Electric polarizability and the solution of an inhomogeneous differential equation

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$$E_e^{rad}=\frac{e}{c}\left[\frac{\mathbf{n}\times\{(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\}}{(1-\boldsymbol{\beta}\cdot\mathbf{n})^3R}\right]_{ret}, \quad \mathbf{B}_e^{rad}=[\mathbf{n}]\times\mathbf{E}_e^{rad}. \quad (19)$$

The energy-flux density associated with $\mathbf{E}_{e-g}^{rad}$ and $\mathbf{B}_{e-g}^{rad}$ is given by the Poynting vector,

$$\mathbf{S}_{e-g}=\frac{c}{4\pi}\mathbf{E}_{e-g}^{rad}\times\mathbf{B}_{e-g}^{rad}. \quad (20)$$

Substituting Eqs. (18) into (20) we get

$$\mathbf{S}_{e-g}=\{1+(g/e)^2\}\mathbf{S}_e, \quad (21)$$

where $\mathbf{S}_e=(c/4\pi)\mathbf{E}_e^{rad}\times\mathbf{B}_e^{rad}$ is the Poynting vector associated with the radiation fields of the electric charge. In the particular case of a nonrelativistic accelerated charge (the terms involving $\boldsymbol{\beta}$ are negligible) the vector $\mathbf{S}_e$ is given by[6]

$$\mathbf{S}_e=\frac{c}{4\pi}|\mathbf{E}_e^{rad}|^2\mathbf{n}, \quad (22)$$

where now

$$\mathbf{E}_e^{rad}=\frac{e}{c}\left[\frac{\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})}{R}\right]_{ret}. \quad (23)$$

Using Eq. (22) into (21) we obtain

$$\mathbf{S}_{e-g}=\{1+(g/e)^2\}\frac{c}{4\pi}|\mathbf{E}_e^{rad}|^2\mathbf{n}, \quad (24)$$

and thus the power radiated per unit solid angle is

$$\frac{dP_{e-g}}{d\Omega}=\mathbf{S}_{e-g}\cdot\mathbf{n}R^2=\{1+(g/e)^2\}\frac{c}{4\pi}|R\mathbf{E}_e^{rad}|^2 =\frac{(e^2+g^2)}{4\pi c}|\mathbf{n}\times(\mathbf{n}\times\dot{\boldsymbol{\beta}})|^2. \quad (25)$$

If $\mathbf{n}$ makes angle ϑ with $\dot{\boldsymbol{\beta}}$, then the power radiated can be written as

$$\frac{dP_{e-g}}{d\Omega}=\frac{(e^2+g^2)}{4\pi c^3}|\dot{\mathbf{v}}|^2\sin^2\vartheta. \quad (26)$$

The total instantaneous power radiated is found by integrating Eq. (26) over all solid angle. Thus we obtain the *Larmor* formula for a nonrelativistic, accelerated dual-charged particle:

$$P_{e-g}=\frac{2}{3}\frac{(e^2+g^2)}{c^3}|\dot{\mathbf{v}}|^2. \quad (27)$$

We challenge the reader to derive the *Larmor* formula for a relativistic, accelerated dual-charged particle.

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I am indebted to Professor John D. Jackson for suggesting the use of the duality transformations in the problem of finding the fields of particles possessing both magnetic and electric charge. I would also like to thank Eduardo Roa for a careful reading of the manuscript.

[1]See, e.g., John D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), Sec. 6.12; Jack Vanderlinde, *Classical Electromagnetic Theory* (Wiley, New York, 1993), p. 89.
[2]Julian Schwinger, "A magnetic model of matter," Science **165**, 757–761 (1969).
[3]José A. Heras, "Jefimenko's formulas with magnetic monopoles and the Liénard–Wiechert fields of a dual-charged particle," Am. J. Phys. **62**, 525–531 (1994).
[4]See, e.g., Jackson in Ref. 1, p. 657.
[5]See, e.g., Jackson in Ref. 1, p. 252; see, also, Vanderlinde in Ref. 1, p. 89; David J. Griffiths, *Introduction to Electrodynamics*, 2nd ed. (Prentice-Hall, Englewood Cliffs, NJ, 1989), p. 308.
[6]See, e.g., Jackson in Ref. 1, p. 659.

Electric polarizability and the solution of an inhomogeneous differential equation

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In this paper, we are going to illustrate the effectiveness of the method of the inhomogeneous differential equation in obtaining the energy shift of a quantum level in second order perturbation theory. The energy shift will be used to calculate the electric polarizability due to the interaction between a static electric field and a charged particle moving under the influence of a one-dimensional delta potential. Both relativistic and nonrelativistic problems will be treated. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

Perturbation theory is one of the most important tools in solving a large variety of physics problems. Second order perturbation is used to calculate some basic quantities such as the electric polarizability and the magnetic susceptibility of a quantum system. The electric polarization of a system due to an interaction with some external electric field is an important application at both the introductory physics level and in advanced physics research. In our general physics courses, we study the electric polarization of a dielectric in relation to induced charges, dielectric constants and indices of refraction. At the advanced level, electric polarizability has been studied and used extensively to test the validity of physics models and different regions of a system of energy levels. For example, Schafer, Muller, Vasak, and Greiner[1] calculated the electric and magnetic polarizability of the nucleon in the M.I.T. bag model. Another example is the study of the contribution of the quark sea to the electric polarizability of the π^- and k mesons.[2]

In this paper, we are going to concentrate on the beautiful method of replacing the conventional algorithm used to calculate the energy shift in second order perturbation theory with the solution of an inhomogeneous differential equation devised by Dalgarno and Lewis[3] and discussed by Schwartz.[4] As is known, the conventional method involves an infinite sum or an integral that contains all possible states allowed by the transition. Some of these states, for example, scattering states, can be very difficult or impossible to obtain in a large number of problems. The knowledge of the unperturbed state will be all that we need for calculating the exact energy of that particular state to second order when we apply the technique of the inhomogeneous differential equation.[3,4]

A number of articles published more than 20 years ago recognized the effectiveness of such a technique in dealing with a variety of interesting problems. Among these problems is the calculation of the long range forces between a proton and a hydrogen atom.[3] Another one is the study of nuclear quadrupole coupling in polar molecules done by Foley and Tycko.[5] Sternheimer used the method of the inhomogeneous differential equation in studying the electronic polarizability of a number of ions[6] and alkali atoms.[7]

As indicated in Ref. 8, the method that we cited in Refs. 3 and 4 did not receive sufficient attention in recent years in solving for the energy shift in second or higher order perturbation theory. For this reason, we present the solution of a simple problem using the method of the inhomogeneous differential equation. The reader who has knowledge of basic calculus courses, in addition to a standard modern physics course, can reproduce without difficulty the results of our work. In addition, our simple model will allow us to extend the use of the same technique to solve for the electric polarizability of a relativistic system.

In the next section, we give a brief summary of the method of the inhomogeneous differential equation. This will be followed by a presentation of the model we adopt as illustration of the electric polarizability, both relativistic and nonrelativistic. Finally, we close with some concluding remarks.

II. THE METHOD OF THE INHOMOGENEOUS DIFFERENTIAL EQUATION

In this section, we give a brief summary of the method. Interested readers can find the details in Refs. 3 and 4, Schiff[9] and Merzbacher.[10] The ground state energy shift to the second order ΔE_0 is given by

$$\Delta E_0=\Sigma_n'\left|\int \psi_n^* H' \psi_0 \, d\tau\right|^2 \Big/ \{E_0-E_n\}. \tag{1}$$

ψ_0 is the ground state and is occupied by a charged particle as we assume in our problem. E_0 is the ground state energy. The functions ψ_n's represent all states allowed by the transition due to the interaction H'. The summation in Eq. (1) excludes the state ψ_0.

The first step in producing the alternative expression for ΔE_0 is accomplished by the introduction of an operator F satisfying the following equation:

$$\left\{\int \psi_n^* H' \psi_0 \, d\tau\right\} \Big/ \{E_0-E_n\}=\int \psi_n^* F \psi_0 \, d\tau. \tag{2}$$

With some simple manipulations and the use of the completeness relation, ΔE_0 can be rewritten as

$$\Delta E_0=\int \psi_0^* H' F \psi_0 \, d\tau-\left[\int \psi_0^* H' \psi_0 \, d\tau\right]\left[\int \psi_0^* F \psi_0 \, d\tau\right]. \tag{3}$$

To obtain a differential equation that eliminates the need of performing the infinite summation in Eq. (1), we use the following property:

$$\int \psi_n^* [F,H_0] \psi_0 \, d\tau=(E_0-E_n)\int \psi_n^* F \psi_0 \, d\tau. \tag{4}$$

Now with the use of Eq. (4), Eq. (2) can be written as[9]

$$(E_0-H_0)\phi=H'\psi_0-\left[\int \psi_0^* H' \psi_0 \, d\tau\right]\psi_0, \tag{5}$$

where

$$H_0\psi_0=E_0\psi_0, \tag{6}$$

and

$$\phi=F\psi_0. \tag{7}$$

If ϕ is taken to be orthogonal to ψ_0,[3,4,9,10] Eq. (3) can be written as

$$\Delta E_0=\int \psi_0^* H' \phi \, d\tau. \tag{8}$$

At this point we have to concentrate on Eqs. (5) and (8). When we start the problem, we have the knowledge of H_0, ψ_0, and E_0. We solve Eq. (5) for ϕ as a first step. The second and final step is to use ϕ in Eq. (8) to find ΔE_0. It is clear then that the only stationary state needed for such a calculation is the ground state ψ_0. The infinite summation in Eq. (1) is completely avoided.

III. ELECTRIC POLARIZABILITY OF A NONRELATIVISTIC PARTICLE

As stated before, the model we are presenting in this article is a simple one. It consists of a particle bound by an attractive potential $V(x)$ for which we choose the simple form

$$V(x)=-g\delta(x). \tag{9}$$

The strength of the potential is represented by g and $\delta(x)$ is the usual Dirac delta function. The Schrödinger equation in one dimension can then be written as

$$\left[-(\hbar^2/2m)\left(\frac{\partial^2}{\partial x^2}\right)-g\delta(x)\right]\psi=E\psi, \tag{10}$$

where m is the mass of the particle in our problem. Equation (10) produces a bound state in addition to a continuum of states. The normalized wave function is easily found to be,

$$\psi_0=\{(mg)/\hbar^2\}^{1/2}\exp(-k_0|x|), \tag{11}$$

where $k_0=(mg)/\hbar^2$. The bound state energy E_0 which follows from Eq. (10) is given by

$$E_0=-(mg^2)/(2\hbar^2)=-(\hbar^2k_0^2)/(2m). \quad (12)$$

When we use the method of the inhomogeneous differential equation, we do not have to have an explicit expression for the continuum.

Now let us assume the interaction H' is due to the presence of a static electric field which is applied in the vicinity of the particle. To simplify the geometry of the problem without any loss of generality, we take the electric field $\mathscr{E}$ to be parallel to the x axis. When the electric field is applied, the particle, which is assumed to have an electric charge q, is in the state ψ_0. The electric dipole Hamiltonian which represents the interaction of the electric field and the charge of the particle is given by

$$H'=-q\mathscr{E}x. \quad (13)$$

The final result for the electric polarizability is, of course, independent of the sign of the charge. This happens because the electric polarizability is proportional to the energy shift ΔE. It is clear from Eq. (1) that ΔE_0 is a second order correction which will include only the square of the electric charge. Now we are in a position to calculate the energy shift ΔE_0. The first step is to find ϕ. This can be done by the aid of Eq. (5) and the available expression of E_0, H_0, and ψ_0. Equation (5) in our problem can then be written as

$$\left[E_0+(\hbar^2/2m)\left(\frac{\partial^2}{\partial x^2}\right)+g\,\delta(x)\right]\phi=-q\mathscr{E}x\psi_0. \quad (14)$$

The second term on the right-hand side of Eq. (5) vanishes because the Hamiltonian, H', has odd parity. With the use of the properties of the delta function and substituting by Eqs. (11) and (12) in Eq. (14), we obtain the following expression for the region where $x>0$

$$\left[-(\hbar^2k_0^2/2m)+(\hbar^2/2m)\left(\frac{\partial^2}{\partial x^2}\right)\right]\phi(x>0)$$
$$=-q\mathscr{E}x\{(mg/\hbar^2)\}^{1/2}\exp(-k_0x). \quad (15)$$

For the region $x<0$, Eq. (14) becomes

$$\left[-(\hbar^2k_0^2/2m)+(\hbar^2/2m)\left(\frac{\partial^2}{\partial x^2}\right)\right]\phi(x<0)$$
$$=-q\mathscr{E}x\{(mg)/\hbar^2\}^{1/2}\exp(k_0x), \quad (16)$$

where $\phi(x>0)$ and $\phi(x<0)$ refer to the expressions of ϕ in the regions $x>0$ and $x<0$, respectively.

Equations (15) and (16) are very simple and can be solved essentially by inspection. The expressions for ϕ are given by

$$\phi(x>0)=[(mq\mathscr{E}/2\hbar^2k_0)x^2+(mq\mathscr{E}/2\hbar^2k_0^2)x]$$
$$\times(k_0)^{1/2}\exp(-k_0x), \quad (17)$$

and

$$\phi(x<0)=[-(mq\mathscr{E}/2\hbar^2k_0)x^2+(mq\mathscr{E}/2\hbar^2k_0^2)x]$$
$$\times(k_0)^{1/2}\exp(k_0x). \quad (18)$$

These expressions of $\phi(x>0)$ and $\phi(x<0)$ can easily be checked by substituting them in Eqs. (15) and (16).

At this point we can calculate the energy shift ΔE_0 by performing the integration given in Eq. (8). The integration is very simple and leads to the following result for ΔE_0:

$$\Delta E_0=-5/8[(q^2m^2g)/(\hbar^4k_0^5)]\mathscr{E}^2. \quad (19)$$

The electric polarizability α_{pol} is defined by

$$\Delta E_0=-(1/2)\alpha_{\text{pol}}\mathscr{E}^2. \quad (20)$$

With the use of the Eqs. (19) and (20), we obtain the familiar expression for α_{pol} which is given by

$$\alpha_{\text{pol}}=5/4[(q^2m)/(\hbar^2k_0^4)]. \quad (21)$$

IV. ELECTRIC POLARIZABILITY OF A RELATIVISTIC PARTICLE

In this problem, our potential $V(z)$ is given by

$$V(z)=g_s\beta\delta(z), \quad (22)$$

where g_s assumes a negative value and represents the strength of the scalar potential $V(z)$. β is given by[11]

$$\beta=\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (23)$$

The Dirac equation in this case can be reduced to[12]

$$[c\alpha_zP_z+\beta mc^2+\beta g_s\delta(z)]\psi(z)=E\psi(z), \quad (24)$$

where α_z is given by[11]

$$\alpha_z=\begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix}, \quad (25)$$

and σ_z is the z part of the Pauli matrices.

The expression for the ground state of a positive energy particle is given by[12]

$$\psi_0(z)=\theta(z)\psi_0^>(0)\exp[-k_0z]+\theta(-z)\psi_0^<(0)\exp[k_0z], \quad (26)$$

where

$$\psi_0^>(0)=[(k_0(E_0+mc^2))/(2mc^2)]^{(1/2)}$$
$$\times\begin{pmatrix} 1 \\ (i\hbar ck_0\sigma_z)/(E_0+mc^2) \end{pmatrix}, \quad (27)$$

and

$$\psi_0^<(0)=[(k_0(E_0+mc^2))/(2mc^2)]^{(1/2)}$$
$$\times\begin{pmatrix} 1 \\ (-i\hbar ck_0\sigma_z)/(E_0+mc^2) \end{pmatrix}, \quad (28)$$

$\theta(z)$ is 0, 1 or 1/2 for $z<0$, $z>0$, and $z=0$, respectively. $\psi_0^>(0)$ and $\psi_0^<(0)$ are the values of $\psi_0(z)$ just above and below the xy plane.[12] E_0 is the energy of the bound particle and it is related to k_0 by the following expression:[12]

$$\hbar^2c^2k_0^2=m^2c^4-E_0^2, \quad (29)$$

where m is the mass of the particle.

Now, our first step is to solve the inhomogeneous differential equation which is given by

$$\left[E_0+i\hbar c\alpha_z\frac{\partial}{\partial z}-\beta mc^2-\beta g_s\delta(z)\right]\phi(z)$$
$$=-q\mathscr{E}z\psi_0(z). \quad (30)$$

The general solution can be written as

$$\phi(z)=[(k_0(E_0+mc^2))/(2mc^2)]^{1/2}[\theta(z)G_1(z)\exp(-k_0z)$$
$$+\theta(-z)G_2(z)\exp(k_0z)], \quad (31)$$

where $G_1(z)$ and $G_2(z)$ are given by

$$G_1(z)=\begin{pmatrix} A_1z^2+B_1z+D_1 \\ A_2z^2+B_2z+D_2 \end{pmatrix}, \quad (32)$$

$$G_2(z)=\begin{pmatrix} A_1'z^2+B_1'z+D_1' \\ A_2'z^2+B_2'z+D_2' \end{pmatrix}. \quad (33)$$

Now substituting Eqs. (31) in Eq. (30) and considering separately the regions $z>0$, $z<0$, and $z=0$, we can determine the coefficients A_1, A_2, B_1, B_2, D_1, D_2, A_1', A_2', B_1', B_2', D_1', D_2'. $A_1, A_2, B_1, B_2, D_1, D_2$ are given by

$$A_1=(E_0q\mathscr{E})/(2c^2\hbar^2k_0). \quad (34)$$

$$A_2=(i\sigma_zE_0q\mathscr{E})/(2c\hbar(E_0+mc^2)), \quad (35)$$

$$B_1=(mq\mathscr{E})/(2\hbar^2k_0^2), \quad (36)$$

$$B_2=(-i\sigma_zmcq\mathscr{E})/(2\hbar k_0(E_0+mc^2)), \quad (37)$$

$$D_1=(-mc^2q\mathscr{E})/(4E_0k_0(E_0+mc^2)), \quad (38)$$

$$D_2=(-i\sigma_zmcq\mathscr{E})/(4\hbar k_0^2E_0). \quad (39)$$

ϕ should have an opposite parity to ψ_0 for the integration in Eq. (8) to survive. Due to this, the following relationships exist:

$$A_1'=-A_1, \quad A_2'=A_2, \quad B_1'=B_1, \quad B_2'=-B_2,$$

$$D_1'=-D_1, \quad D_2'=D_2. \quad (40)$$

Now substituting by ψ_0 and ϕ in Eq. (8), we find the following expression for ΔE_0:

$$\Delta E_0=(-R\mathscr{E}^2)[((6E_0)/8)-((m^2c^4)/(8E_0))], \quad (41)$$

where $R=q^2/(c^2\hbar^2k_0^4)$. Using the definition in Eq. (20), we finally arrive at the expression of electric polarizability. α_{pol}, in this case given by

$$\alpha_{\mathrm{pol}}=R[(6E_0)/4-(m^2c^4)/(4E_0)]. \quad (42)$$

By taking the nonrelativistic limit in Eq. (42) ($E_0\to mc^2$) we get the expression appearing in Eq. (21).

The relativistic problem has many significant results. One of them is the existence of negative electric polarizability. This phenomenon occurs when $E_0<mc^2/\sqrt{6}$. It is very clear that Eq. (29) implies that E_0 will always be less than mc^2. The range of the values E_0 can assume and the corresponding explanation is given in Ref. 12. A separate paper will be devoted to define, discuss, and demonstrate the physics of the electric polarizability of a relativistic particle.

V. CONCLUSION

We calculated the relativistic and nonrelativistic electric polarizabilities of a particle under the influence of a one-dimensional delta potential. Instead of using the conventional method in calculating the energy shift in the second order perturbation, we applied a method based on the solution of an inhomogeneous differential equation.[3,4] We hope that we have illustrated the advantage of using such a method in avoiding a number of mathematical difficulties. More important is our goal of emphasizing the efficiency of this technique which can be applied to a variety of problems. We believe the simplicity of the model that we presented in this paper is helpful in accomplishing our goal. The interested reader can find more applications of this technique in the work published by Mavromatis.[8]

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EDITORS AND WRITERS

Shawn...would take a failed text, saw it up into a dozen chunks, excise the vacant musings, straighten out the thinking, fix the grammar, oil the transitions, and put it back together as an acceptable piece of work—all without leaving a trace of his presence. Only the writer ever knew that Shawn had been there, and the writer wasn't likely to tell. When I started work as an editor, in 1945, Shawn gave me as a manual of instruction a manuscript he had edited, by a writer now long and deservedly forgotten. The thing was a revelation. Every page of the typescript was black with Shawn's crabbed little corrections and transpositions, and the margins were littered with requests for more facts or better explanations. He had, it became clear, completely rewritten the piece—but always carefully using the lame writer's own language and constructions. What was astonishing was that by the time the job was done Shawn himself had vanished. There was nobody left but the writer—but now a writer who had a competent piece to offer.

Gardner Botsford, "Remembering Mr. Shawn," *The New Yorker*, December 28, 1992/January 4, 1993, p. 139.