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Tunneling ionization of atoms

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We discuss the theory for the ionization of atoms by tunneling due to a strong external static electric field or an intense low frequency laser field.
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I. INTRODUCTION

Strong field physics, that is, the interaction of strong fields with matter, is one of the current topics in atomic, molecular, and optical physics and brings together the newest laser technology and most advanced theoretical approaches. During the last ten years, developments in laser technology have been in the direction of shorter and shorter pulses of higher and higher intensities.[1] Pulses with a duration as short as a few femtoseconds ($1\ \text{fs}=10^{-15}\ \text{s}$) are now routinely produced, and peak intensities around 10^{14}–$10^{15}\ \text{W/cm}^2$ are not unusual.[2] Note that if we combine the atomic units of energy ($\alpha^2 m_e c^2$), time ($a_0/\alpha c$), and length (a_0), where α is the fine structure constant, m_e is the mass of the electron, c is the speed of light, and a_0 is the Bohr radius, we obtain the atomic unit of intensity $I_0=3.51\times 10^{16}\ \text{W/cm}^2$, which is within reach of experiment. The field strength corresponding to an intensity of I_0 is 1 a.u. of field strength $F_0=5.14\times 10^9$ V/cm. The new light sources are typically based on a Ti:sapphire system operating at a wavelength of 800 nm corresponding to a photon energy of ~ 1.5 eV. Due to their high intensities, other wavelengths can be produced by using nonlinear optics. In this way femtosecond pulses covering the spectrum from the infrared to the ultraviolet can be produced. Such laser systems have recently been applied to study the behavior of atoms and molecules under intense field ionization.[3]

Although we will not discuss it here, we mention the phenomenon of rescattering, which is one of the key ideas in strong field physics involving the tunneling ionization of atoms.[4] To model this phenomenon, a three-step process is considered: first the atom is ionized by the field, then the freed electron propagates in the electric field of the laser, and finally it may be driven back to the parent ion. Here it may either scatter to produce high energy electrons, recombine with the emission of a high energy photon, or scatter inelastically, producing a doubly charged ion.[5] In this three-step rescattering model the ionization event is typically described by a tunneling formula. The ionization rate as a function of the phase of the field and the momentum distribution of the liberated electrons serve as inputs for the propagation of the electrons in the field. Hence, a good description of the initial ionization process is needed in order to obtain accurate models for whatever process the propagating electron subsequently undergoes.

The ionization step is fairly simple because it involves the process of tunneling. As we shall return to shortly, the tunneling picture is well-justified for intense low-frequency laser fields. Figure 1 shows the combined potential felt by the electron from an external electric field F along the z direction and from a nucleus of charge Z:

$$V=-\frac{Z}{r}+Fz. \tag{1}$$

(We shall use atomic units $\hbar=e=m=1$ throughout unless otherwise indicated.) The electron may tunnel through the barrier and escape along the negative z axis. The field strength and ionization potential in Fig. 1 are arbitrary but representative values. The parameter κ introduced in the caption of Fig. 1 is defined as $\kappa=\sqrt{2I_p}$, where I_p is the ionization potential.

The tunneling ionization rate in a static, electric field was derived by Landau and Lifshitz for the ground state of the hydrogen atom,[6] and later generalized to any asymptotic Coulomb wave function by Smirnov and Chibisov.[7] There are some misprints in Ref. 7 which make the derivation difficult to follow and results from Ref. 7 should be taken with caution. Perelomov, Popov, and Terent'ev[8] corrected the result from Ref. 7 and generalized it to electromagnetic fields of low frequency by taking the appropriate time average. It was shown[8,9] that a generalized theory, which also covers multiphoton processes, simplifies to the time-average of the tunneling result[7] when the Keldysh parameter, $\gamma=\omega/\omega_t$, is much less than unity. The tunneling frequency ω_t is defined by $\omega_t=F/(2I_p)^{1/2}$, and ω is the optical frequency; ω_t^{-1} is the tunneling time, that is, the time the electron spends under the barrier (see Fig. 1).

The requirement for the validity of the tunneling model for an oscillating field is that the width of the barrier does not change during the time the electron spends traversing it, that is, the electron adiabatically follows the changes in the external field. In terms of these quantities, γ can be expressed as

$$\gamma=\sqrt{2I_p}\frac{\omega}{F}. \tag{2}$$

We see that the requirement that the Keldysh parameter be small is fulfilled when the intensity is high and the frequency is low, precisely the parameter range of present day intense lasers.

In this note we present a self-contained derivation of the tunneling theory for ionization of atoms in laser fields. The derivation is a beautiful combination of physical insight and technical manipulations, and it brings together many different aspects of theoretical physics, including separation of coordinates, asymptotic forms of wave functions, and semiclassical solutions.

II. DERIVING THE IONIZATION RATE

The starting point in the derivation of the tunneling theory for an atom in an external field is the observation that there

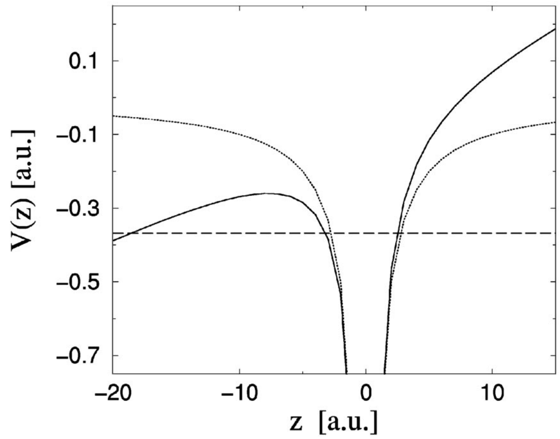

Fig. 1. Full curve: the potential of Eq. (1) along the z axis with $Z=1$ and $F=0.0169$ a.u. This field strength corresponds to 8.69×10^7 V/cm (1 a.u. of field strength is 5.14×10^9 V/cm). Dashed curve: the pure Coulomb potential. The horizontal line corresponds to $\kappa=0.86$, which is equivalent to a ionization potential of 10 eV.

are regions in configuration space where the electron is sufficiently far away from the core that it effectively feels a Coulomb attraction from the nucleus screened by all the other electrons. We will show that the ionization rate can be obtained for systems with one general property: the asymptotic wave function must behave as the asymptotic Coulomb wave function.

The derivation involves the following steps.

(a) Separate Schrödinger's equation in parabolic coordinates in the presence of an external field.
(b) Find the asymptotic solution to the pure Coulomb problem.
(c) Combine these two results to obtain an expression for the ionization rate which only requires the knowledge of the wave function on the outside of the barrier.
(d) Obtain a semiclassical expression for the wave function outside the barrier and in the classically forbidden region under the barrier.
(e) Write the rate in terms of the semiclassical wave function outside the barrier and find an expression that depends only on the normalization of the wave function. This normalization is determined by requiring the semiclassical wave function in a certain region inside the barrier to match the asymptotic Coulomb wave function. The rate is then described completely by parameters characterizing the asymptotic Coulomb wave function.
(f) Perform the appropriate time average to obtain the rate in a low frequency laser field.

In a teaching context we suggest that the lecturer provide an overview of the derivation and leave some parts of the derivation as student exercises. As will become clear, any of the points in Secs. II A and II B may be formulated as problems.

A. Separation in parabolic coordinates: Static field

Parabolic coordinates are defined as[6]

$$\xi=r+z,\quad \eta=r-z,\quad \phi=\arctan(y/x),\tag{3}$$

with $\xi,\eta\in[0,\infty[$ and $\phi\in[0,2\pi]$. It follows that $r=(\xi+\eta)/2$. We may express the Cartesian coordinates in terms of the parabolic ones as

$$x=\sqrt{\xi\eta}\cos\phi,\quad y=\sqrt{\xi\eta}\sin\phi,\quad z=(\xi-\eta)/2.\tag{4}$$

The azimuthal angle ϕ is defined as in spherical coordinates. In parabolic coordinates the Laplace operator reads

$$\nabla^2=\frac{4}{\xi+\eta}\left[\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right)\right]+\frac{1}{\xi\eta}\frac{\partial^2}{\partial\phi^2},\tag{5}$$

and the potential of Eq. (1) is

$$V=-\frac{2Z}{\xi+\eta}+\frac{1}{2}F(\xi-\eta).\tag{6}$$

If we write the Schrödinger equation, $(-\frac{1}{2}\nabla^2+V)\psi=E\psi$, in parabolic coordinates, and express the wave function as a product of functions of each coordinate $\psi=f_1(\xi)f_2(\eta)e^{im\phi}/\sqrt{2\pi}$, with m the magnetic quantum number, we obtain

$$\begin{aligned}&\frac{1}{f_1(\xi)}\left[\frac{d}{d\xi}\left(\xi\frac{df_1}{d\xi}\right)+\left(\frac{E\xi}{2}-\frac{m^2}{4\xi}-\frac{F\xi^2}{4}\right)f_1(\xi)\right]\\&\quad+\frac{1}{f_2(\eta)}\left[\frac{d}{d\eta}\left(\eta\frac{df_2}{d\eta}\right)+\left(\frac{E\eta}{2}-\frac{m^2}{4\eta}+\frac{F\eta^2}{4}\right)f_2(\eta)\right]\\&\quad=-Z,\end{aligned}\tag{7}$$

where we have multiplied through by $-(\xi+\eta)/2$ and divided by ψ.

If we introduce $f_1(\xi)=\chi_1(\xi)/\sqrt{\xi}$ and $f_2(\eta)=\chi_2(\eta)/\sqrt{\eta}$, the differential operators simplify, and we are led to

$$-\frac{1}{2}\frac{d^2\chi_1}{d\xi^2}+U_1(\xi)\chi_1=\frac{E}{4}\chi_1,\tag{8a}$$

$$-\frac{1}{2}\frac{d^2\chi_2}{d\eta^2}+U_2(\eta)\chi_2=\frac{E}{4}\chi_2,\tag{8b}$$

with the constraint

$$\beta_1+\beta_2=Z.\tag{9}$$

The effective one-dimensional potentials are

$$U_1(\xi)=-\frac{\beta_1}{2\xi}+\frac{m^2-1}{8\xi^2}+\frac{F\xi}{8},\tag{10a}$$

$$U_2(\eta)=-\frac{\beta_2}{2\eta}+\frac{m^2-1}{8\eta^2}-\frac{F\eta}{8}.\tag{10b}$$

As two generic examples, we have sketched these potentials in Fig. 2 for $m=0,2$.

As seen from the potential of Eq. (1), the ionization will occur in the $-z$ direction, which in parabolic coordinates is along the η coordinate [see Eq. (3)].

The derivation of the results (8) and (10) may be formulated as a problem. For example, the opening question could be (a) obtain V in parabolic coordinates. Equation (5) could be provided by the teacher, and part (b) could be to perform the separation of coordinates.

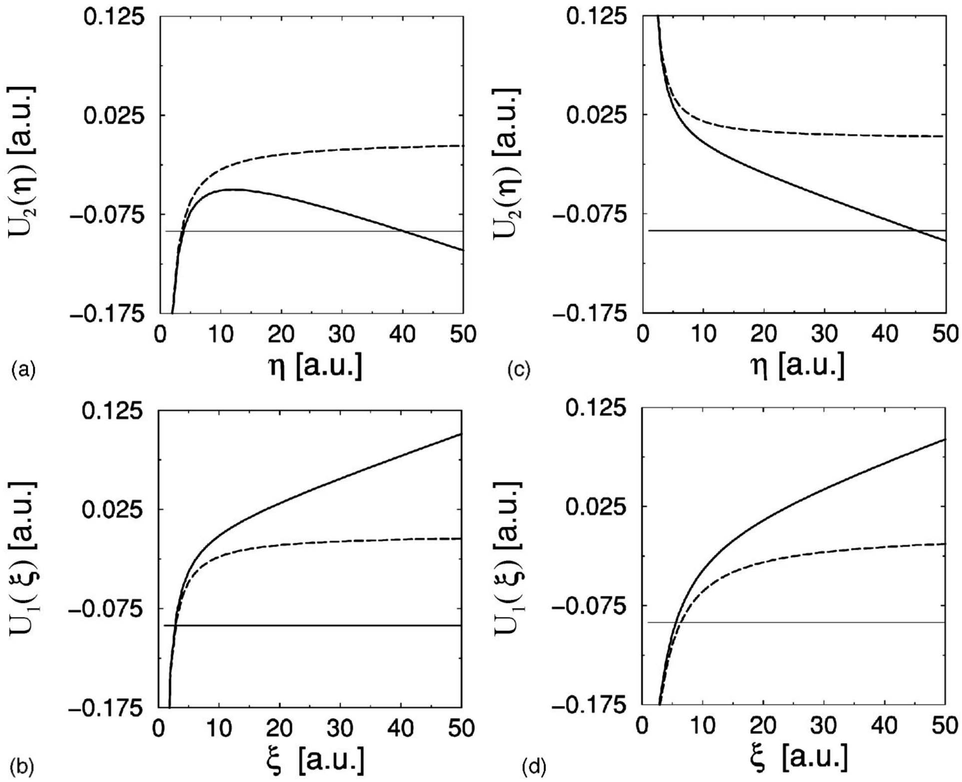

Fig. 2. Full curves: the potentials (a) $U_2(\eta)$, $m=0$; (b) $U_1(\xi)$, $m=0$; (c) $U_2(\eta)$, $m=2$; and (d) $U_1(\xi)$, $m=2$. The separation constant β_2 has been chosen according to Eq. (19). For all the plots the parameters are $Z=1$, $\kappa=0.86$, and $F=0.0169$, as in Fig. 1. Dashed curves: the pure Coulomb potentials. The horizontal line corresponds to the effective binding energy in the η and ξ coordinates, $-I_p/4$.

B. Asymptotic Coulomb wave function and separation constants

Finding the asymptotic wave function in spherical coordinates may be formulated as a brief problem.

Problem: Show that the asymptotic form (large r) of the Coulomb wave function with energy $E=-\kappa^2/2$ is given by

$$\psi_C=R(r)Y_{lm}(\theta,\phi)=Dr^{(Z/\kappa)-1}e^{-\kappa r}Y_{lm}(\theta,\phi). \tag{11}$$

Solution: Substitute the expression of $R(r)$ from Eq. (11) into the radial equation[6]

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)-\frac{l(l+1)}{r^2}R+2(E-V)R=0. \tag{12}$$

For $V=-Z/r$ it is then readily verified that $R(r)$ is the correct solution to leading order in $1/r$.

In Eq. (11) D is a normalization constant that is only known in analytical form for the pure Coulomb problem (see, for example, Ref. 6). Ammosov, Delone, and Krainov[10] obtained a general, analytical expression in terms of effective quantum numbers by considering the semiclassical solutions to the radial wave equation.

Ionization occurs near the axis of the field, that is, along the negative z axis. We leave it as an exercise to verify the following approximation for spherical harmonics close to the $+z$ direction ($\theta\approx 0$):

$$Y_{lm}(\theta,\phi)\approx Q(l,m)\frac{\sin^{|m|}\theta}{2^{|m|}|m|!}\frac{e^{im\phi}}{\sqrt{2\pi}}, \tag{13}$$

with

$$Q(l,m)\equiv(-1)^{(m+|m|)/2}\sqrt{\frac{2l+1}{2}\frac{(l+|m|)!}{(l-|m|)!}}. \tag{14}$$

The expansion in the $-z$ direction has the same magnitude, but the sign should be changed according to $Y_{lm}(\theta\approx 0,\phi)/Y_{lm}(\theta\approx\pi,\phi)=(-1)^{l-|m|}$. The sign of the spherical harmonics turns out to be unimportant because $Q(l,m)$ only enters the final expression for the ionization rate as $|Q|^2$.

In the region of interest z is large and negative ($r\approx -z$). In parabolic coordinates this region is equivalent to $\eta\gg\xi$. We expand the left-hand side of the equality $\cos\theta'=-(\xi-\eta)/(\xi+\eta)$ to lowest order in θ' and the right-hand side to lowest order in ξ/η and find that

$$\sin\theta=\sin\theta'\approx\theta'\approx 2\sqrt{\xi/\eta}, \tag{15}$$

where θ (θ') is the angle with the positive (negative) z axis. We may thus write the asymptotic wave function as

$$\psi_C\approx B\frac{2^{-(Z/\kappa)+1}}{|m|!}\xi^{|m|/2}e^{-\kappa\xi/2}\eta^{Z/\kappa-|m|/2-1}e^{-\kappa\eta/2}\frac{e^{im\phi}}{\sqrt{2\pi}}, \tag{16}$$

where we have introduced

$$B\equiv Q(l,m)D, \tag{17}$$

for the asymptotic constant. In general, D should be calculated numerically by matching the solution of the Schrödinger equation to the form (11).

The separation constants, β_1 and β_2 [see Eq. (9)] are determined by requiring the asymptotic solution (16) to solve Eq. (8) in the limit of vanishing field strength $F\to 0$. This approach is valid for $F/\kappa^3\ll 1$ (see Ref. 6, p. 292). The inequality $F/\kappa^3\ll 1$ will also be used in Sec. II F where an expression for the rate in an oscillating field is obtained.

From Eq. (16) we read off the ξ-dependence:

$$\frac{\chi_1(\xi)}{\sqrt{\xi}} = \xi^{|m|/2} e^{-\kappa\xi/2}. \tag{18}$$

We leave it as an exercise to substitute Eq. (18) into Eq. (8a) to obtain

$$\beta_1 = \frac{\kappa}{2}(|m|+1), \tag{19}$$

which fixes β_2 through Eq. (9).

C. Expression for the ionization rate

In Sec. II A we found that the wave function could be written as

$$\psi = \frac{\chi_1(\xi)}{\sqrt{\xi}} \frac{\chi_2(\eta)}{\sqrt{\eta}} \frac{e^{im\phi}}{\sqrt{2\pi}}. \tag{20}$$

The ionization rate is found by integrating the probability current density in the z direction, j_z, over a surface orthogonal to it

$$W = -\int j_z \, dS. \tag{21}$$

The probability current density for z large and negative ($z \approx -\eta/2$) is

$$j_z \equiv \frac{i}{2}\left(\psi \frac{\partial \psi^*}{\partial z} - \psi^* \frac{\partial \psi}{\partial z}\right) \approx i \frac{|\chi_1|^2}{2\pi\xi\eta}\left(\chi_2^* \frac{d\chi_2}{d\eta} - \chi_2 \frac{d\chi_2^*}{d\eta}\right). \tag{22}$$

For ionization near the z axis the differential surface element is written as

$$dS = \rho \, d\phi \, d\rho = \sqrt{\xi\eta} d\phi \frac{1}{2}\sqrt{\frac{\eta}{\xi}} d\xi \approx \frac{1}{2}\eta \, d\phi \, d\xi, \tag{23}$$

so the rate can be expressed as

$$W = \frac{i}{2}\left[\chi_2 \frac{d\chi_2^*}{d\eta} - \chi_2^* \frac{d\chi_2}{d\eta}\right] \int_0^\infty d\xi \frac{|\chi_1(\xi)|^2}{\xi}. \tag{24}$$

For $\chi_1(\xi)$ we use Eq. (18), and express the rate in terms of the η part of the wave function:

$$W = \frac{i}{2} \frac{|m|!}{\kappa^{|m|+1}}\left[\chi_2 \frac{d\chi_2^*}{d\eta} - \chi_2^* \frac{d\chi_2}{d\eta}\right]. \tag{25}$$

Problem: Why is it accurate to use Eq. (18) for $\chi_1(\xi)$ in the evaluation of the integral in Eq. (24)?

D. Semiclassical solutions

To obtain a closed, analytic expression for the rate, the semiclassical solutions for the function $\chi_2(\eta)$ are needed under and outside the barrier. Such solutions may be applied if[6]

$$\left|\frac{1}{p^3} \frac{dU_2}{d\eta}\right| \ll 1, \tag{26}$$

where the potential $U_2(\eta)$ is given by Eq. (10b). The momentum corresponding to this potential follows from Eq. (8b),

$$p = \left[-\left(\frac{\kappa}{2}\right)^2 + \frac{\beta_2}{\eta} - \frac{m^2-1}{4\eta^2} + \frac{1}{4}F\eta\right]^{1/2}, \tag{27}$$

which shows that Eq. (26) is fulfilled in the ranges

$$\frac{\beta_2}{\kappa^2} \ll \eta \ll \frac{\kappa^2}{F}, \quad \eta \gg \frac{\kappa^2}{F}. \tag{28}$$

It is seen from Eq. (26) that the semiclassical solutions do not apply near the classical turning points ($p=0$). For our purposes the interesting turning point is the one on the outer side of the barrier, which from Eq. (27) for $\eta \gg 1$ is seen to be

$$\eta_0 \approx \kappa^2/F. \tag{29}$$

Therefore, when relating the semiclassical solutions in the classically allowed ($\eta > \eta_0$) and forbidden regions ($\eta < \eta_0$), care should be taken around $\eta \approx \eta_0$.

The asymptotic Coulomb wave function is an exponentially decreasing function of η. Therefore, the semiclassical wave function in the classically forbidden region is of the form

$$\chi_2(\eta) = \frac{C}{\sqrt{|p|}} \exp\left(-\int_{\eta_0}^{\eta} |p| d\eta\right) \quad (\eta \ll \eta_0). \tag{30}$$

Equation (30) must be related to the semiclassical solution in the classically allowed region, $\eta > \eta_0$ representing a running-wave solution

$$\chi_2(\eta) = \frac{C_1}{\sqrt{p}} \exp\left(i \int p \, d\eta\right) \quad (\eta > \eta_0). \tag{31}$$

To fix the relation between the exponents and the normalization constants C and C_1, we may match directly to the quantum mechanical solutions in the two regions or, as will be done here, use purely semiclassical methods. To this end, the first step is to consider the semiclassical wave function in a region $\eta < \eta_0$ with $|\eta - \eta_0|$ small enough that we can make a Taylor expansion of the potential around $\eta = \eta_0$:

$$U_2(\eta) \approx U_2(\eta_0) + (\eta - \eta_0) \left.\frac{dU_2}{d\eta}\right|_{\eta_0}. \tag{32}$$

Note that this expansion is not necessarily in contradiction with the requirement that η should be far enough away from η_0 for the semiclassical solutions to apply. If we use the approximation in Eq. (32), the momentum becomes $p \approx (2F_0)^{1/2}(\eta - \eta_0)^{1/2}$, where $F_0 = -dU_2/d\eta$, and we find

$$\int_{\eta_0}^{\eta} |p| d\eta = \frac{2}{3}(2F_0)^{1/2}(\eta_0 - \eta)^{3/2}. \tag{33}$$

To get around the classical turning point, we apply an analytical continuation of the wave function into the complex η plane. This continuation allows us to get from the classically forbidden to the classically allowed region through a semicircle in the complex η plane, and it enables us to connect the solutions through a region where both the Taylor expansion and the semiclassical solutions (formally) apply. We note that for $\eta < \eta_0$, the momentum is purely imaginary, giving $|p| = -ip$. If we introduce

$$\eta - \eta_0 = \rho e^{i\varphi}, \tag{34}$$

and use Eq. (33), we may express the semiclassical wave function as

$$\chi_2(\eta)=\frac{C}{[2F_0\rho e^{i(\varphi+\pi)}]^{1/4}} \times\exp(-\tfrac{2}{3}(2F_0)^{1/2}\rho^{3/2}e^{i(3/2)(\varphi+\pi)}). \qquad (35)$$

Passing from the classically forbidden to the classically allowed region then corresponds to letting the angle φ in the complex plane undergo the change $\varphi=\pi\rightarrow 0$, so the semiclassical wave function for $\eta>\eta_0$ becomes

$$\chi_2(\eta)=\frac{C}{[2F_0\rho e^{i\pi}]^{1/4}}\exp\left(-\frac{2}{3}(2F_0)^{1/2}\rho^{3/2}e^{i(3/2)\pi}\right). \qquad (36)$$

If we write this result as

$$\chi_2(\eta)=\frac{Ce^{-i\pi/4}}{[2F_0\rho]^{1/4}}\exp\left(i\frac{2}{3}(2F_0)^{1/2}\rho^{3/2}\right), \qquad (37)$$

and take it as the Taylor expansion of the general form (31), we fix C_1 in terms of C, and obtain

$$\chi_2(\eta)=\frac{C}{\sqrt{p}}\exp\left(i\int p\,d\eta-i\frac{\pi}{4}\right)\quad(\eta>\eta_0). \qquad (38)$$

E. Putting it all together

We are now ready to put all the pieces together to obtain the final expression for the ionization rate. We use Eq. (38) and neglect variations in p in the prefactor and find

$$\frac{d\chi_2}{d\eta}=\frac{C}{\sqrt{p}}ip\exp\left(i\int p\,d\eta-i\frac{\pi}{4}\right). \qquad (39)$$

If we substitute Eq. (39) into Eq. (25), the rate becomes

$$W=\frac{|m|!}{\kappa^{|m|+1}}|C|^2. \qquad (40)$$

The semiclassical wave function under the barrier is given by Eq. (30) with the momentum given in Eq. (27). In the complete range of integration we have $\eta\gg 1$, and at the left end point (where we will match to the asymptotic Coulomb wave function) we further use the approximation necessary for the semiclassical wave function to be valid, that is, Eq. (28). To obtain the wave function we therefore introduce some approximations. In the prefactor all terms depending on η will be neglected according to Eq. (28), whereas in the integrand in the exponent the term proportional to η^{-2} will be neglected, and the Coulomb term will only be included through a first-order Taylor expansion:

$$|p(\eta)|\approx\left[\left(\frac{\kappa}{2}\right)^2-\frac{\beta_2}{\eta}-\frac{F\eta}{4}\right]^{1/2} \approx\frac{\kappa}{2}\left[1-\frac{F\eta}{\kappa^2}\right]^{1/2}-\frac{\beta_2}{(\kappa^2-F\eta)^{1/2}\eta}. \qquad (41)$$

The integral in the exponent therefore contains two terms. The first term is

$$\int_{\eta_0}^{\eta}\frac{\kappa}{2}\left(1-\frac{F\eta'}{\kappa^2}\right)^{1/2}d\eta'\approx-\frac{\kappa^3}{3F}+\frac{\kappa\eta}{2}. \qquad (42)$$

The second term is [see Ref. 11, Eq. (2.211)]

$$\int_{\eta_0}^{\eta}\frac{\beta_2\,d\eta'}{(\kappa^2-F\eta')^{1/2}\eta'}=\frac{\beta_2}{\kappa}\ln\left(\frac{\left(1-\frac{F\eta'}{\kappa^2}\right)^{1/2}-1}{\left(1-\frac{F\eta'}{\kappa^2}\right)^{1/2}+1}\right)\Bigg|_{\eta_0}^{\eta} \approx\ln\left(\frac{F\eta}{4\kappa^2}\right)^{\beta_2/\kappa}. \qquad (43)$$

In these evaluations we used that $\eta_0\approx\kappa^2/F$ [see Eq. (29)]. Finally, the semiclassical wave function corresponding to the one-dimensional Schrödinger equation in the η coordinate is approximately

$$\chi_2(\eta)\approx C\left(\frac{2}{\kappa}\right)^{1/2}\left(\frac{F\eta}{4\kappa^2}\right)^{\beta_2/\kappa}e^{-\kappa\eta/2}e^{\kappa^3/3F}. \qquad (44)$$

If we substitute Eqs. (18) and (44) into Eq. (20), the total semiclassical wave function becomes

$$\psi=C\kappa^{-1/2}2^{-Z/\kappa+|m|/2+1}\left(\frac{F}{2\kappa^2}\right)^{Z/\kappa-|m|/2-1/2} \times e^{\kappa^3/3F}\xi^{|m|/2}e^{-\kappa\xi/2}\eta^{Z/\kappa-|m|/2-1}e^{-\kappa\eta/2}\frac{e^{im\phi}}{\sqrt{2\pi}}. \qquad (45)$$

If we compare this expression for the asymptotic Coulomb wave function in parabolic coordinates given in Eq. (16), we find

$$C=\kappa^{1/2}\frac{B}{2^{|m|/2}|m|!}\left(\frac{2\kappa^2}{F}\right)^{Z/\kappa-|m|/2-1/2}e^{-\kappa^3/3F}, \qquad (46)$$

and the ionization rate can hereby be expressed in terms of the parameters B [Eq. (17)], κ, and m from the asymptotic Coulomb wave function as

$$W=\frac{|B|^2}{2^{|m|}|m|!}\frac{1}{\kappa^{2Z/\kappa-1}}\left(\frac{2\kappa^3}{F}\right)^{2Z/\kappa-|m|-1}e^{-2\kappa^3/3F}. \qquad (47)$$

Equation (47) is exactly the expression stated in Ref. 8. This expression is very general. For a given nuclear charge Z, ionization potential $I_p=\kappa^2/2$, field strength F, and magnetic quantum number m, the only unknown in Eq. (47) is the constant B. This constant in turn is determined by the analytical function $Q(l,m)$ from Eq. (14) and the normalization constant D, determined by matching the quantum mechanical wave function of the atom to the asymptotic Coulomb form (11).

Usually $F/2\kappa^3$ is much smaller than unity, and hence it follows from Eq. (47) that atoms prepared in a state with $m\neq 0$ will ionize much more slowly than atoms with $m=0$.

F. Ionization in a slowly varying field

We now consider the situation when the external field is oscillating,

$$\mathbf{F}(t)=F\cos(\omega t)\hat{\mathbf{z}}. \qquad (48)$$

We assume the adiabatic approximation to be valid, that is, the time the electron uses to tunnel through the barrier is much shorter than an optical period corresponding to the regime with $\gamma\ll 1$ [see Eq. (2)]. Because the atom adiabatically follows the changes in the external field, we can obtain the instantaneous ionization rate by inserting $|\mathbf{F}(t)|$ in the solution obtained with a static field. Because typical frequencies will correspond to wavelengths in the infrared region, the observed ionization rate will be an average of the instantaneous ionization rate over one period of the field.

We obtain the observed ionization rate from

$$W_{\rm obs}=\frac{1}{\pi}\int_{-\pi/2}^{\pi/2}W(t)d(\omega t), \tag{49}$$

where $W(t)$ is the instantaneous rate. Here the exponential dependence on the field strength clearly shows that the dominant contribution to the rate will be when $|\cos(\omega t)|$ is near its maximum, that is, $\omega t\sim 0$. Because the exponential is much more rapidly changing than any power, the time dependence of the preexponential factor will be neglected. For $\omega t\sim 0$, we make a Taylor expansion of $1/\cos(\omega t)$ and the time-dependent rate is approximated by

$$W(t)\approx W_{\rm stat}\exp\left(-\frac{\kappa^3}{3F}(\omega t)^2\right), \tag{50}$$

where $W_{\rm stat}$ is the rate in a static field as in Eq. (47). The observed rate is obtained as

$$W_{\rm obs}\approx\frac{1}{\pi}W_{\rm stat}\int_{-\pi/2}^{\pi/2}\exp\left(-\frac{\kappa^3}{3F}(\omega t)^2\right)d(\omega t)$$
$$=\left(\frac{3F}{\pi\kappa^3}\right)^{1/2}W_{\rm stat}, \tag{51}$$

where the integration limits were extended to infinity assuming that $\kappa^3/F\gg 1$.

For a slightly elliptical field

$$\mathbf{F}=F[\cos(\omega t)\hat{\mathbf{x}}\pm\varepsilon\sin(\omega t)\hat{\mathbf{y}}], \tag{52}$$

with $\varepsilon\ll 1$, the result is

$$W_{\rm obs}\approx\left(\frac{3F}{\pi(1-\varepsilon^2)\kappa^3}\right)^{1/2}W_{\rm stat}, \tag{53}$$

assuming that $\kappa^3/F\gg 1-\varepsilon^2$.

When the light is circularly polarized, the magnitude of the electric field is constant. In this case the ionization rate will be equal to the static rate, and thereby be much larger than the rate for linearly polarized light.

Equations (47) and (51) are the main results of the present paper. As an application of these expressions, we suggest the calculation of the ionization rates for different intensities for Ar and Xe. In Ref. 10 the relevant values for the coefficient D can be found by making the identification $D=C_{n^\star l^\star}\kappa^{Z/\kappa+1/2}$, where $C_{n^\star l^\star}$ is the normalization constant of the Coulomb wave function used in Ref. 10 with $n^\star=Z/\kappa$ and $l^\star$ the effective principal and angular momentum quantum numbers. Ammosov, Delone, and Krainov (ADK)[10] list experimental results that can be used for comparison with the theory.

III. SUMMARY

Now that we have obtained these relatively simple expressions for the ionization rate, it is worthwhile to consider the reliability of our approximations. This question has been addressed both theoretically and experimentally, implying that within the approximations outlined in this paper ($\gamma\ll 1$ and $F/\kappa^3\ll 1$), the model gives a good description of the ionization of atoms in strong fields. In fact these limits are too strict; the model is considered to be valid for $\gamma<0.5$ and for fields weak enough that we can maintain the picture of a barrier through which the electron tunnels. An example of the experimental tests of the tunneling theory can be found in Ref. 12. Note, however, that to compare the obtained rate with experimental data, we must integrate the rate equations using the appropriate pulse profile both temporally and spatially. For a brief summary of the theoretical comparison between the static field ionization rates obtained from the ADK model with those obtained by directly solving the Schrödinger equation, see Ref. 2.

We have presented a coherent derivation of a much used tunneling formula for strong field ionization of atoms. The derivation consists of six parts, Secs. II A–II F. Each of these parts may be formulated as an exercise. We hope that the present note will be a useful supplement for teaching atomic, molecular, and optical physics.

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