24)$$

H_2 will be a supersymmetric partner of H_1 if it can be expressed as

$$H_2 = A^+A + E_{1,1}. \quad (25)$$

In this case, we find, using Eqs. (18), (24), (19), (25), and (17),

$$H_1 - H_2 = V_1 - V_2 = [A, A^+] = v'. \quad (26)$$

Also, using the expression for v given in Eq. (22), we have

$$V_2 = V_1 - (\ln \Psi_{1,1})''. \quad (27)$$

Therefore, we obtain the following property corresponding to the particular solution v of the differential Eq. (20),

$$A^+|\Psi_{1,1}\rangle = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + \frac{\Psi'_{1,1}}{\Psi_{1,1}} \right) |\Psi_{1,1}\rangle = 0. \quad (28)$$

Finally, we may state that the Hamiltonians H_1 and H_2 , whose potentials are V_1 and V_2 , respectively, are supersymmetric partners when

$$V_2 - V_1 = -(\ln \Psi_{1,1})''$$

where $|\Psi_{1,1}\rangle$ is the ground eigenstate of H_1 , and H_1 and H_2 have the same spectrum except the ground state of H_1 is missing in H_2 (proposition 4).

This procedure may be iterated to generate a Hamiltonian hierarchy with the property that the N th member of the hierarchy has the same eigenvalue spectrum as the first member H_1 , except for missing the $(N-1)$ eigenvalues of H_1 .⁴ In particular, the N th excited state of H_1 is degenerate with the ground state of H_{N+1} , and the corresponding wavefunctions are simply related.

III. SOME ELEMENTARY RESULTS ABOUT THE COULOMB POTENTIAL

Let us recall some useful results about the Coulomb potential and then we shall point out the analogy with SSQM Hamiltonians in Sec. IV.

The radial equation is usually written in a. u. as

$$\left(-\frac{1}{2} \frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{l(l+1)}{2r^2} - \frac{1}{r} \right) R_{n,l}(r) = E_{n,l} R_{n,l}(r). \quad (29)$$

We can simplify the differential operator to be studied by defining

$$R_{n,l}(r) = (1/r) u_{n,l}(r). \quad (30)$$

Multiplying both sides of (29) by r , we obtain for $u_{n,l}(r)$ the following differential equation:

$$\left(-\frac{1}{2} \frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} - \frac{1}{r} \right) u_{n,l}(r) = E_{n,l} u_{n,l}(r). \quad (31)$$

This equation is analogous to the one-dimensional problem of a particle moving in an effective potential $V_e(r)$ such that

$$V_e = - (1/r) + [l(l+1)/2r^2]. \quad (32)$$

Nevertheless, we must not lose sight of the fact that the variable r can take on only nonnegative real values. In fact, the whole potential is given by Eq. (32) for $r \geq 0$ and

$$V_e = \infty \text{ for } r < 0. \quad (33)$$

We recall that for a one-dimensional potential, if any bound states exist, they are not degenerate.⁵ To demonstrate the above result, one needs only the definition of a bound state, i.e., the wavefunction must go to zero as $|r| \rightarrow \infty$. We note that we can demonstrate the same result for the central potential $V(r)$ defined by both Eqs. (32) and (33). Then it will be quite justified to apply proposition 2 to the one-dimensional radial Schrödinger equation, since the spectrum is not degenerate.

Therefore, denoting h_l as the one-dimensional Hamiltonian operator

$$h_l = -\frac{1}{2} \frac{d^2}{dr^2} + \frac{l(l+1)}{2r^2} - \frac{1}{r}, \quad (34)$$

and the corresponding eigenequation as

$$h_l |i, l\rangle = E_{i,l} |i, l\rangle, \quad (35)$$

where $i = \{1, 2, \dots\}$ labels the eigenstates for each l -given Hamiltonian, and $l = \{0, 1, 2, \dots\}$. The behavior of the eigenfunctions of h_l are

$$u_{i,l}(r) \underset{r \rightarrow 0}{\sim} r^{l+1}, \quad (36)$$

$$u_{i,l}(r) \underset{r \rightarrow \infty}{\sim} \exp(-\sqrt{2E_{i,l}} r). \quad (37)$$

Now it is easy to obtain the ground-state wavefunctions for each h_l , since they must be nodeless,

$$u_{1,l}(r) = N_{1,l} r^{l+1} \exp(\sqrt{-2E_{1,l}} r). \quad (38)$$

Before using SSQM arguments to generate the hydrogen atom spectrum, we need one more result deduced from the elementary study of eigenequation (35) when $l=0$. Deducing $u_{1,0}(r)$ from Eq. (38) and then solving Eq. (31), we obtain

$$E_{1,0} = -\frac{1}{2} (\text{a. u.}). \quad (39)$$

We emphasize here that only Eqs. (38) and (39), which are obtained via a simple analysis of the mathematical properties of the Schrödinger equation, will be used in Sec. IV to solve the Coulombic potential via SSQM relations.

IV. SSQM AND THE COULOMB POTENTIAL

Our study will be different than the usual treatment of the Coulomb-potential energy degeneracy. In this section we utilize the SSQM propositions presented in Sec. II and the Coulomb potential results of Eqs. (38) and (39) presented in Sec. III.

First, we wish to prove that the different h_l operators produce a supersymmetric pattern. Second, we wish to generate the exact values of the energy spectrum, i.e., $E_{n,l} = E_n = -1/2n^2$. Finally, we give a method to generate the wavefunctions of all states using only SSQM arguments and the ground-state wavefunctions given by Eq. (38).

A. Supersymmetric partners

We are going to show that two adjacent "Hamiltonians" h_l and h_{l+1} are supersymmetric partners. Starting with a given h_l , the supersymmetric partner h_s is obtained via Eqs. (26) and (27):

$$h_s - h_l = V_s - V_l = -\frac{d^2}{dr^2} \ln[u_{1,l}(r)], \quad (40)$$

where $v_l = -1/r + l(l+1)/2r^2$ is the effective potential and $u_{1,l}$ is given by (38). Then,

$$\begin{aligned} V_s &= \frac{-1}{r} + \frac{l(l+1)}{2r^2} + \frac{l+1}{r^2} \\ &= -\frac{1}{r} + \frac{1}{2r^2} [(l+1)(l+2)], \end{aligned} \quad (41)$$

showing that V_s is equal to V_{l+1} . Therefore, the h_s supersymmetric partner of h_l is h_{l+1} .

Since the ground state of h_l is missing in h_{l+1} (proposition 4), by repeating the above procedure for l varying from 0 to ∞ , we obtain the energy spectra displayed in Fig. 1. The formula linking adjacent energy levels of supersymmetric partners is

$$E_{n,l} = E_{n-1,l+1} = E_{n-2,l+2} = \dots = E_{1,l+n-1} \quad (42)$$

with $n = \{1, 2, \dots\}$, $l = \{0, 1, 2, \dots\}$.

Therefore, we have found via SSQM the accidental energy degeneracy of the Coulomb potential hydrogen atom spectrum.

B. Spectra values

We will now find all the energy level values of the spectra using SSQM arguments and the ground-state energy level, i.e., $-\frac{1}{2}$ (a. u.).

Indexing both Eqs. (19) and (25) as follows

$$H_1 = A_1 A_1^+ + E_{1,1},$$

$$H_2 = A_1^+ A_1 + E_{1,1},$$

and noting that H_2 can also be written as

$$H_2 = A_2 A_2^+ + E_{1,2},$$

where A_2 and $E_{1,2}$ are defined according to the usual rela-

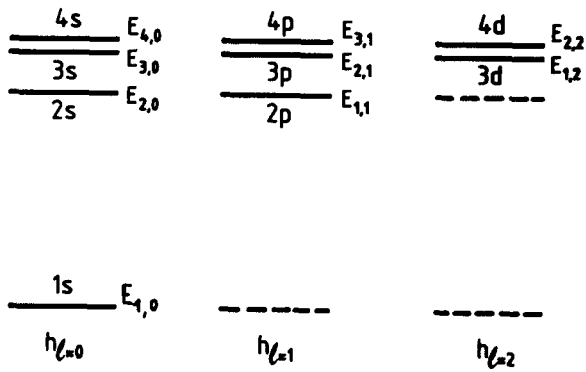


Fig. 1. Energy levels for the hydrogen atom using both the usual notation (nl) and $E_{n,l}$ notation. The correspondence between the different given Hamiltonians h_i appears clearly via the relation (42), $E_{n,l} = E_{n-1,l+1} = \dots = E_{1,l+n-1}$.

tions used in this article, we may write two different general expressions for H_{l+1}

$$h_{l+1} = A_{l+1} A_{l+1}^+ + E_{1,l+1}, \quad (43)$$

and

$$h_{l+1} = A_l^+ A_l + E_{1,l}. \quad (44)$$

Moreover, using Eqs. (12) and (13), we obtain

$$2A_{l+1} A_{l+1}^+ = p^2 + v_{l+1}^2 + v'_{l+1}, \quad (45)$$

$$2A_l^+ A_l = p^2 + v_l^2 - v'_l. \quad (46)$$

Using the last four equations, we obtain

$$v_l^2 - v'_l + 2E_{1,l} = v_{l+1}^2 + v'_{l+1} + 2E_{1,l+1}. \quad (47)$$

Now we can calculate v_j via Eqs. (22) and (38)

$$v_j = (j+1)/r - (-2E_{1,j})^{1/2}. \quad (48)$$

Putting Eq. (48) in Eq. (47) with $j = l$ and $l+1$, we get (after some algebra)

$$[(l+1)(-2E_{1,l})^{1/2} - (l+2)(-2E_{1,l+1})^{1/2}]/r = 0. \quad (49)$$

Since this equation must be true for all positive r values, we obtain the relation linking $E_{1,l}$ and $E_{1,l+1}$, i.e., the formula between the ground-state energy of adjacent spectrum

$$E_{1,l}/E_{1,l+1} = [(l+2)/(l+1)]^2. \quad (50)$$

Using this relation, formula (42), and the value of $E_{1,0}$ [Eq. (39)], we obtain the well-known energy-level formula of the hydrogen atom

$$E_{1,l} = E_n = -1/2n^2. \quad (51)$$

C. Recursion relation for eigenkets

We now show that it is possible to get the eigenfunctions of the hydrogen atom starting with SSQM relations and using only the ground-state wavefunction of each h_i Hamiltonian given by Eq. (38). We want to find a general formula giving any unknown $|n,l\rangle$ ket. By applying Eq. (9) for the situation where $E_{1,l}$ is the energy renormalizing constant, we obtain

$$|n,l\rangle = (E_{n,l} - E_{1,l})^{-1/2} A_l |n-1, l+1\rangle. \quad (52a)$$

By iterating Eq. (52a) until we reach a ground-state ket, we obtain

$$|n-1, l+1\rangle = (E_{n-1, l+1} - E_{1, l+1})^{-1/2} \times A_{l+1} |n-2, l+2\rangle, \quad (52b)$$

$$|2, l+n-2\rangle = (E_{2, l+n-2} - E_{1, l+n-2})^{-1/2} \times A_{l+n-2} |1, l+n-1\rangle. \quad (52c)$$

By combining all $(n-1)$ equations above, we obtain

$$|n,l\rangle = [(E_{n,l} - E_{1,l})(E_{n-1, l+1} - E_{1, l+1}) \dots \times (E_{2, l+n-2} - E_{1, l+n-2})]^{-1/2} \times (A_l A_{l+1} \dots A_{l+n-2}) |1, l+n-1\rangle. \quad (53)$$

Now, noting the relation between the energies: $E_{n,l} = E_{n',l'}$ when $n+l = n'+l'$, and taking the $h_{l=0}$ energy spectrum as reference, we obtain

$$|n,l\rangle = [(E_{n+l,0} - E_{1+l,0})(E_{n+l,0} - E_{2+l,0}) \dots \times (E_{n+l,0} - E_{n-1+l,0})]^{-1/2} \times A_l A_{l+1} \dots A_{l+n-2} |1, l+n-1\rangle. \quad (54)$$

This expression is a general formula that generates the eigenkets of the hydrogen atom knowing only the ground state of each h_l and the A_l operators. The utilization of this formula is illustrated in Fig. 2. Using Eqs. (10) and (48), we obtain the expression of the differential operator A_l ,

$$\sqrt{2} A_l = \left(\frac{d}{dr} + \frac{u'_{l,l}}{u_{l,l}} \right) = \left(\frac{d}{dr} + \frac{l+1}{r} - (-2E_{1,l})^{1/2} \right). \quad (55)$$

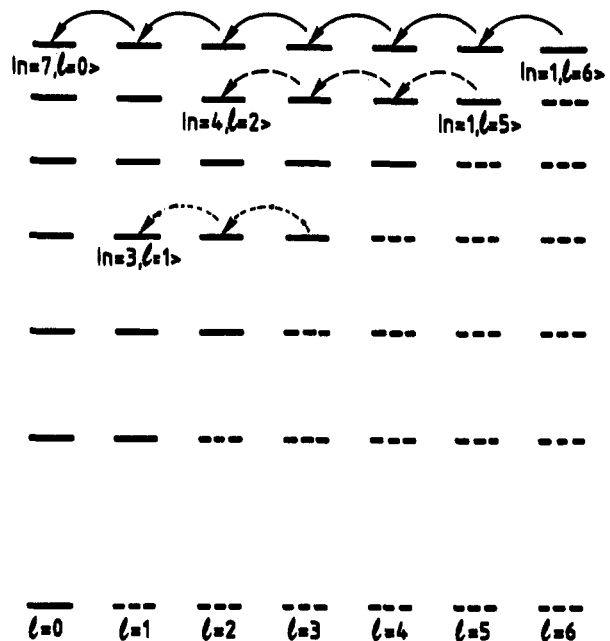


Fig. 2. How to use Eq. (54). The known wavefunctions are the ground state for each l spectrum. Example 1: To obtain $|n=7, l=0\rangle$, follow the arrows; 2: To obtain $|n=4, l=2\rangle$, follow the arrows; 3: To obtain $|n=3, l=1\rangle$, follow the arrows. We note that the iteration number we need is $n-1$.

It is now instructive to illustrate this method by calculating explicitly the first excited-state wavefunction $u_{2,0}(r)$ of the $l=0$ energy spectrum starting from the ground-state wavefunction $u_{1,1}(r)$ of the $l=1$ energy spectrum. Putting $n=2$ and $l=0$ in Eq. (52a), we obtain

$$|2,0\rangle = [E_{2,0} - E_{1,0}]^{-1/2} A_0 |1,1\rangle,$$

where the number of iterations is $n-1=1$. All the terms of the right member of the above equation are known [$E_{1,0} = -\frac{1}{2}$; $E_{2,0} = -\frac{1}{8}$; $\langle r|1,1\rangle = N_{1,1} r^2 \exp(-r/2)$; $N_{1,1} = 1/2\sqrt{6}$ and $A_0 = (d/dr + 1/r - 1)/\sqrt{2}$]. By doing simple algebra, we obtain

$$\langle r|2,0\rangle = r(2-r)\exp(-r/2)/2\sqrt{2},$$

which is the usual result for the $2s$ normalized wavefunction.

V. CONCLUSION

We have studied the Coulomb potential via a method grounded on both supersymmetry in quantum mechanics and elementary properties of the Schrödinger equation. Some useful properties of the supersymmetric Hamiltonian are demonstrated and summarized in four propositions. Two basic results of the Schrödinger equation are obtained via a mathematical analysis of the radial eigen-equation, without solving it.

The hydrogen atom spectrum presents a supersymmetric pattern when it is viewed as several spectra each one corresponding to the different angular momentum values. The accidental degeneracy of the spectrum is then ex-

plained using SSQM arguments. [Of course, we cannot claim that this accidental degeneracy is due to a supersymmetric property, as we know that there exists an invariant (the Runge-Lenz vector) that is responsible for this degeneracy.] Also, the eigenvalues and eigenfunctions of the hydrogen atom spectrum are obtained via this approach.

In short, the method presented here using SSQM arguments and elementary Schrödinger equation properties yields the most important results of the nonrelativistic hydrogen atom. It can be used as an instructive exercise in an undergraduate quantum mechanics course.

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^aPermanent address: LURE (CNRS, CEA, MEN).

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An algebraic approach for solving mechanical problems

Carlos Farina de Souza and Miriam Mendes Gandelman

Instituto de Física, Universidade Federal do Rio de Janeiro, Ilha do Fundão-Cidade Universitária, CEP: 21.944, Rio de Janeiro, Brazil

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An algebraic approach is used to obtain the time evolution of physical systems based on the Poisson bracket formulation of classical mechanics. This method is applied in many examples, including three-dimensional problems.

I. INTRODUCTION

The most obvious approach to a mechanical problem, given a force law, is to solve Newton's differential equation of motion. However, after Newton's work many other approaches have been developed with increasing mathematical sophistication, for instance, the Lagrangian formulation, which stands out for its aesthetic elegance, the Hamiltonian formulation, which provides a natural generalization for many other branches of physics, etc. These two formulations do not even use the concept of a vectorial force as occurs in Newtonian mechanics. Nevertheless, in

all these formalisms we must solve differential equations of motion. The use of differential equations is not restricted to classical mechanics, but is common to all of classical physics. For example, classical electromagnetic fields are described by the Maxwell differential equations,¹ all the wave mechanical phenomena are governed by differential equations, etc.² Even in quantum mechanics, where many different approaches have been developed since its birth in the beginning of this century, the Schrödinger formulation based on a differential equation is one of the most commonly used.³

However, it is sometimes more convenient to develop a