The strange polarization of the classical atom

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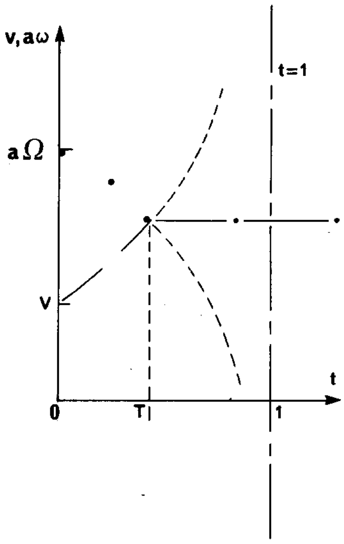

Fig. 5. As for Fig. 4. Graphs plotted against t.

Then Eqs. (16) and (18) become, via (23),

$$v = 1/(1+t), \tag{24}$$

$$\omega = \Omega + 1 - 1/(1+t). \tag{25}$$

Slipping ceases when $v = \omega$, that is when

$$t = T = (1-\Omega)/(1+\Omega). \tag{26}$$

No sketches are given for Eqs. (24) and (25), for they are similar to those for underspin plotted against ψ in Fig. 4. This is the case since t and ψ are both zero together and t increases with ψ while the appearance of the logarithm in Eq. (23) leads to algebraic (rather than exponential) decay or growth of the appropriate terms without affecting the upward or downward concavity of the curves. Inspection of d^2v/dt^2 and $d^2\omega/dt^2$ confirms this.

The overspin counterparts of Eqs. (24) and (25) are

$$v = 1/(1-t), \tag{27}$$

$$\omega = \Omega + 1 - 1/(1-t). \tag{28}$$

Sketches of these results are shown in Fig. 5. Each curve has $t = 1$ as an asymptote.

For overspin it is found that

$$T = (\Omega - 1)/(\Omega + 1). \tag{29}$$

The case where C is a cycloid has also been investigated and curves were obtained with concavity similar to those in Fig. 5, when v and $a\omega$ were plotted against t. Of course, Fig. 4 is valid for the cycloid—up to the stage where more than one point of contact occurs.

III. CONCLUSIONS

Rolling and slipping motions on a rough plane of the kind considered in this article are also discussed or set as problems in various textbooks on physics.[2–4] The treatment presented in this article may lead to a better understanding of such motions through the graphical illustration of results. In a similar way a clear understanding should be gained from the graphs for similar motion on the inside of a rough cylinder in the absence of gravity. In this case, for a given body B with V and Ω specified, the value of T depends on the geometry of C. However, as illustrated by Eqs. (16) and (18), v and ω depend only on ψ, the angle turned by the tangent $\hat{\mathbf{t}}$ (or $\hat{\mathbf{T}}$) irrespective of the curve C considered. A little thought reveals that this is not inconsistent with the physics of the problem. Of course, the position Ψ at which slipping ceases is also independent of C. These features do not appear when gravity is included.

[1] R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964), p. 12-5.

[2] R. A. Becker, *Introduction to Theoretical Mechanics* (McGraw-Hill, New York, 1954), pp. 209–211, pp. 218–219.

[3] F. W. Constant, *Theoretical Physics* (Addison-Wesley, Reading, MA, 1962), p. 177.

[4] G. R. Fowles, *Analytical Mechanics* (Holt, Rinehart and Winston, New York, 1962), pp. 172–175.

The strange polarization of the classical atom

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A classical hydrogen atom will polarize in a direction perpendicular to a uniform electric field in its orbital plane. While some features of this unexpected behavior can be reconciled with physical reality, it illustrates representative difficulties of classical mechanics applied to atomic physics. The exceedingly unintuitive results have instructional value for students.

I. INTRODUCTION

In the teaching of physics, we normally move from classical mechanics to electromagnetism with little connection. The notion of polarization in nonconductors is taken as a matter of faith and, much later, quantum mechanics is introduced to explain the stability of atoms and sharpness of spectral lines. The following curious problem can serve as a

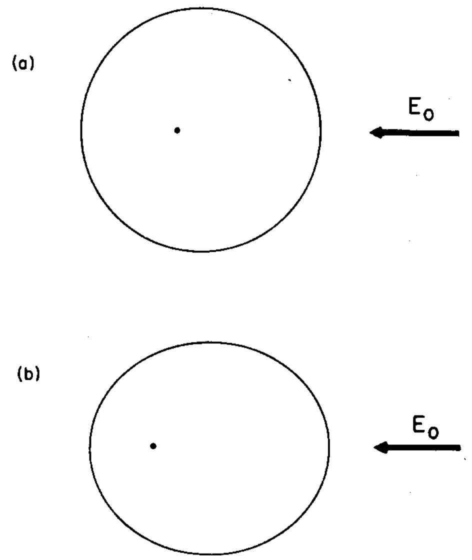

Fig. 1. Intuitive polarization is in the direction of the electric field. If the uniform E_0 is small compared to $1/r_0$, where r_0 is the initial orbit radius, we expect Keplerian orbits for the electron. (a) shows a displacement of the center of the circular orbit. (b) shows an elliptical orbit in the direction of the field.

point of departure from classical physics as well as demonstrate the need for careful analysis of superficially simple problems.

II. THE PROBLEM

Consider a classical hydrogen atom with a fixed nucleus and the electron in an initially circular orbit. Apply a *weak* electric field in the orbital plane. Intuition suggests that the atom will develop an electric dipole moment in the direction of the electric field. From classical mechanics we might expect: (1) a displacement of the center of the circular orbit, as in Fig. 1(a), (2) an evolution into an elliptical orbit extending in the direction of the electric field, as in Fig 1(b), or (3) some combination of these effects.

Wrong! The atom initially polarizes in a direction *perpendicular* to the electric field as illustrated in Fig. 2. Specifically, it polarizes in the direction $\mathbf{L}\times\mathbf{E}$, where $\mathbf{L}$ is the electron's orbital angular momentum, i.e., whether it polarizes up or down depends on the electron's direction of revolution. The elliptical orbits become progressively more eccentric because the electric field applies a torque that reduces the electron's angular momentum. The period-averaged energy does not change, so the major axis remains constant while the minor axis contracts.

III. WHY

Some feel for this strange behavior can be derived from Kepler's second law, which will be approximately correct for a small perturbation. When the electron moves toward the electric field, it accelerates and, at the diametric position in its orbit, it decelerates. If the perturbation is small, the radius vector sweeps out approximately equal area per unit time, so the accelerated part of the orbit moves toward the nucleus and the decelerated part moves away. As the orbit develops more eccentricity, the electron spends more time in the decelerating phase at the greater distance from the nucleus.

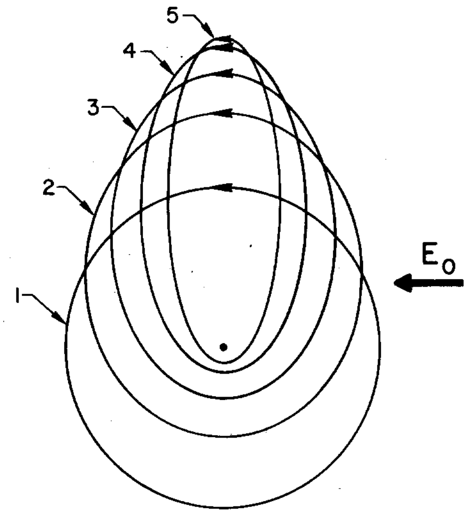

Fig. 2. Actual polarization is perpendicular to the electric field. If $E_0 \ll 1/r_0$, orbits are elliptical with constant major axis and contracting minor axis. The dipole moment limits to $\frac{3}{2}r_0$ in the $\mathbf{L}\times\mathbf{E}$ direction. Numbers show the sequential time progression.

Assuming the nucleus is fixed, the equation of motion is

$$m\ddot{\mathbf{r}} + \mathbf{r}e^2/r^3 = e\mathbf{E}_0,$$

simplified to

$$\ddot{x} = E_0 - (x^2+y^2)^{-3/2}x \qquad (1)$$
$$\ddot{y} = -(x^2+y^2)^{-3/2}y$$

in units where $e = m = 1$, with the electric field in the orbital plane.

A quantitative explanation of the atom's weird behavior is offered by the ancient formalisms of celestial mechanics. The method of variation of parameters[1] can be used to describe the precession of an elliptical orbit in the presence

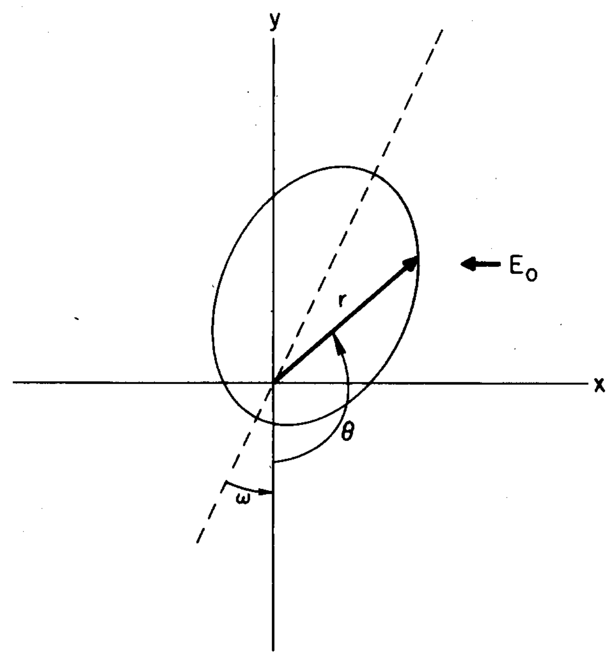

Fig. 3. Electron in an approximately elliptical orbit subjected to an E_0 disturbing field. ω is the argument of the perihelion. $(\omega + \theta)$ is the true anomaly.

of a disturbing force. The equation of motion for the argument of the perihelion (perinuc?), as shown in Fig. 3, is

$$\dot{\omega} = \frac{E_0 L}{q} \frac{q[\cos\theta\cos(\theta+\omega)\sin(\theta+\omega) - \sin\theta\cos^2(\theta+\omega)] - 2\cos\theta\sin(\theta+\omega) + \sin\theta\cos(\theta+\omega)}{q\cos(\theta+\omega) - 1} \tag{2}$$

where $q = \sqrt{1 + 2EL^2}$, E is the total energy, and L is the orbital angular momentum. To find the effective precession rate we must take the time average of $\dot{\omega}(\theta+\omega)$ around an orbital period. If $E_0 \ll 1/r$, then $r^2\dot{\theta}$ is nearly constant, so

$$\begin{aligned}\bar{\omega}(\omega) &= \frac{1}{S}\int_0^{2\pi} \dot{\omega}(\theta,\omega) r^2\, d\theta \\ &= \frac{1}{S}\int_0^{2\pi} \frac{L^4 \dot{\omega}(\theta,\omega) d\theta}{[q\cos(\theta+\omega) - 1]^2} \\ &= \frac{1}{S}\int_0^{2\pi} I(\theta,\omega) d\theta,\end{aligned}$$

where S is the area of the ellipse $= \pi L/2\sqrt{-2E^3} = \pi ab$, where a and b are the major and minor semiaxes. This is a tedious integration, but the strange features of the orbital precession can be extracted without herculean feats of algebraic manipulation.

Assertion: The orbit has two stable orientations, $\omega = 0$ for counterclockwise revolution and $\omega = \pi$ for clockwise revolution.

Proof: The integrand for $\omega = 0$ is

$$I(\theta,0) = \frac{E_0 L^5}{q} \frac{\cos\theta\sin\theta}{(1 - q\cos\theta)^3},$$

which is manifestly odd, i.e., $I(\theta,0) = -I(-\theta,0)$. So

$$\bar{\omega}(0) = \int_0^{2\pi} I(\theta,0) d\theta = 0.$$

Similarly, for $\omega = \pi$

$$I(\theta,\pi) = -\frac{E_0 L^5}{q} \frac{\cos\theta\sin\theta}{(1 + q\cos\theta)^3},$$

so $I(\theta,\pi) = -I(-\theta,\pi)$ and $\bar{\omega}(\pi) = 0$. Thus we conclude that either $\omega = 0$ or $\omega = \pi$ may be stable, depending on the behavior of $\bar{\omega}(\omega)$ in the immediate neighborhood.

To determine this behavior, we examine the integrands for small deviations from 0 and π. Consider $\omega = \delta$, where $\delta \ll 1$. We have from Eq. (2),

$$I(\theta,\delta) \cong \frac{E_0 L^5}{q} \times \frac{q\delta^2\sin\theta - q\delta\cos\theta + \delta(1 + \cos^2\theta) + \cos\theta\sin\theta}{(q\delta\sin\theta - q\cos\theta + 1)^3}$$

by setting $\sin\delta = \delta$ and $\cos\delta = 1$. By expanding the denominator and dropping δ^2 and δ^3 terms, we obtain

$$I(\theta,\delta) \cong I(\theta,0) + (E_0 L^5/q) F(\theta,q)\delta,$$

where

$$F(\theta,q) = \frac{1 + \cos^2\theta - q\cos\theta}{(1 - q\cos\theta)^3} - \frac{3q\sin^2\theta\cos\theta}{(1 - q\cos\theta)^4}.$$

But $0 < q < 1$, so

$$\int_0^{2\pi} F(\theta,q) d\theta = 3\pi/(1 - q^2)^{3/2} > 0$$

and

$$\bar{\omega}(\delta) = -\bar{\omega}(-\delta) > 0.$$

Thus for counterclockwise motion, the orbit precesses back to $\omega = 0$ if slightly nudged in either direction.

On the other hand, for $\omega = \pi + \delta$, where $|\delta| \ll 1$, we obtain

$$I(\theta,\pi+\delta) \cong I(\theta,\pi) - (E_0 L^5/q) F(\theta,-q)\delta,$$

but

$$\int_0^{2\pi} F(\theta,-q) d\theta = \int_0^{2\pi} F(\theta+\pi,q) d\theta = \int_0^{2\pi} F(\theta,q) d\theta,$$

so

$$\bar{\omega}(\pi+\delta) = -\bar{\omega}(\pi-\delta) < 0$$

and the $\omega = \pi$ orientation is unstable.

If the direction of revolution is reversed, $\dot{\omega}$ becomes $-\dot{\omega}$ because the argument of the perihelion is defined in the direction of revolution. The right-hand side of Eq. (2) remains the same because both the defined directions of the tangential force and the true anomaly are reversed. Thus the $\omega = 0$ orientation becomes unstable and the $\omega = \pi$ orientation becomes stable for clockwise revolution. *Quod erat demonstrandum.*

The evolution of the orbital angular momentum can be obtained by time averaging the torque. Assuming that the ellipse is in the $\omega = 0$ stable orientation,

$$\bar{\tau} = -\frac{E_0\sqrt{2E^3}}{\pi L} \int_0^{2\pi} r^3 \cos\theta\, d\theta$$

or

$$\dot{L} \cong (3E_0\sqrt{2EL^2} + 1)/4E. \tag{3}$$

This suggests that the angular momentum will eventually

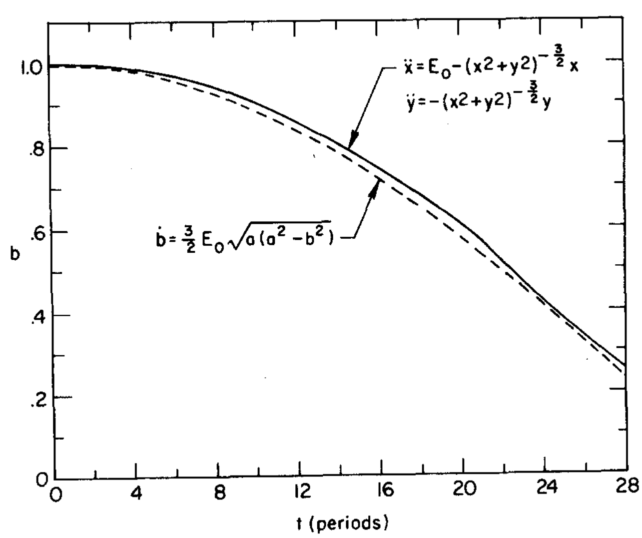

Fig. 4. Comparison of Eq. (4) for the collapse rate of the semiminor axis with numerical integration of Eq. (1). Initial state is circular orbit of unit radius. $E_0 = 1/200$.

go to zero and for $E_0 \ll 1/r_0$, the dipole moment will limit to $\frac{3}{2}r_0$ in a direction perpendicular to $\mathbf{E}_0$. From Eq. (3), the semiminor axis collapses at the rate

$$\dot{b} \cong \tfrac{3}{2} E_0 \sqrt{a(a^2 - b^2)}. \tag{4}$$

Figure 4 compares Eq. (4) with a numerical integration of the equations of motion in Eqs. (1).

The perpendicular polarization can almost be derived from inspection of Eqs. (1). The equations of motion represent a pair of coupled anharmonic oscillators, whose frequencies are dependent on mass ($m = 1$), amplitude, and time-averaged restoring force. The only way we can have the frequencies $\nu_x = \nu_y$ is for the electron to make symmetrical excursions in the $+x$ and $-x$ directions. Assuming $E_0 \ll 1/r_0$, the orbits are approximately ellipses and, because of the symmetrical excursions in the x direction, the major axis must be in the y direction.

IV. PARADOX RESOLVED

This exceedingly odd behavior must relate to familiar physics. It is all too easy to sweep the paradox under the quantum rug and haul out the shopworn lecture on Stark splitting. But deeper analysis reveals a time-averaged behavior resembling dipolarization of the usual sort. However, the analysis involves still more curious and unintuitive behavior.

Equation (3) suggests that the angular momentum will go to zero, not asymptotically, but actually pass through zero. Equations (1) are invariant under time reversal. So the orbits become narrower and narrower until the electron comes to a stop and reverses its direction. It then retraces its path, acquiring angular momentum all the way back to its starting position. It then follows a mirror-symmetric path in the opposite direction. This oddity is most dramatically illustrated by integrating Eqs. (1) on a personal computer and displaying the electron location graphically. The time-averaged dipolarization in the y direction in fact becomes zero and a small displacement of the orbits in the x direction corresponds to the familiar dipolarization that does not average away with time. However, time averaging produces a large electric quadrupole oriented perpendicular to the electric field.

Extension of the problem to three dimensions is straightforward. When the electric field is out of the orbital plane, a steady state of precession can be achieved, which demonstrates some of the more familiar aspects of atomic polarization. However, the tendency of the atom to distort in the perpendicular direction is still quite apparent.

V. CONCLUSIONS

I have yet to find a physicist who could predict this curious behavior at first blush. Some aspects can be reconciled with common sense: Time averaging removes the perpendicular dipolarization and results in a parallel dipolarization like that with which we are familiar. But at the same time, averaging produces an enormous perpendicular quadrupole moment, which appears to be unphysical. To my knowledge, there is no experimental evidence to verify the nonexistence of this quadrupole moment. Perhaps such an experiment should be undertaken.

ACKNOWLEDGMENT

The author gratefully acknowledges the tenacious skepticism of Albert Petschek whose deep physical insight and nearly infallible intuition inspired this article.

[1]See A. E. Roy, *Orbital Motion* (Wiley, New York, 1978), p. 184, or F. R. Moulton, *An Introduction to Celestial Mechanics* (Dover, New York, 1970), p. 404.

Energy in the center of mass

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In the collision of a particle (A) with a system of bonded particles (B–C), only the energy in the center of mass reference frame is available for breaking the bond. This is because some of the initial energy must be used in conserving the momentum of the system. This article describes a simple air-track experiment illustrating this well-known fact. Two carts on a horizontal air track are bonded together by small magnets; a sudden impulse is provided to one of the carts by a falling weight. A feature of the experiment is the direct determination of magnetic bond energy by numerical integration of $F\,dx$. For cases where the energy in the center of mass is sufficient to break the magnetic bond, the ensuing motion of the system can be described in terms of an effective mass.

I. INTRODUCTION

Collisional interactions in the gas phase are of importance in many branches of science, including, for example, astrophysics (stellar atmosphere studies) and physical chemistry research involving the excitation and dissociation of diatomic (and larger) molecules. A feature of all such interactions is that only the kinetic energy in the cen-