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Some considerations about thermodynamic cycles

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Abstract
After completing their introductory studies on thermodynamics at the university level, typically in a second-year university course, most students show a number of misconceptions. In this work, we identify some of those erroneous ideas and try to explain their origins. We also give a suggestion to attack the problem through a systematic and detailed study of various thermodynamic cycles. In the meantime, we derive some useful relations.

1. Introduction

In university physics textbooks at the introductory level [1–3], the analysis of heat engines and heat pumps/refrigerators is typically included in the study of the second law of thermodynamics. Heat engines are usually used to introduce the Kelvin–Planck statement of the second law; heat pumps/refrigerators are used to introduce the Clausius statement of the second law. The schematic diagrams of these machines, based on the conservation of energy (first law of thermodynamics) and the exchange of heat and work, allow us to explain, in a satisfactory manner, the concepts of efficiency of a heat engine, coefficient of performance (COP) of a heat pump and coefficient of performance of a refrigerator.

In particular, the Carnot cycle is discussed in detail, and the Carnot theorem, which links this cycle with heat engines of greatest efficiency and heat pumps/refrigerators of greatest coefficients of performance, is proved. The student also works out a number of other cycles (Otto cycle, Diesel cycle, etc) while studying heat engines, sometimes directly in the text [1], sometimes through proposed problems [2, 3]. Nevertheless, those cycles do not appear during the study of heat pumps/refrigerators.

At the same time, and gradually throughout this study, the concept of entropy is introduced, and the difference between reversible and irreversible processes is explained.

My teaching experience has shown that the student concludes the study of these topics with some important misconceptions, namely

- Erroneous comprehension of the idea of cycle inversion.
- Difficulty in understanding the importance of the Carnot cycle; specifically, difficulty in understanding the true meaning of the sentence ‘Carnot cycle is reversible, while other cycles are irreversible’. The student frequently *thinks* in this way: ‘If, after each cycle, the system returns to its initial state, and the two reservoirs have the same original temperatures, where can be the irreversibility?’ Or in this way: ‘If a cycle could be inverted, then it is automatically reversible!’
- Erroneous belief in that the unique schematic diagrams allowed by the two laws of thermodynamics are the diagram corresponding to a heat engine and the diagram corresponding to a heat pump/refrigerator.
- Profound belief that it is enough to invert the cycle of any heat engine to automatically transform it into the corresponding heat pump/refrigerator.

The first misconception has a clearly identifiable source: only the Carnot cycle is explicitly inverted in textbooks. This difficulty is thus relatively easy to solve, by showing other examples of inversion to the student.

The second misconception has another origin: the student rarely calculates entropy changes while dealing with heat engines in specific cycles; the Carnot cycle is, one more time, the only exception. So the confusion between the entropy of the system and the entropy of the universe is very usual in this context.

The last two misconceptions directly result from the lack of various examples of inversion of thermodynamic cycles in most texts.

This work has a primary goal to show how these misconceptions can be solved. We will proceed to a systematic and careful analysis of various specific cycles, all well known in the study of heat engines; naturally, we will begin with the Carnot cycle in order to show its unique property. In all cases, we will study the cycle of the heat engine, and then we will proceed to its inversion. Whenever possible we will get relations between the relevant parameters.

We will only consider the simplest versions of those cycles, that is, the ideal versions. Here and there we will refer to some engineering applications, but only as secondary information. We will also ignore all practical difficulties in the implementation of specific cycles as well as the corresponding solutions. Finally, little historical digressions will be included as footnotes, for the sake of completeness.

2. Theory and notation

We write the first law of thermodynamics in the form

$$\Delta U = Q + W, \tag{1}$$

where ΔU represents the change of internal energy of the system, Q is the heat absorbed by the system and W is the work done on the system. We will focus our attention to hydrostatic systems—systems described by thermodynamic variables V (volume), p (pressure) and T (temperature)—following cycles, so

$$\Delta U = 0, \qquad Q = \oint T\,\mathrm{d}S, \qquad W = -\oint p\,\mathrm{d}V; \tag{2}$$

here, S represents the entropy of the system.

We admit the existence of only two thermal reservoirs (hot body and cold body), at different but constant temperatures, in contact with the system; the first law of thermodynamics assumes the form

$$0 = Q_{\mathrm{cycle}} + W_{\mathrm{cycle}} \iff Q_{\mathrm{h}} + Q_{\mathrm{c}} + W_{\mathrm{cycle}} = 0\,, \tag{3}$$

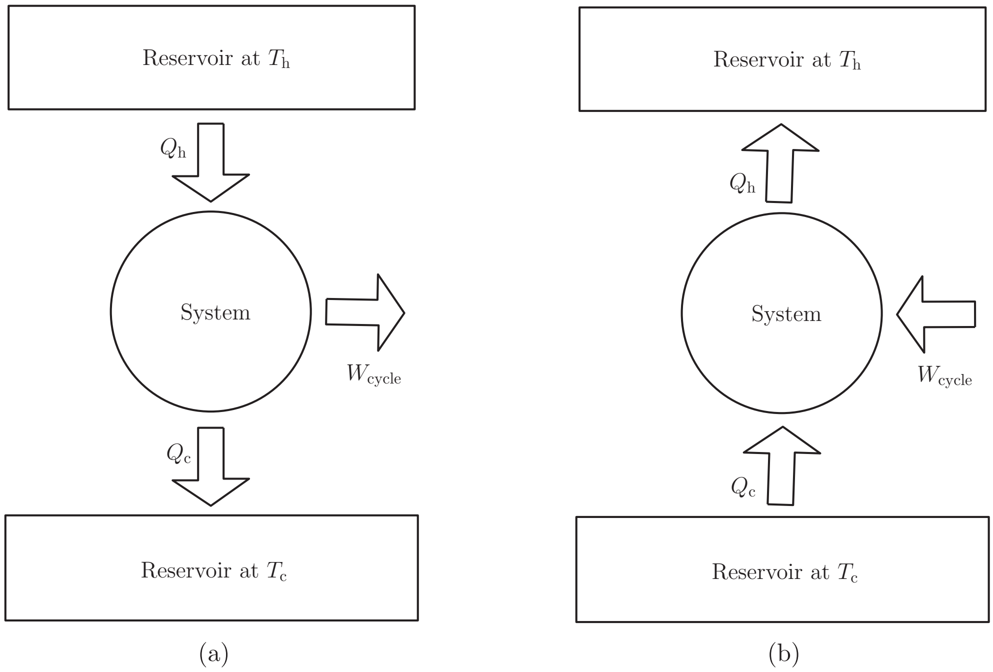

Figure 1. Schematic diagram of (a) a heat engine and (b) a heat pump/refrigerator.

where Q_h represents the heat absorbed by the system, during the cycle, from the reservoir at higher temperature (labelled T_h) and Q_c represents the heat absorbed by the system, during the cycle, from the reservoir at lower temperature (labelled T_c).

We will designate as a heat engine any machine working in cycles that allows us to extract work from the two reservoirs. The schematic representation of a heat engine is shown in figure 1(a). Since in this case $W_{cycle} < 0$, $Q_h > 0$ and $Q_c < 0$, we have

$$|Q_h| - |Q_c| - |W_{cycle}| = 0 \implies |W_{cycle}| = |Q_h| - |Q_c| \ . \tag{4}$$

The efficiency of the heat engine is given by

$$\epsilon \equiv \frac{|W_{cycle}|}{|Q_h|} = \frac{|Q_h| - |Q_c|}{|Q_h|} = 1 - \frac{|Q_c|}{|Q_h|} < 1. \tag{5}$$

Since $W_{cycle} < 0$ and $Q_{cycle} > 0$ in a heat engine, its cycle, when represented in a volume–pressure (V–p) diagram or in an entropy–temperature (S–T) diagram, must be performed in a clockwise direction, on account of relations (2).

By inverting all arrows of figure 1(a), we get the traditional schematic representation of a heat pump/refrigerator, shown in figure 1(b). Since in this case $W_{cycle} > 0$, $Q_h < 0$ and $Q_c > 0$, we have

$$-|Q_h| + |Q_c| + |W_{cycle}| = 0 \implies |W_{cycle}| = |Q_h| - |Q_c|. \tag{6}$$

The COPs of the heat pump (HP) and the refrigerator (R) are

$$\mathrm{COP}^{\mathrm{HP}} \equiv \frac{|Q_h|}{|W_{cycle}|} = \frac{|Q_h|}{|Q_h| - |Q_c|} > 1 \ , \tag{7}$$

$$\mathrm{COP}^{\mathrm{R}} \equiv \frac{|Q_c|}{|W_{cycle}|} = \frac{|Q_c|}{|Q_h| - |Q_c|} \ , \tag{8}$$

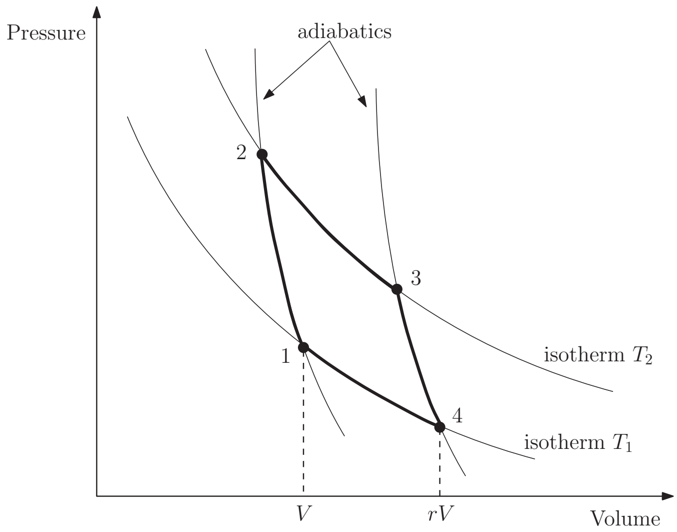

Figure 2. Volume–pressure diagram of the Carnot cycle.

and it is clear that

$$\mathrm{COP}^{\mathrm{HP}} - \mathrm{COP}^{\mathrm{R}} = 1 \ . \tag{9}$$

Since $W_{\text{cycle}} > 0$ and $Q_{\text{cycle}} < 0$ in a heat pump/refrigerator, its cycle, when represented in a (V–p) diagram or in an (S–T) diagram, must be realized in a counterclockwise direction, on account of relations (2).

In any case, the change of the entropy of the universe after one cycle is

$$\Delta S^{\text{universe}} = \Delta S^{\text{system}} + \Delta S^{\text{reservoir at } T_{\text{h}}} + \Delta S^{\text{reservoir at } T_{\text{c}}} = 0 + \frac{-Q_{\text{h}}}{T_{\text{h}}} + \frac{-Q_{\text{c}}}{T_{\text{c}}}$$
$$= -\left(\frac{Q_{\text{h}}}{T_{\text{h}}} + \frac{Q_{\text{c}}}{T_{\text{c}}}\right) . \tag{10}$$

We will assume that the system (or working substance) is always an ideal gas, and that phase changes do not occur. We will represent the number of moles by n, the gas constant by R, the molar heat capacities at constant volume and at constant pressure by c_V and c_p, respectively, and the adiabatic coefficient by γ. Some relations between these quantities are

$$pV = nRT \ , \quad c_p - c_V = R \ , \quad \gamma \equiv \frac{c_p}{c_V} > 1 \ , \quad c_V = \frac{R}{\gamma - 1} \ , \quad c_p = \frac{\gamma R}{\gamma - 1} \ . \tag{11}$$

3. The Carnot cycle

Let us consider the Carnot cycle[1] shown in figure 2, with two isothermal processes and two adiabatic processes, where $r > 1$ and $T_1 < T_2$. We define $\Delta T \equiv T_2 - T_1 > 0$.

Since each adiabatic curve satisfies the equation $TV^{\gamma-1} = \text{constant}$, we can write

$$\begin{cases} T_2 V_2^{\gamma-1} = T_1 V_1^{\gamma-1} \\ T_3 V_3^{\gamma-1} = T_4 V_4^{\gamma-1} \end{cases} \Longrightarrow \left(\frac{V_3}{V_2}\right)^{\gamma-1} = \left(\frac{V_4}{V_1}\right)^{\gamma-1} \Longrightarrow \frac{V_3}{V_2} = \frac{V_4}{V_1} \equiv r \ , \tag{12}$$

[1] Introduced by Nicolas Léonard Sadi Carnot (1796–1832) in 1824.

because $T_3 = T_2$ and $T_4 = T_1$. The parameter r is called compression ratio and relates the initial and final volumes of the gas during isothermal processes.[2]

Let us suppose that the cycle is done in the clockwise direction. In processes $1 \to 2$ and $3 \to 4$, the gas is thermally isolated, in process $2 \to 3$, the gas is in contact with a thermal reservoir at temperature $T_\mathrm{h} = T_2$, and in process $4 \to 1$, the gas is in contact with a thermal reservoir at temperature $T_\mathrm{c} = T_1$.

We have

$$Q_{12} = 0\,; \qquad Q_{23} = -W_{23} = nRT_2 \ln\left(\frac{V_3}{V_2}\right) = nRT_2 \ln r\,;$$

$$Q_{34} = 0\,; \qquad Q_{41} = -W_{41} = -nRT_1 \ln\left(\frac{V_4}{V_1}\right) = -nRT_1 \ln r\,, \quad \text{so}$$

$$Q_\mathrm{h} = Q_{23} = nRT_2 \ln r > 0\,, \qquad Q_\mathrm{c} = Q_{41} = -nRT_1 \ln r < 0\,,$$

$$Q_\mathrm{cycle} = Q_\mathrm{h} + Q_\mathrm{c} = nR\Delta T \ln r > 0\,, \qquad W_\mathrm{cycle} = -Q_\mathrm{cycle} = -nR\Delta T \ln r < 0\,.$$

The gas is absorbing heat from the reservoir at higher temperature, is rejecting heat to the reservoir at lower temperature and is doing work; this behaviour defines a heat engine.

According to (5), the efficiency of the Carnot engine is given by

$$\epsilon_\text{Carnot cycle} = 1 - \frac{|Q_\mathrm{c}|}{|Q_\mathrm{h}|} = 1 - \frac{nRT_1 \ln r}{nRT_2 \ln r} = 1 - \frac{T_1}{T_2} = \frac{\Delta T}{T_2} \tag{13}$$

and it depends only on the temperatures of the two reservoirs.

Using (10), the entropy change of the universe during a cycle of the Carnot engine is

$$\Delta S^\text{universe}_{\text{Carnot cycle},\circlearrowright} = -\frac{Q_\mathrm{h}}{T_\mathrm{h}} - \frac{Q_\mathrm{c}}{T_\mathrm{c}} = -\frac{nRT_2 \ln r}{T_2} + \frac{nRT_1 \ln r}{T_1} = 0\,, \tag{14}$$

that is, the clockwise Carnot cycle is reversible. This property makes the Carnot engine a very special engine: it is the unique reversible heat engine functioning with only two reservoirs. (Any other cycle could be performed in a reversible way, but that would require a virtually infinite number of thermal reservoirs.)

Let us try to transform this heat engine into a heat pump/refrigerator. In order to do so, we invert the cycle to the counterclockwise direction. In process $1 \to 4$, the gas is in contact with a reservoir at temperature $T'_\mathrm{c} = T_\mathrm{c} = T_1$, in processes $4 \to 3$ and $2 \to 1$, the gas is thermally isolated, and in process $3 \to 2$, the gas is in contact with a reservoir at temperature $T'_\mathrm{h} = T_\mathrm{h} = T_2$. (The primed quantities refer to the counterclockwise cycle. We will adopt this notation from now on.) We have

$$Q_{14} = -Q_{41} = nRT_1 \ln r\,; \quad Q_{43} = 0\,; \quad Q_{32} = -Q_{23} = -nRT_2 \ln r\,; \quad Q_{21} = 0\,; \quad \text{so}$$

$$Q'_\mathrm{h} = Q_{32} = -nRT_2 \ln r < 0\,, \quad Q'_\mathrm{c} = Q_{14} = nRT_1 \ln r > 0\,,$$

$$Q'_\mathrm{cycle} = Q'_\mathrm{h} + Q'_\mathrm{c} = -nR\Delta T \ln r < 0\,, \quad W'_\mathrm{cycle} = -Q'_\mathrm{cycle} = nR\Delta T \ln r > 0\,.$$

Naturally, $Q'_\mathrm{cycle} = -Q_\mathrm{cycle}$ and $W'_\mathrm{cycle} = -W_\mathrm{cycle}$. Now the gas is rejecting heat to the reservoir at higher temperature, is absorbing heat from the reservoir at lower temperature, and it is doing work on the gas; this behaviour defines a heat pump/refrigerator. In short, we see

[2] Conventionally, we think of a clockwise cycle, representing the heat engine. As the name indicates, the compression ratio relates the initial and final volumes during the isothermal compression process, namely the volumes V_4 and V_1. The expansion ratio is defined as the ratio between the final and initial volumes during the isothermal expansion process; in this case $r' \equiv \frac{V_3}{V_2}$. We can also define the pressure ratio as the ratio between the final and initial pressures during the isothermal compression process; here, $r_\mathrm{p} \equiv \frac{p_1}{p_4}$. Since $p_1V_1 = p_4V_4$ and $p_2V_2 = p_3V_3$, we have $r_\mathrm{p} \equiv \frac{p_1}{p_4} = \frac{V_4}{V_1} = r$ and $\frac{p_2}{p_3} = \frac{V_3}{V_2} = r' = r$, on account of (12). Thus in the Carnot cycle, the compression ratio r, the expansion ratio r' and the pressure ratio r_p are the same.

that the straightforward inversion of the cycle of a Carnot engine *automatically* generates a (Carnot) heat pump/refrigerator.

According to (7) and (8), the COP of the Carnot heat pump and the Carnot refrigerator are

$$\mathrm{COP}^{\mathrm{HP}}_{\text{Carnot cycle}} = \frac{|Q'_{\mathrm{h}}|}{|W'_{\text{cycle}}|} = \frac{nRT_2 \ln r}{nR\Delta T \ln r} = \frac{T_2}{\Delta T} , \tag{15}$$

$$\mathrm{COP}^{\mathrm{R}}_{\text{Carnot cycle}} = \frac{|Q'_{\mathrm{c}}|}{|W'_{\text{cycle}}|} = \frac{nRT_1 \ln r}{nR\Delta T \ln r} = \frac{T_1}{\Delta T} . \tag{16}$$

We can verify the general relation (9), and we also observe that

$$\mathrm{COP}^{\mathrm{HP}}_{\text{Carnot cycle}} = \frac{1}{\epsilon_{\text{Carnot cycle}}} . \tag{17}$$

Finally, the change in the entropy of the universe during a cycle of this Carnot heat pump/refrigerator is given by

$$\Delta S^{\text{universe}}_{\text{Carnot cycle},\circlearrowleft} = -\frac{Q'_{\mathrm{h}}}{T_{\mathrm{h}}} - \frac{Q'_{\mathrm{c}}}{T_{\mathrm{c}}} = \frac{nRT_2 \ln r}{T_2} - \frac{nRT_1 \ln r}{T_1} = 0 , \tag{18}$$

which shows that the counterclockwise Carnot cycle is also reversible; this property makes the Carnot heat pump/refrigerator a very special heat pump/refrigerator.

4. Stirling cycle

Let us consider the Stirling cycle[3] shown in figure 3. This cycle has a clear resemblance to the Carnot cycle[4]; it has two isothermal processes and two isochoric processes (replacing the adiabatic processes), where $r > 1$ and $T_1 < T_2$. We define $\Delta T \equiv T_2 - T_1 > 0$, as before.

Let us assume that the cycle operates in the clockwise direction. In processes $1 \to 2 \to 3$, the gas is in contact with a thermal reservoir at temperature $T_{\mathrm{h}} = T_2$, and in processes $3 \to 4 \to 1$, the gas is in contact with a thermal reservoir at temperature $T_{\mathrm{c}} = T_1$. We have

$$Q_{12} = nc_V \Delta T_{12} = n\frac{R}{\gamma - 1}\Delta T ; \quad Q_{23} = -W_{23} = nRT_2 \ln\left(\frac{V_3}{V_2}\right) = nRT_2 \ln r ;$$

$$Q_{34} = nc_V \Delta T_{34} = -n\frac{R}{\gamma - 1}\Delta T ; \quad Q_{41} = -W_{41} = -nRT_1 \ln\left(\frac{V_4}{V_1}\right) = -nRT_1 \ln r , \quad \text{so}$$

$$Q_{\mathrm{h}} = Q_{12} + Q_{23} = nR\left(T_2 \ln r + \frac{\Delta T}{\gamma - 1}\right) > 0 ,$$

$$Q_{\mathrm{c}} = Q_{34} + Q_{41} = -nR\left(T_1 \ln r + \frac{\Delta T}{\gamma - 1}\right) < 0 ,$$

$$Q_{\text{cycle}} = Q_{\mathrm{h}} + Q_{\mathrm{c}} = nR\Delta T \ln r > 0 , \quad W_{\text{cycle}} = -Q_{\text{cycle}} = -nR\Delta T \ln r < 0 .$$

Note that the values of Q_{cycle} and W_{cycle} in the Stirling cycle are the same as the corresponding values in the Carnot cycle. Since $Q_{\mathrm{h}} > 0$, $Q_{\mathrm{c}} < 0$ and $W_{\text{cycle}} < 0$, the gas is absorbing heat from the reservoir at higher temperature, is rejecting heat to the reservoir at lower temperature and is doing work; these properties characterize a heat engine.

[3] Used by Robert Stirling (1790–1878) in 1816 in his air engine; that heat engine was improved in 1840 by Robert's brother, James Stirling (1800–76).

[4] In this cycle the compression ratio, the expansion ratio and the pressure ratio also coincide. The equality of the first two ratios is automatic, and the equality with the third ratio is proved as in the previous section.

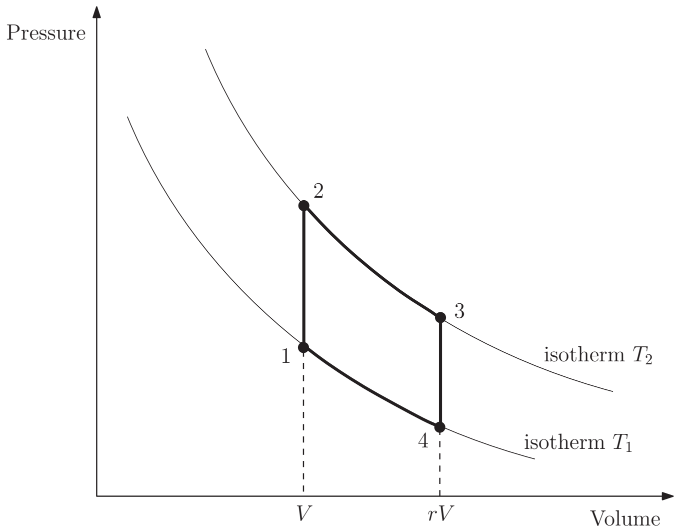

Figure 3. Volume–pressure diagram of the Stirling cycle.

The efficiency of the Stirling engine is given by

$$\epsilon_{\text{Stirling cycle}} = \frac{|W_{\text{cycle}}|}{|Q_{\text{h}}|} = \frac{nR\Delta T \ln r}{nR\left(T_2 \ln r + \frac{\Delta T}{\gamma-1}\right)} = \frac{\Delta T \ln r}{T_2 \ln r\left[1 + \frac{\Delta T}{(\gamma-1)T_2 \ln r}\right]}$$
$$= \frac{\frac{\Delta T}{T_2}}{1 + \frac{\frac{\Delta T}{T_2}}{(\gamma-1)\ln r}} = \frac{\epsilon_{\text{Carnot cycle}}}{1 + \frac{\epsilon_{\text{Carnot cycle}}}{(\gamma-1)\ln r}}, \tag{19}$$

where we have used relation (13) for the efficiency of the Carnot engine. This expression clearly shows that $\epsilon_{\text{Stirling cycle}} < \epsilon_{\text{Carnot cycle}}$ and that $\epsilon_{\text{Stirling cycle}} \to \epsilon_{\text{Carnot cycle}}$ if $r \to \infty$. It can also be written in the form

$$\frac{1}{\epsilon_{\text{Stirling cycle}}} - \frac{1}{\epsilon_{\text{Carnot cycle}}} = \frac{1}{(\gamma-1)\ln r}\,. \tag{20}$$

The entropy change of the universe during a cycle of the Stirling engine is

$$\Delta S^{\text{universe}}_{\text{Stirling cycle}, \circlearrowright} = -\frac{Q_{\text{h}}}{T_{\text{h}}} - \frac{Q_{\text{c}}}{T_{\text{c}}} = -\frac{nR\left(T_2 \ln r + \frac{\Delta T}{\gamma-1}\right)}{T_2} + \frac{nR\left(T_1 \ln r + \frac{\Delta T}{\gamma-1}\right)}{T_1}$$
$$= \frac{nR}{\gamma-1}\Delta T\left(\frac{1}{T_1} - \frac{1}{T_2}\right) = \frac{nR}{\gamma-1}\frac{(\Delta T)^2}{T_1 T_2} > 0\,, \tag{21}$$

that is, the clockwise Stirling cycle is irreversible.

Let us try to transform this heat engine into a heat pump/refrigerator, as in the previous section. In order to do so, we invert the cycle to the counterclockwise direction. We must be careful here when identifying the reservoirs that are in contact with the gas in the various processes. In processes $1 \to 4$ and $2 \to 1$, the gas should be in contact with a reservoir at temperature $T'_{\text{c}} = T_{\text{c}} = T_1$, and in processes $4 \to 3 \to 2$, the gas should be in contact with a reservoir at temperature $T'_{\text{h}} = T_{\text{h}} = T_2$. Then we have

$$Q_{14} = -Q_{41} = nRT_1 \ln r\,; \quad Q_{43} = -Q_{34} = n\frac{R}{\gamma-1}\Delta T\,;$$

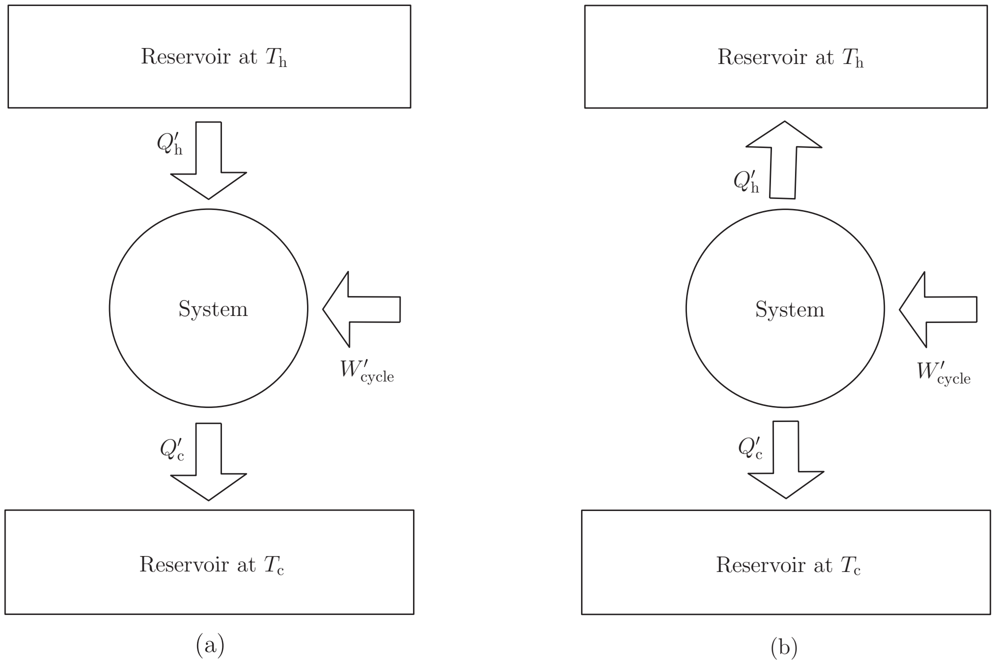

Figure 4. Schematic diagrams of situations (a) and (b) described in sections 4 and 5.

$$Q_{32} = -Q_{23} = -nRT_2 \ln r\,; \quad Q_{21} = -Q_{12} = -n\frac{R}{\gamma - 1}\Delta T\,, \quad \text{so}$$

$$Q'_h = Q_{43} + Q_{32} = -nR\left(T_2 \ln r - \frac{\Delta T}{\gamma - 1}\right), \quad Q'_c = Q_{14} + Q_{21} = nR\left(T_1 \ln r - \frac{\Delta T}{\gamma - 1}\right),$$

$$Q'_{cycle} = Q'_h + Q'_c = -nR\Delta T \ln r < 0, \quad W'_{cycle} = -Q'_{cycle} = nR\Delta T \ln r > 0\,.$$

It is very important to compare Q'_h and Q'_c with Q_h and Q_c, respectively, and understand the origin of the difference. The next observation to do is this: although work is being done on the gas, there is no warranty that $Q'_h < 0$ (condition characterizing a heat pump) or $Q'_c > 0$ (condition characterizing a refrigerator). Actually, we have three possible situations.

(a) $T_2 \ln r - \frac{\Delta T}{\gamma-1} < 0 \Longleftrightarrow \ln r < \frac{\Delta T}{(\gamma-1)T_2} \Longleftrightarrow r < \exp\left[\frac{\Delta T}{(\gamma-1)T_2}\right]$.
In this case, $Q'_h > 0$ and $Q'_c < 0$, that is, the gas is absorbing heat from the reservoir at higher temperature and rejects heat to the reservoir at lower temperature. This situation is represented by the diagram shown in figure 4(a), and it does not correspond to a heat engine, nor a heat pump/refrigerator; it is described by the relation $|W'_{cycle}| = |Q'_c| - |Q'_h|$.

(b) $T_2 \ln r - \frac{\Delta T}{\gamma-1} > 0$ and $T_1 \ln r - \frac{\Delta T}{\gamma-1} < 0 \Longleftrightarrow \frac{\Delta T}{(\gamma-1)T_2} < \ln r < \frac{\Delta T}{(\gamma-1)T_1}$
$\Longleftrightarrow \exp\left[\frac{\Delta T}{(\gamma-1)T_2}\right] < r < \exp\left[\frac{\Delta T}{(\gamma-1)T_1}\right]$.
In this case, $Q'_h < 0$ and $Q'_c < 0$, that is, the gas rejects heat to the reservoir at higher temperature and to the reservoir at lower temperature. This situation is represented by the diagram shown in figure 4(b), and it does not correspond to a heat engine, nor a refrigerator; it is described by the relation $|W'_{cycle}| = |Q'_c| + |Q'_h|$. We could say that it

achieves the same goal as that of a heat pump because it sends heat to the reservoir at higher temperature; nevertheless, its coefficient of performance is low because

$$\mathrm{COP}^{\mathrm{HP}}_{\text{Stirling cycle,(b)}} = \frac{|Q'_{\mathrm{h}}|}{|W'_{\text{cycle}}|} = \frac{nR\left(T_2 \ln r - \frac{\Delta T}{\gamma-1}\right)}{nR\Delta T \ln r} = \frac{T_2}{\Delta T} - \frac{1}{(\gamma-1)\ln r},$$

and we see that $0 < \mathrm{COP}^{\mathrm{HP}}_{\text{Stirling cycle, (b)}} < 1$.

(c) $T_1 \ln r - \frac{\Delta T}{\gamma-1} > 0 \Longleftrightarrow \ln r > \frac{\Delta T}{(\gamma-1)T_1} \Longleftrightarrow r > \exp\left[\frac{\Delta T}{(\gamma-1)T_1}\right]$.
In this case, $Q'_{\mathrm{h}} < 0$ and $Q'_{\mathrm{c}} > 0$, that is, the gas absorbs heat from the reservoir at lower temperature, and rejects heat to the reservoir at higher temperature; these properties define a heat pump/refrigerator. The corresponding coefficients of performance are

$$\mathrm{COP}^{\mathrm{HP}}_{\text{Stirling cycle, (c)}} = \frac{|Q'_{\mathrm{h}}|}{|W'_{\text{cycle}}|} = \frac{nR\left(T_2 \ln r - \frac{\Delta T}{\gamma-1}\right)}{nR\Delta T \ln r} = \frac{T_2}{\Delta T} - \frac{1}{(\gamma-1)\ln r}, \tag{22}$$

$$\mathrm{COP}^{\mathrm{R}}_{\text{Stirling cycle, (c)}} = \frac{|Q'_{\mathrm{c}}|}{|W'_{\text{cycle}}|} = \frac{nR\left(T_1 \ln r - \frac{\Delta T}{\gamma-1}\right)}{nR\Delta T \ln r} = \frac{T_1}{\Delta T} - \frac{1}{(\gamma-1)\ln r}, \tag{23}$$

with $1 < \mathrm{COP}^{\mathrm{HP}}_{\text{Stirling cycle, (c)}} < \mathrm{COP}^{\mathrm{HP}}_{\text{Carnot cycle}}$ and $0 < \mathrm{COP}^{\mathrm{R}}_{\text{Stirling cycle, (c)}} < \mathrm{COP}^{\mathrm{R}}_{\text{Carnot cycle}}$. These two coefficients verify the general relation (9), and using equations (13) and (20), we obtain

$$\begin{aligned}\mathrm{COP}^{\mathrm{HP}}_{\text{Stirling cycle, (c)}} &= \frac{1}{\epsilon_{\text{Carnot cycle}}} - \left(\frac{1}{\epsilon_{\text{Stirling cycle}}} - \frac{1}{\epsilon_{\text{Carnot cycle}}}\right)\\ &= \frac{2}{\epsilon_{\text{Carnot cycle}}} - \frac{1}{\epsilon_{\text{Stirling cycle}}};\end{aligned} \tag{24}$$

a similar relation is also satisfied by the Carnot heat pump, as is easy to prove.

Thus, we can say that the inversion of the cycle of the Stirling engine does not necessarily represent a heat pump/refrigerator; it only occurs if certain conditions are satisfied.

The change in the entropy of the universe during the inverted (counterclockwise) Stirling cycle is

$$\begin{aligned}\Delta S^{\text{universe}}_{\text{Stirling cycle, }\circlearrowleft} &= -\frac{Q'_{\mathrm{h}}}{T_{\mathrm{h}}} - \frac{Q'_{\mathrm{c}}}{T_{\mathrm{c}}} = \frac{nR\left(T_2 \ln r - \frac{\Delta T}{\gamma-1}\right)}{T_2} - \frac{nR\left(T_1 \ln r - \frac{\Delta T}{\gamma-1}\right)}{T_1}\\ &= \frac{nR}{\gamma-1}\Delta T\left(\frac{1}{T_1} - \frac{1}{T_2}\right) = \frac{nR}{\gamma-1}\frac{(\Delta T)^2}{T_1 T_2} > 0;\end{aligned} \tag{25}$$

this result confirms the irreversibility of this cycle and coincides with the result obtained for the clockwise cycle.

It is worth mentioning that conditions describing the previously referred situations (a), (b) and (c) can be written in an alternate form; if we look at the expressions of W'_{cycle} and $\Delta S^{\text{universe}}_{\text{Stirling cycle}}$, we see that, for situation (a),

$$\ln r < \frac{\Delta T}{(\gamma-1)T_2} \Longleftrightarrow nR\Delta T \ln r < \frac{nR}{\gamma-1}\frac{(\Delta T)^2}{T_2},$$

that is,

$$W'_{\text{cycle}} < T_1 \Delta S^{\text{universe}}_{\text{Stirling cycle}} \quad \text{(situation (a))}.$$

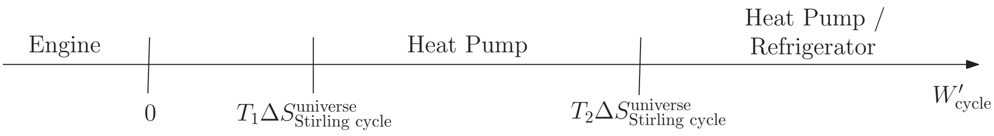

Figure 5. Behaviour of the Stirling cycle according to W'_{cycle} values.

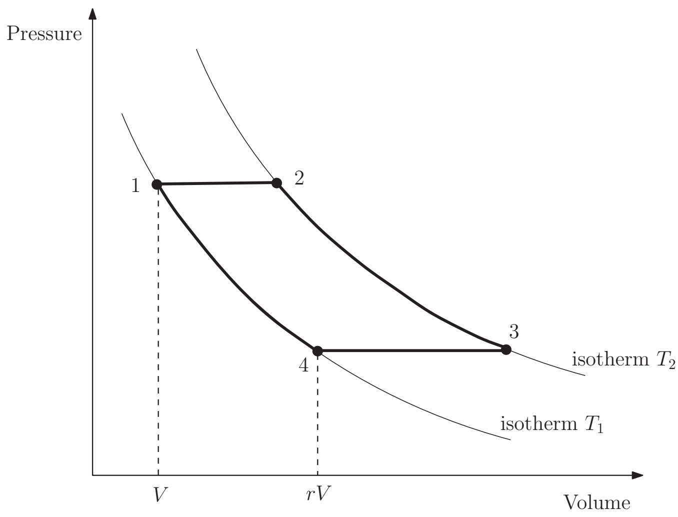

Figure 6. Volume−pressure diagram of the Ericsson cycle.

In a similar way, we obtain

$$T_1\Delta S^{\text{universe}}_{\text{Stirling cycle}} < W'_{\text{cycle}} < T_2\Delta S^{\text{universe}}_{\text{Stirling cycle}} \quad \text{(situation (b))}.$$

$$T_2\Delta S^{\text{universe}}_{\text{Stirling cycle}} < W'_{\text{cycle}} \quad \text{(situation (c))}.$$

Thus, we can interpret $T_1\Delta S^{\text{universe}}_{\text{Stirling cycle}}$ as the minimum work we have to do on the gas to implement a Stirling heat pump, and $T_2\Delta S^{\text{universe}}_{\text{Stirling cycle}}$ as the minimum work we have to do on the gas to implement a Stirling refrigerator. Figure 5 summarizes all these conclusions.

5. Ericsson cycle

Let us now consider the Ericsson cycle[5], shown in figure 6. This cycle also shows strong similarities with the Carnot cycle[6], and has two isothermal processes and two isobaric processes (which replace the adiabatic processes), where $r > 1$ and $T_1 < T_2$. Let $\Delta T \equiv T_2 - T_1 > 0$.

Since isothermal processes are described by the equation $pV = \text{constant}$, we have

$$\begin{cases} p_1V_1 = p_4V_4 & \Longrightarrow \quad \frac{p_1}{p_4} = \frac{V_4}{V_1} \\ p_2V_2 = p_3V_3 & \Longrightarrow \quad \frac{p_2}{p_3} = \frac{V_3}{V_2} \end{cases} \quad \Longrightarrow \quad \frac{V_3}{V_2} = \frac{V_4}{V_1} = r \tag{26}$$

[5] Used by John Ericsson (1803–89) in his external combustion engine in 1852.
[6] Once more, the compression ratio, the expansion ratio and the pressure ratio are equal in this cycle, as is clear in (26).

because $p_1 = p_2$ and $p_3 = p_4$. Let us assume that the cycle occurs in the clockwise direction. In processes $1 \to 2 \to 3$, the gas is in contact with a thermal reservoir at temperature $T_{\rm h} = T_2$, and in processes $3 \to 4 \to 1$, the gas is in contact with a thermal reservoir at temperature $T_{\rm c} = T_1$.

We have

$$Q_{12} = nc_p \Delta T_{12} = n\frac{\gamma R}{\gamma - 1}\Delta T\,; \quad Q_{23} = -W_{23} = nRT_2 \ln\left(\frac{V_3}{V_2}\right) = nRT_2 \ln r\,;$$
$$Q_{34} = nc_p \Delta T_{34} = -n\frac{\gamma R}{\gamma - 1}\Delta T\,; \quad Q_{41} = -W_{41} = -nRT_1 \ln\left(\frac{V_4}{V_1}\right) = -nRT_1 \ln r\,, \quad \text{so}$$
$$Q_{\rm h} = Q_{12} + Q_{23} = nR\left(T_2 \ln r + \frac{\gamma \Delta T}{\gamma - 1}\right) > 0\,,$$
$$Q_{\rm c} = Q_{34} + Q_{41} = -nR\left(T_1 \ln r + \frac{\gamma \Delta T}{\gamma - 1}\right) < 0\,,$$
$$Q_{\rm cycle} = Q_{\rm h} + Q_{\rm c} = nR\Delta T \ln r > 0\,, \quad W_{\rm cycle} = -Q_{\rm cycle} = -nR\Delta T \ln r < 0\,.$$

Note that the values of $Q_{\rm cycle}$ and $W_{\rm cycle}$ in the Ericsson cycle are the same values calculated in the Carnot and Stirling cycles. Since $Q_{\rm h} > 0$, $Q_{\rm c} < 0$ and $W_{\rm cycle} < 0$, the gas is absorbing heat from the reservoir at higher temperature, is rejecting heat to the reservoir at lower temperature, and is doing work, so this cycle behaves as a heat engine.

The efficiency of the Ericsson engine is given by

$$\begin{aligned}\epsilon_{\rm Ericsson\ cycle} &= \frac{|W_{\rm cycle}|}{|Q_{\rm h}|} = \frac{nR\Delta T \ln r}{nR\left(T_2 \ln r + \frac{\gamma\Delta T}{\gamma-1}\right)} = \frac{\Delta T \ln r}{T_2 \ln r\left[1 + \frac{\gamma \Delta T}{(\gamma-1)T_2 \ln r}\right]} \\ &= \frac{\frac{\Delta T}{T_2}}{1 + \frac{\gamma \frac{\Delta T}{T_2}}{(\gamma-1)\ln r}} = \frac{\epsilon_{\rm Carnot\ cycle}}{1 + \frac{\gamma\, \epsilon_{\rm Carnot\ cycle}}{(\gamma-1)\ln r}}\,,\end{aligned} \tag{27}$$

where we have used relation (13) for the efficiency of the Carnot engine. Comparing this expression with (19), we see that $0 < \epsilon_{\rm Ericsson\ cycle} < \epsilon_{\rm Stirling\ cycle} < \epsilon_{\rm Carnot\ cycle} < 1$ and also that $\epsilon_{\rm Ericsson\ cycle} \to \epsilon_{\rm Carnot\ cycle}$ if $r \to \infty$.

This can also be written as

$$\frac{1}{\epsilon_{\rm Ericsson\ cycle}} - \frac{1}{\epsilon_{\rm Carnot\ cycle}} = \frac{\gamma}{(\gamma - 1)\ln r}\,. \tag{28}$$

The entropy change of the universe during a cycle of the Ericsson engine is

$$\begin{aligned}\Delta S^{\rm universe}_{\rm Ericsson\ cycle,\,\circlearrowright} &= -\frac{Q_{\rm h}}{T_{\rm h}} - \frac{Q_{\rm c}}{T_{\rm c}} = -\frac{nR\left(T_2 \ln r + \frac{\gamma\Delta T}{\gamma-1}\right)}{T_2} + \frac{nR\left(T_1 \ln r + \frac{\gamma\Delta T}{\gamma-1}\right)}{T_1} \\ &= \frac{nR\gamma}{\gamma - 1}\Delta T\left(\frac{1}{T_1} - \frac{1}{T_2}\right) = \frac{nR\gamma}{\gamma - 1}\frac{(\Delta T)^2}{T_1 T_2} > 0\,,\end{aligned} \tag{29}$$

so the clockwise Ericsson cycle is irreversible.

Let us try, once again, to transform this heat engine into a heat pump/refrigerator. In order to do so, we invert the cycle to the counterclockwise direction and carefully identify the reservoirs that are in contact with the gas in the various processes. In processes $1 \to 4$ and $2 \to 1$ the gas is in contact with a reservoir at temperature $T'_{\rm c} = T_{\rm c} = T_1$, and in processes $4 \to 3 \to 2$ the gas is in contact with a reservoir at temperature $T'_{\rm h} = T_{\rm h} = T_2$. So

$$Q_{14} = -\,Q_{41} = nRT_1 \ln r; \qquad Q_{43} = -\,Q_{34} = n\frac{\gamma R}{\gamma - 1}\Delta T\,;$$

$$Q_{32} = -Q_{23} = -nRT_2 \ln r\,; \quad Q_{21} = -Q_{12} = -n\frac{\gamma R}{\gamma - 1}\Delta T\,, \quad \text{so}$$

$$Q'_{\rm h} = Q_{43} + Q_{32} = -nR\left(T_2 \ln r - \frac{\gamma \Delta T}{\gamma - 1}\right), \quad Q'_{\rm c} = Q_{14} + Q_{21} = nR\left(T_1 \ln r - \frac{\gamma \Delta T}{\gamma - 1}\right),$$

$$Q'_{\rm cycle} = Q'_{\rm h} + Q'_{\rm c} = -nR\Delta T \ln r < 0\,, \quad W'_{\rm cycle} = -Q_{\rm cycle} = nR\Delta T \ln r > 0\,.$$

Once more, it is instructive to compare $Q'_{\rm h}$ and $Q'_{\rm c}$ with $Q_{\rm h}$ and $Q_{\rm c}$, respectively.

As in the previous section, work is done on the gas but there is no warranty that $Q'_{\rm h} < 0$ (the main condition representing a heat pump) or $Q'_{\rm c} > 0$ (the main condition representing a refrigerator). We have three possible scenarios:

(a) $r < \exp\left[\frac{\gamma\Delta T}{(\gamma-1)T_2}\right]$; (b) $\exp\left[\frac{\gamma\Delta T}{(\gamma-1)T_2}\right] < r < \exp\left[\frac{\gamma\Delta T}{(\gamma-1)T_1}\right]$; (c) $r > \exp\left[\frac{\gamma\Delta T}{(\gamma-1)T_1}\right]$.

Scenario (a) is shown in figure 4(a) and does not correspond to a heat pump, nor to a refrigerator. Scenario (b) is shown in figure 4(b); we can say that it behaves as a heat pump with a low coefficient of performance, given by

$$\mathrm{COP}^{\rm HP}_{\text{Ericsson cycle, (b)}} = \frac{|Q'_{\rm h}|}{|W'_{\rm cycle}|} = \frac{nR\left(T_2 \ln r - \frac{\gamma\Delta T}{\gamma-1}\right)}{nR\Delta T \ln r} = \frac{T_2}{\Delta T} - \frac{\gamma}{(\gamma-1)\ln r},$$

and we can write $0 < \mathrm{COP}^{\rm HP}_{\text{Ericsson cycle, (b)}} < \mathrm{COP}^{\rm HP}_{\text{Stirling cycle, (b)}} < 1$. Scenario (c) is unique that corresponds to a heat pump/refrigerator, with the coefficients of performance

$$\mathrm{COP}^{\rm HP}_{\text{Ericsson cycle, (c)}} = \frac{|Q'_{\rm h}|}{|W'_{\rm cycle}|} = \frac{nR\left(T_2 \ln r - \frac{\gamma\Delta T}{\gamma-1}\right)}{nR\Delta T \ln r} = \frac{T_2}{\Delta T} - \frac{\gamma}{(\gamma-1)\ln r}\,, \tag{30}$$

$$\mathrm{COP}^{\rm R}_{\text{Ericsson cycle, (c)}} = \frac{|Q'_{\rm c}|}{|W'_{\rm cycle}|} = \frac{nR\left(T_1 \ln r - \frac{\gamma\Delta T}{\gamma-1}\right)}{nR\Delta T \ln r} = \frac{T_1}{\Delta T} - \frac{\gamma}{(\gamma-1)\ln r}\,. \tag{31}$$

These satisfy relations $1 < \mathrm{COP}^{\rm HP}_{\text{Ericsson cycle, (c)}} < \mathrm{COP}^{\rm HP}_{\text{Stirling cycle,(c)}} < \mathrm{COP}^{\rm HP}_{\text{Carnot cycle}}$ and $0 < \mathrm{COP}^{\rm R}_{\text{Ericsson cycle, (c)}} < \mathrm{COP}^{\rm R}_{\text{Stirling cycle,(c)}} < \mathrm{COP}^{\rm R}_{\text{Carnot cycle}}$, and verify the general relation (9) and still another relation similar to (24):

$$\begin{aligned} \mathrm{COP}^{\rm HP}_{\text{Ericsson cycle, (c)}} &= \frac{1}{\epsilon_{\text{Carnot cycle}}} - \left(\frac{1}{\epsilon_{\text{Ericsson cycle}}} - \frac{1}{\epsilon_{\text{Carnot cycle}}}\right) \\ &= \frac{2}{\epsilon_{\text{Carnot cycle}}} - \frac{1}{\epsilon_{\text{Ericsson cycle}}}\,. \end{aligned} \tag{32}$$

Thus, the inversion of the cycle does not necessarily represent a heat pump/refrigerator.

The entropy change of the universe after one counterclockwise Ericsson cycle is

$$\begin{aligned} \Delta S^{\rm universe}_{\text{Ericsson cycle, }\circlearrowleft} &= -\frac{Q'_{\rm h}}{T_{\rm h}} - \frac{Q'_{\rm c}}{T_{\rm c}} = \frac{nR\left(T_2 \ln r - \frac{\gamma\Delta T}{\gamma-1}\right)}{T_2} - \frac{nR\left(T_1 \ln r - \frac{\gamma\Delta T}{\gamma-1}\right)}{T_1} \\ &= \frac{nR\gamma}{\gamma-1}\Delta T\left(\frac{1}{T_1} - \frac{1}{T_2}\right) = \frac{nR\gamma}{\gamma-1}\frac{(\Delta T)^2}{T_1 T_2} > 0\,; \end{aligned} \tag{33}$$

this is the same result as that of the clockwise cycle.

Finally, and in a similar way as we did for the Stirling cycle, previous scenarios (a)–(c) can be written as

$$W'_{\rm cycle} < T_1 \Delta S^{\rm universe}_{\text{Ericsson cycle}} \quad \text{(scenario (a))},$$

$$T_1 \Delta S^{\text{universe}}_{\text{Ericsson cycle}} < W'_{\text{cycle}} < T_2 \Delta S^{\text{universe}}_{\text{Ericsson cycle}} \quad \text{(scenario (b))},$$

$$T_2 \Delta S^{\text{universe}}_{\text{Ericsson cycle}} < W'_{\text{cycle}} \qquad \text{(scenario (c))},$$

so we can interpret $T_1 \Delta S^{\text{universe}}_{\text{Ericsson cycle}}$ as the minimum work we have to do on the gas to get an Ericsson heat pump, and $T_2 \Delta S^{\text{universe}}_{\text{Ericsson cycle}}$ as the minimum work we have to do on the gas to get an Ericsson refrigerator. A diagram similar to figure 5 would show these results.

It is worth observing that $Q_{12} + Q_{34} = 0$ in both Stirling and Ericsson cycles, that is, the heats exchanged by the system during the non-isothermal processes are given by opposite numbers. This fact has been explored in engineering through the concept of regeneration: we include in the system a device (called the regenerator) which stores the heat rejected by the gas during process $3 \to 4$ and transfers it back to the gas in process $1 \to 2$. It is easy to prove that the Stirling and Ericsson cycles with regeneration have the same efficiency and the same coefficients of performance as the Carnot cycle. The reader interested in this topic could find it in most engineering thermodynamics books [4].

In this section, and in the previous section, we have considered cycles that maintain the two isothermal processes of the Carnot cycle and replace the two adiabatic processes by isobaric and isochoric processes, respectively. In the following sections, we will maintain the adiabatic processes of the Carnot cycle and replace the isothermal processes.

6. Otto cycle

Let us consider the Otto cycle[7] shown in figure 7, with two adiabatic processes and two isochoric processes (which replace the isothermal processes in the Carnot cycle), where $r > 1$ is the compression ratio. Observe that here the parameter r relates the extreme volumes of the gas along the adiabatic processes instead of along the isothermal processes[8]. The points of the cycle with the highest and lowest temperatures are also shown. We define $\Delta T \equiv T_3 - T_2 > 0$. (The reason for choosing $T_3 - T_2$ instead of $T_3 - T_1$ will be clear soon.)

Since one of the equations of adiabatic curves is $T V^{\gamma-1} = \text{constant}$, we can write

$$\begin{cases} T_2 V_2^{\gamma-1} = T_1 V_1^{\gamma-1} \\ T_3 V_3^{\gamma-1} = T_4 V_4^{\gamma-1} \end{cases} \implies \begin{cases} \dfrac{T_2}{T_1} = \left(\dfrac{V_1}{V_2}\right)^{\gamma-1} = r^{\gamma-1} \\ \dfrac{T_3}{T_4} = \left(\dfrac{V_4}{V_3}\right)^{\gamma-1} = r^{\gamma-1} \end{cases} \implies \frac{T_2}{T_3} = \frac{T_1}{T_4} < 1\,. \qquad (34)$$

Let us assume that the cycle is realized in the clockwise direction. In processes $1 \to 2$ and $3 \to 4$, the gas is thermally isolated; in process $2 \to 3$, the gas is in contact with a thermal reservoir at temperature $T_{\text{h}} = T_3$, and in process $4 \to 1$, the gas is in contact with a thermal reservoir at temperature $T_{\text{c}} = T_1$.

We have

$$Q_{12} = 0\,; \quad Q_{23} = nc_V \Delta T_{23} = n\frac{R}{\gamma - 1}(T_3 - T_2) = \frac{nR\Delta T}{\gamma - 1}\,; \quad Q_{34} = 0\,;$$

$$Q_{41} = nc_V \Delta T_{41} = -n\frac{R}{\gamma - 1}(T_4 - T_1) = -\frac{nRT_4}{\gamma - 1}\left(1 - \frac{T_1}{T_4}\right) = -\frac{nRT_4}{\gamma - 1}\left(1 - \frac{T_2}{T_3}\right)$$

[7] In honour of Nikolaus August Otto (1832–91), who designed an internal combustion engine with this cycle in 1862. The Otto cycle is also known as the Beau de Rochas cycle because Alphonse Beau de Rochas (1815–93) actually was the first to patent a four-cycle engine using this cycle, in 1861.

[8] The same observation applies to the other parameters, r' and r_{p}. In the Otto cycle the expansion ratio and the compression ratio are automatically equal (r), but the pressure ratio r_{p} is higher: $r_{\text{p}} \equiv \frac{p_2}{p_1} = \frac{p_3}{p_4} = r^\gamma > r$; this relation is easily proved by using the equation $pV^\gamma = \text{constant}$ of adiabatic curves.

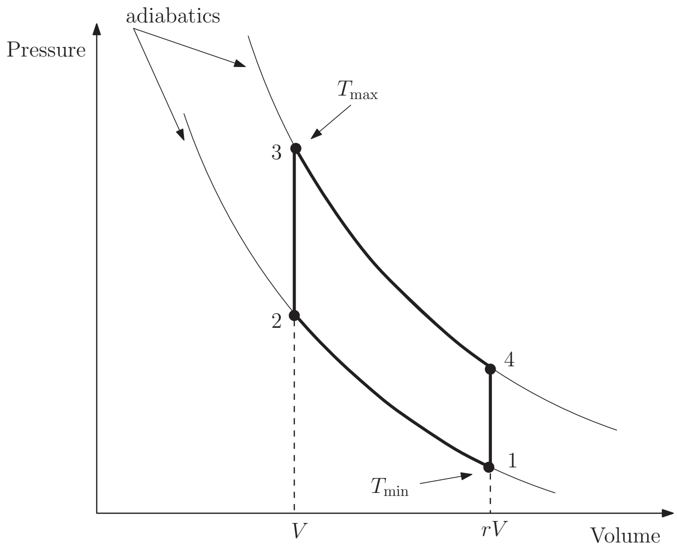

Figure 7. Volume–pressure diagram of the Otto cycle.

$$= -\frac{nRT_4}{\gamma - 1}\frac{T_3 - T_2}{T_3} = -\frac{nR\Delta T}{\gamma - 1}\frac{T_4}{T_3} = -\frac{nR\Delta T}{\gamma - 1}\frac{1}{r^{\gamma-1}}, \quad \text{so}$$

$$Q_\mathrm{h} = Q_{23} = \frac{nR\Delta T}{\gamma - 1} > 0\,, \qquad Q_\mathrm{c} = Q_{41} = -\frac{nR\Delta T}{\gamma - 1}\frac{1}{r^{\gamma-1}} < 0\,,$$

$$Q_\mathrm{cycle} = Q_\mathrm{h} + Q_\mathrm{c} = \frac{nR\Delta T}{\gamma - 1}\left(1 - \frac{1}{r^{\gamma-1}}\right) > 0,$$

$$W_\mathrm{cycle} = -Q_\mathrm{cycle} = -\frac{nR\Delta T}{\gamma - 1}\left(1 - \frac{1}{r^{\gamma-1}}\right) < 0\,.$$

Thus, the gas is absorbing heat from the reservoir at higher temperature, is rejecting heat to the reservoir at lower temperature and is doing work; this defines a heat engine.

The efficiency of the Otto engine is given by

$$\epsilon_\text{Otto cycle} = \frac{|W_\mathrm{cycle}|}{|Q_\mathrm{h}|} = \frac{\frac{nR\Delta T}{\gamma-1}\left(1 - \frac{1}{r^{\gamma-1}}\right)}{\frac{nR\Delta T}{\gamma-1}} = 1 - \frac{1}{r^{\gamma-1}}\ , \tag{35}$$

and it depends only on the compression ratio and on the nature of the gas. Since T_1 and T_3 are the temperatures of the two reservoirs, it is clear that $\epsilon_\text{Otto cycle} < \epsilon_\text{Carnot cycle}$ because $\epsilon_\text{Carnot cycle} = 1 - \frac{T_1}{T_3}$, whilst $\epsilon_\text{Otto cycle} = 1 - \frac{T_4}{T_3}$, and $T_4 > T_1$.

The entropy change of the universe after one cycle of the Otto engine is

$$\Delta S^\text{universe}_{\text{Otto cycle},\,\circlearrowright} = -\frac{Q_\mathrm{h}}{T_\mathrm{h}} - \frac{Q_\mathrm{c}}{T_\mathrm{c}} = -\frac{\frac{nR\Delta T}{\gamma-1}}{T_3} + \frac{\frac{nR\Delta T}{\gamma-1}\frac{T_4}{T_3}}{T_1} = \frac{nR\Delta T}{(\gamma-1)T_3}\left(\frac{T_4}{T_1} - 1\right)$$

$$= \frac{nR\Delta T}{(\gamma-1)T_3}\left(\frac{T_3}{T_2} - 1\right) = \frac{nR}{\gamma - 1}\frac{(\Delta T)^2}{T_2 T_3} > 0\,, \tag{36}$$

so the clockwise Otto cycle is also irreversible. It should be noted that, opposite to previous cycles, ΔT does not represent here the temperature difference between the two reservoirs, but the range of temperatures over which the irreversible heating process occurs. This new interpretation can be extended to previous cycles. Also note that $\Delta S^{\text{universe}}_{\text{Otto cycle},\circlearrowright}$ can be written in the form

$$\Delta S^{\text{universe}}_{\text{Otto cycle},\circlearrowright} = \frac{nR}{\gamma-1}\left(\frac{T_4-T_1}{T_1}\right)\left(\frac{T_3-T_2}{T_3}\right) = \frac{nR}{\gamma-1}\left(\frac{\Delta T_{\text{c}}}{T_{\text{c}}}\right)\left(\frac{\Delta T_{\text{h}}}{T_{\text{h}}}\right), \tag{37}$$

where $\Delta T_{\text{c}} \equiv T_4 - T_1 = |T_1 - T_4|$ is the range of temperatures over which the irreversible cooling process occurs, and $\Delta T_{\text{h}} \equiv T_3 - T_2 = |T_3 - T_2|$ is the range of temperatures over which the irreversible heating process occurs.

Let us try to transform this heat engine into a heat pump/refrigerator. In order to do so, we invert the cycle to the counterclockwise direction and carefully identify the reservoirs that are in contact with the gas in the various processes.

In this case, we have an interesting situation: there is no need to employ the same thermal reservoirs used in the clockwise cycle[9] (temperatures T_3 and T_1); we can use reservoirs at temperatures T_4 and T_2. Thus, in process $1 \to 4$, the gas is in contact with a reservoir at temperature T_4; in processes $4 \to 3$ and $2 \to 1$, the gas is thermally isolated; and in process $3 \to 2$, the gas is in contact with a reservoir at temperature T_2. We have

$$Q_{14} = -Q_{41} = \frac{nR\Delta T}{\gamma-1}\frac{1}{r^{\gamma-1}}\,; \quad Q_{43} = 0\,; \quad Q_{32} = -Q_{23} = -\frac{nR\Delta T}{\gamma-1}\,; \quad Q_{21} = 0\,.$$

Since $Q_{14} > 0$ and $Q_{32} < 0$, if we want this cycle to represent a heat pump/refrigerator we have to be sure that $Q_{14} = Q'_{\text{c}}$ and $Q_{32} = Q'_{\text{h}}$, that is, T_4 should be the temperature of the reservoir at lower temperature, and T_2 must be the temperature of the reservoir at higher temperature; that is possible only if $T_4 < T_2$ (it should be noted, looking at figure 7, that this condition is not guaranteed *a priori*). Thus we additionally assume that $T'_{\text{c}} = T_4 < T_2 = T'_{\text{h}}$ and we will have

$$Q'_{\text{h}} = Q_{32} = -\frac{nR\Delta T}{\gamma-1} < 0, \quad Q'_{\text{c}} = Q_{14} = \frac{nR\Delta T}{\gamma-1}\frac{1}{r^{\gamma-1}} > 0\,,$$

$$Q'_{\text{cycle}} = Q'_{\text{h}} + Q'_{\text{c}} = -\frac{nR\Delta T}{\gamma-1}\left(1-\frac{1}{r^{\gamma-1}}\right) < 0\,,$$

$$W'_{\text{cycle}} = -Q'_{\text{cycle}} = \frac{nR\Delta T}{\gamma-1}\left(1-\frac{1}{r^{\gamma-1}}\right) > 0.$$

The coefficients of performance of this inverted Otto cycle are

$$\text{COP}^{\text{HP}}_{\text{Otto cycle}} = \frac{|Q'_{\text{h}}|}{|W'_{\text{cycle}}|} = \frac{\frac{nR\Delta T}{\gamma-1}}{\frac{nR\Delta T}{\gamma-1}\left(1-\frac{1}{r^{\gamma-1}}\right)} = \frac{1}{1-\frac{1}{r^{\gamma-1}}} > 1\,, \tag{38}$$

$$\text{COP}^{\text{R}}_{\text{Otto cycle}} = \frac{|Q'_{\text{c}}|}{|W'_{\text{cycle}}|} = \frac{\frac{nR\Delta T}{\gamma-1}\frac{1}{r^{\gamma-1}}}{\frac{nR\Delta T}{\gamma-1}\left(1-\frac{1}{r^{\gamma-1}}\right)} = \frac{\frac{1}{r^{\gamma-1}}}{1-\frac{1}{r^{\gamma-1}}} > 0\,. \tag{39}$$

These two coefficients verify the general relation (9) and

$$\text{COP}^{\text{HP}}_{\text{Otto cycle}} = \frac{1}{\epsilon_{\text{Otto cycle}}}\,. \tag{40}$$

The entropy change of the universe during the counterclockwise Otto cycle is

$$\Delta S^{\text{universe}}_{\text{Otto cycle},\circlearrowleft} = -\frac{Q'_{\text{h}}}{T'_{\text{h}}} - \frac{Q'_{\text{c}}}{T'_{\text{c}}} = \frac{\frac{nR\Delta T}{\gamma-1}}{T_2} - \frac{\frac{nR\Delta T}{\gamma-1}\frac{T_4}{T_3}}{T_4} = \frac{nR\Delta T}{\gamma-1}\left(\frac{1}{T_2}-\frac{1}{T_3}\right)$$
$$= \frac{nR}{\gamma-1}\frac{(\Delta T)^2}{T_2T_3} > 0\,, \tag{41}$$

[9] Actually, if we try to use the same reservoirs we will get a situation of the type shown in figure 4(a), which does not represent a heat engine, nor a heat pump/refrigerator. We leave the details to the reader.

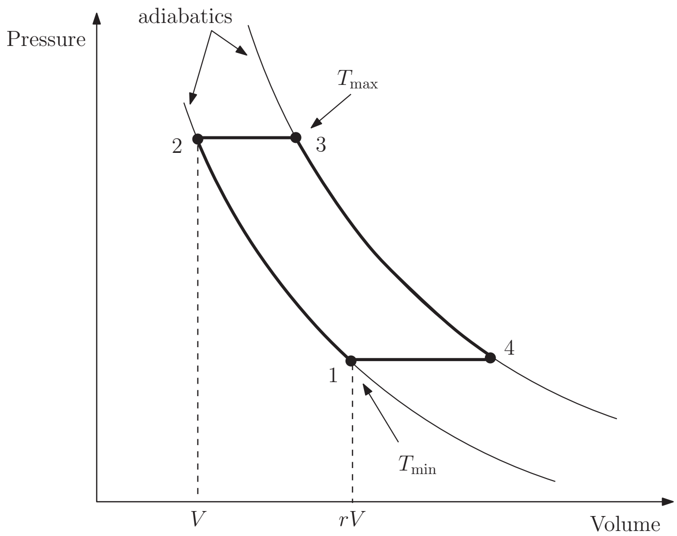

Figure 8. Volume–pressure diagram of the Joule cycle.

the same result obtained for the clockwise cycle, so we can write it in the form (37).

Finally, comparing the expressions for W'_{cycle} and $\Delta S^{\text{universe}}_{\text{Otto cycle}}$, we can readily show that condition $T_4 < T_2$ is equivalent to condition $W'_{\text{cycle}} > T_2 \Delta S^{\text{universe}}_{\text{Otto cycle}}$; this allows us, as in previous sections, to interpret the value of $T_2 \Delta S^{\text{universe}}_{\text{Otto cycle}}$ as the minimum work that we have to do on the gas to implement an Otto heat pump/refrigerator. Note that T_2 represents the temperature T'_{h} of the hot reservoir used in the heat pump/refrigerator.

7. Joule cycle

Let us consider the Joule cycle shown in figure 8, with two adiabatic processes and two isobaric processes (which substitute the isothermal processes in the Carnot cycle), where $r > 1$ is the compression ratio[10]. Depending on the context[11], this cycle is also known as the Brayton cycle, Stoddard cycle, Rankine cycle or Bell–Coleman cycle. The points of the cycle with the highest and lowest temperatures are also shown. We define, as before, $\Delta T \equiv T_3 - T_2 > 0$.

[10] In the Joule cycle, as in the Otto cycle, the expansion and compression ratios have the same value (r) and the pressure ratio $r_{\text{p}} = r^{\gamma}$ has a higher value, as we can see in (42).

[11] The designations Joule cycle and Brayton cycle have historical reason: although the cycle was used by Ericsson in 1833, it was not successful at the time; James Prescott Joule (1818–89) proposed it in 1851 and George Brayton (1830–92) was the first to implement it with success in 1872. In both cases, the cycle is related to a heat engine whose working fluid is a gas, with no phase changes (e.g. gas turbine). The name Stoddard cycle is in honour of Elliott Joseph Stoddard (1859–?), who used it in his 1919 and 1933 external combustion engines. The designation Rankine cycle, in honour of William John Macquorn Rankine (1820–72), applies when the working fluid used in the heat engine (typically steam) suffers a phase change during the cycle. The name Bell–Coleman cycle refers to the counterclockwise (refrigeration) cycle used by Henry Bell (1848–1931), John Bell (1850–1929) and Joseph James Coleman (1838–88).

One equation for adiabatics is pV^γ = constant; then, since $p_2 = p_3$ and $p_1 = p_4$, we have

$$\begin{cases} p_2 V_2^\gamma = p_1 V_1^\gamma \\ p_3 V_3^\gamma = p_4 V_4^\gamma \end{cases} \Longrightarrow \begin{cases} \dfrac{p_2}{p_1} = \left(\dfrac{V_1}{V_2}\right)^\gamma = r^\gamma \\ \dfrac{p_3}{p_4} = \left(\dfrac{V_4}{V_3}\right)^\gamma \end{cases} \Longrightarrow \frac{V_4}{V_3} = \frac{V_1}{V_2} = r\,. \tag{42}$$

Adiabatics are also described by $TV^{\gamma-1}$ = constant or $T^\gamma p^{1-\gamma}$ = constant; then

$$\begin{cases} T_2 V_2^{\gamma-1} = T_1 V_1^{\gamma-1} \quad \Longrightarrow \quad \frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1} = r^{\gamma-1} \\ \begin{cases} T_2^\gamma p_2^{1-\gamma} = T_1^\gamma p_1^{1-\gamma} \\ T_3^\gamma p_3^{1-\gamma} = T_4^\gamma p_4^{1-\gamma} \end{cases} \Longrightarrow \left(\frac{T_2}{T_3}\right)^\gamma = \left(\frac{T_1}{T_4}\right)^\gamma \Longrightarrow \frac{T_2}{T_3} = \frac{T_1}{T_4} \end{cases} \Longrightarrow \frac{T_3}{T_4} = \frac{T_2}{T_1} = r^{\gamma-1}\,. \tag{43}$$

Let us assume that the cycle operates in the clockwise direction. In processes $1 \to 2$ and $3 \to 4$, the gas is thermally isolated; in process $2 \to 3$, the gas is in contact with a thermal reservoir at temperature $T_\mathrm{h} = T_3$; and in process $4 \to 1$, the gas is in contact with a thermal reservoir at temperature $T_\mathrm{c} = T_1$. We have

$$Q_{12} = 0\,; \quad Q_{23} = nc_p \Delta T_{23} = n\frac{R}{\gamma-1}(T_3 - T_2) = \frac{nR\Delta T\gamma}{\gamma-1}\;; \quad Q_{34} = 0\,;$$

$$\begin{aligned} Q_{41} &= nc_p\Delta T_{41} = -n\frac{\gamma R}{\gamma-1}(T_4 - T_1) = -\frac{nR\gamma T_4}{\gamma-1}\left(1 - \frac{T_1}{T_4}\right) = -\frac{nR\gamma T_4}{\gamma-1}\left(1 - \frac{T_2}{T_3}\right) \\ &= -\frac{nR\gamma T_4}{\gamma-1}\frac{T_3 - T_2}{T_3} = -\frac{nR\Delta T\gamma}{\gamma-1}\frac{T_4}{T_3} = -\frac{nR\Delta T\gamma}{\gamma-1}\frac{1}{r^{\gamma-1}}, \quad \text{so} \end{aligned}$$

$$Q_\mathrm{h} = Q_{23} = \frac{nR\Delta T\gamma}{\gamma-1} > 0\,, \quad Q_\mathrm{c} = Q_{41} = -\frac{nR\Delta T\gamma}{\gamma-1}\frac{1}{r^{\gamma-1}} < 0\,,$$

$$Q_\mathrm{cycle} = Q_\mathrm{h} + Q_\mathrm{c} = \frac{nR\Delta T\gamma}{\gamma-1}\left(1 - \frac{1}{r^{\gamma-1}}\right) > 0\,,$$

$$W_\mathrm{cycle} = -Q_\mathrm{cycle} = -\frac{nR\Delta T\gamma}{\gamma-1}\left(1 - \frac{1}{r^{\gamma-1}}\right) < 0\,.$$

Thus, the gas is absorbing heat from the reservoir at higher temperature, is rejecting heat to the reservoir at lower temperature and is doing work; thus we have a heat engine.

The efficiency of the Joule engine is given by[12]

$$\epsilon_\text{Joule cycle} = \frac{|W_\mathrm{cycle}|}{|Q_\mathrm{h}|} = \frac{\frac{nR\Delta T\gamma}{\gamma-1}\left(1 - \frac{1}{r^{\gamma-1}}\right)}{\frac{nR\Delta T\gamma}{\gamma-1}} = 1 - \frac{1}{r^{\gamma-1}}\;, \tag{44}$$

and is equal to the efficiency of the Otto engine; it depends only on the compression ratio and on the nature of the gas.

The entropy change of the universe after one cycle of the Joule engine is

$$\begin{aligned} \Delta S^\text{universe}_{\text{Joule cycle}, \circlearrowright} &= -\frac{Q_\mathrm{h}}{T_\mathrm{h}} - \frac{Q_\mathrm{c}}{T_\mathrm{c}} = -\frac{\frac{nR\Delta T\gamma}{\gamma-1}}{T_3} + \frac{\frac{nR\Delta T\gamma}{\gamma-1}\frac{T_1}{T_2}}{T_1} = \frac{nR\Delta T\gamma}{\gamma-1}\left(\frac{1}{T_2} - \frac{1}{T_3}\right) \\ &= \frac{nR\gamma}{\gamma-1}\frac{(\Delta T)^2}{T_2 T_3} > 0\,, \end{aligned} \tag{45}$$

so the clockwise Joule cycle is also irreversible.

[12] Sometimes this efficiency is expressed as a function of the pressure ratio r_p; since $r = r_\mathrm{p}^{1/\gamma}$, we can write $\epsilon_\text{Joule cycle} = 1 - \frac{1}{r_p^{(\gamma-1)/\gamma}}$.

As in the previous section, ΔT does not represent the temperature difference between the two reservoirs, but the range of temperatures over which the irreversible heating process occurs. And it is also possible to rewrite this entropy change in the form

$$\Delta S^{\text{universe}}_{\text{Joule cycle, }\circlearrowright} = \frac{nR\gamma}{\gamma-1}\left(\frac{\Delta T_{\text{c}}}{T_{\text{c}}}\right)\left(\frac{\Delta T_{\text{h}}}{T_{\text{h}}}\right). \tag{46}$$

In order to transform this heat engine into a heat pump/refrigerator, we proceed as we did in the Otto cycle: we invert the cycle, realizing it in the counterclockwise direction, and we use two new reservoirs at temperatures T_4 and T_2. In process $1 \to 4$, the gas is in contact with a reservoir at temperature T_4; in processes $4 \to 3$ and $2 \to 1$, the gas is thermally isolated; in process $3 \to 2$, the gas is in contact with a reservoir at temperature T_2. We have

$$Q_{14} = -Q_{41} = \frac{nR\Delta T\gamma}{\gamma-1}\frac{1}{r^{\gamma-1}}\,; \quad Q_{43} = 0\,; \quad Q_{32} = -Q_{23} = -\frac{nR\Delta T\gamma}{\gamma-1}\,; \quad Q_{21} = 0\,.$$

Imposing now the condition $T'_{\text{c}} = T_4 < T_2 = T'_{\text{h}}$, we will have

$$Q'_{\text{h}} = Q_{32} = -\frac{nR\Delta T\gamma}{\gamma-1} < 0\,, \quad Q'_{\text{c}} = Q_{14} = \frac{nR\Delta T\gamma}{\gamma-1}\frac{1}{r^{\gamma-1}} > 0\,,$$

$$Q'_{\text{cycle}} = Q'_{\text{h}} + Q'_{\text{c}} = -\frac{nR\Delta T\gamma}{\gamma-1}\left(1-\frac{1}{r^{\gamma-1}}\right) < 0\,,$$

$$W'_{\text{cycle}} = -Q'_{\text{cycle}} = \frac{nR\Delta T\gamma}{\gamma-1}\left(1-\frac{1}{r^{\gamma-1}}\right) > 0\,.$$

The coefficients of performance of this inverted Joule cycle are[13]

$$\text{COP}^{\text{HP}}_{\text{Joule cycle}} = \frac{|Q'_{\text{h}}|}{|W'_{\text{cycle}}|} = \frac{\frac{nR\Delta T\gamma}{\gamma-1}}{\frac{nR\Delta T\gamma}{\gamma-1}\left(1-\frac{1}{r^{\gamma-1}}\right)} = \frac{1}{1-\frac{1}{r^{\gamma-1}}} > 1\,, \tag{47}$$

$$\text{COP}^{\text{R}}_{\text{Joule cycle}} = \frac{|Q'_{\text{c}}|}{|W'_{\text{cycle}}|} = \frac{\frac{nR\Delta T\gamma}{\gamma-1}\frac{1}{r^{\gamma-1}}}{\frac{nR\Delta T\gamma}{\gamma-1}\left(1-\frac{1}{r^{\gamma-1}}\right)} = \frac{\frac{1}{r^{\gamma-1}}}{1-\frac{1}{r^{\gamma-1}}} > 0\,. \tag{48}$$

These two coefficients verify the general relation (9) and

$$\text{COP}^{\text{HP}}_{\text{Joule cycle}} = \frac{1}{\epsilon_{\text{Joule cycle}}}\,. \tag{49}$$

The entropy change of the universe after one counterclockwise Joule cycle is

$$\begin{aligned}\Delta S^{\text{universe}}_{\text{Joule cycle, }\circlearrowleft} &= -\frac{Q'_{\text{h}}}{T'_{\text{h}}} - \frac{Q'_{\text{c}}}{T'_{\text{c}}} = \frac{\frac{nR\Delta T\gamma}{\gamma-1}}{T_2} - \frac{\frac{nR\Delta T\gamma}{\gamma-1}\frac{T_4}{T_3}}{T_4} = \frac{nR\Delta T\gamma}{\gamma-1}\left(\frac{1}{T_2}-\frac{1}{T_3}\right)\\ &= \frac{nR\gamma}{\gamma-1}\frac{(\Delta T)^2}{T_2T_3} > 0\,,\end{aligned} \tag{50}$$

the same result as the clockwise cycle; thus it can also be expressed in the form (46).

Finally, as in the Otto cycle, condition $T_4 < T_2$ is equivalent to $W'_{\text{cycle}} > T'_{\text{h}}\Delta S^{\text{universe}}_{\text{Joule cycle}}$, so we can make the same interpretation as the previous sections.

[13] These coefficients of performance can also be written as $\text{COP}^{\text{HP}}_{\text{Joule cycle}} = \frac{1}{1-\frac{1}{r_p^{(\gamma-1)/\gamma}}}$ and $\text{COP}^{\text{R}}_{\text{Joule cycle}} = \frac{\frac{1}{r_p^{(\gamma-1)/\gamma}}}{1-\frac{1}{r_p^{(\gamma-1)/\gamma}}} = \frac{1}{r_p^{(\gamma-1)/\gamma}-1}$.

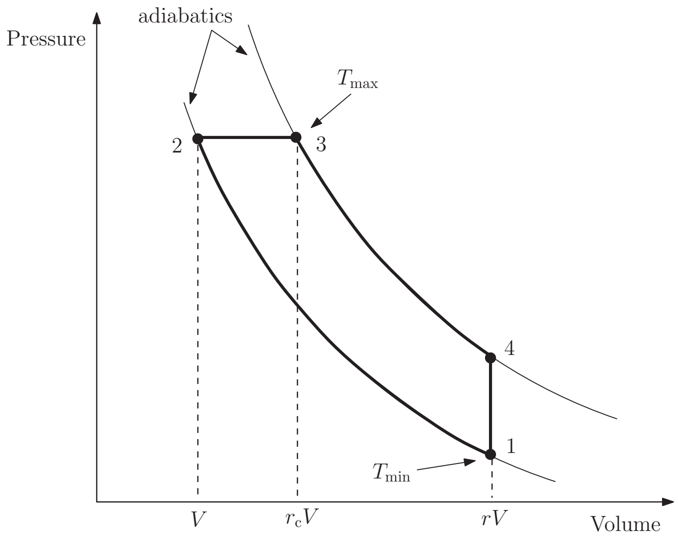

Figure 9. Volume–pressure diagram of the Diesel cycle.

8. Diesel cycle

Let us now consider the Diesel cycle[14] shown in figure 9, with two adiabatic processes, one isobaric process and one isochoric process, where $r > r_c > 1$. The parameter r_c is called the cut-off ratio[15]. Note that, in contrast to all previous cycles, this cycle contains three different kinds of processes instead of two. The points of the cycle with the highest and lowest temperatures are also shown.

Since $p_2 = p_3$, we can write

$$\frac{T_2}{V_2} = \frac{T_3}{V_3} \implies \frac{T_3}{T_2} = \frac{V_3}{V_2} = r_c \ . \tag{51}$$

Since adiabatics can be described by $T V^{\gamma-1} =$ constant, we have

$$\begin{cases} T_2 V_2^{\gamma-1} = T_1 V_1^{\gamma-1} & \implies \frac{T_2}{T_1} = \left(\frac{V_1}{V_2}\right)^{\gamma-1} = r^{\gamma-1} & \implies \frac{T_3}{T_1} = \frac{T_3}{T_2}\frac{T_2}{T_1} = r_c r^{\gamma-1} \\ T_3 V_3^{\gamma-1} = T_4 V_4^{\gamma-1} & \implies \frac{T_4}{T_3} = \left(\frac{V_3}{V_4}\right)^{\gamma-1} = \left(\frac{r_c}{r}\right)^{\gamma-1} & \implies \frac{T_4}{T_1} = \frac{T_4}{T_3}\frac{T_3}{T_1} = r_c^{\gamma} \,. \end{cases} \tag{52}$$

Thus,

$$\begin{cases} T_3 - T_2 = T_1\left(\frac{T_3}{T_1} - \frac{T_2}{T_1}\right) = T_1\left(r_c r^{\gamma-1} - r^{\gamma-1}\right) = T_1\left(r_c - 1\right) r^{\gamma-1} \\ T_4 - T_1 = T_1\left(\frac{T_4}{T_1} - 1\right) = T_1\left(r_c^{\gamma} - 1\right). \end{cases} \tag{53}$$

Let us assume that the cycle follows the clockwise direction. In processes $1 \to 2$ and $3 \to 4$, the gas is thermally isolated; in process $2 \to 3$, the gas is in contact with a thermal reservoir at temperature $T_h = T_3$, and in process $4 \to 1$, the gas is in contact with a thermal reservoir

[14] Used by Rudolph Christian Karl Diesel (1858–1913) in 1893, in his internal combustion engine.

[15] Here the compression ratio is $\frac{V_1}{V_2} = r$ and the expansion ratio is $r' = \frac{V_4}{V_3} = \frac{r}{r_c} < r$. For this cycle we can define two pressure ratios: $r_p = \frac{p_2}{p_1} = \left(\frac{V_1}{V_2}\right)^{\gamma} = r^{\gamma}$ and $r_p' = \frac{p_3}{p_4} = \left(\frac{V_4}{V_3}\right)^{\gamma} = \left(\frac{r}{r_c}\right)^{\gamma} = r'^{\gamma}$.

at temperature $T_c = T_1$. We have

$$Q_{12} = 0\,; \quad Q_{23} = nc_p \Delta T_{23} = n\frac{\gamma R}{\gamma - 1}(T_3 - T_2) = \frac{nRT_1\gamma}{\gamma - 1}\,(r_c - 1)\,r^{\gamma-1};$$

$$Q_{34} = 0\,; \quad Q_{41} = nc_V \Delta T_{41} = -n\frac{R}{\gamma - 1}(T_4 - T_1) = -\frac{nRT_1}{\gamma - 1}\left(r_c^\gamma - 1\right);$$

$$Q_h = Q_{23} = \frac{nRT_1\gamma}{\gamma - 1}\,(r_c - 1)\,r^{\gamma-1} > 0\,, \quad Q_c = Q_{41} = -\frac{nRT_1}{\gamma - 1}\left(r_c^\gamma - 1\right) < 0\,,$$

$$Q_{\text{cycle}} = Q_h + Q_c = \frac{nRT_1}{\gamma - 1}\left[\gamma\,(r_c - 1)\,r^{\gamma-1} - \left(r_c^\gamma - 1\right)\right]\,,$$

$$W_{\text{cycle}} = -Q_{\text{cycle}} = -\frac{nRT_1}{\gamma - 1}\left[\gamma\,(r_c - 1)\,r^{\gamma-1} - \left(r_c^\gamma - 1\right)\right]\,.$$

It is not difficult to show[16] that $Q_{\text{cycle}} > 0$, so $W_{\text{cycle}} < 0$. Thus the gas is absorbing heat from the reservoir at higher temperature, is rejecting heat to the reservoir at lower temperature and is doing work; these properties characterize a heat engine.

The efficiency of a Diesel engine is given by

$$\epsilon_{\text{Diesel cycle}} = \frac{|W_{\text{cycle}}|}{|Q_h|} = \frac{\frac{nRT_1}{\gamma-1}\left[\gamma\,(r_c - 1)\,r^{\gamma-1} - \left(r_c^\gamma - 1\right)\right]}{\frac{nRT_1\gamma}{\gamma-1}\,(r_c - 1)\,r^{\gamma-1}} = 1 - \frac{r_c^\gamma - 1}{\gamma\,(r_c - 1)\,r^{\gamma-1}}\,, \tag{54}$$

and can be put in the form[17]

$$\epsilon_{\text{Diesel cycle}} = 1 - \frac{1}{r^{\gamma-1}}\left[\frac{r_c^\gamma - 1}{\gamma\,(r_c - 1)}\right] < 1 - \frac{1}{r^{\gamma-1}} = \epsilon_{\text{Otto cycle}}\;;$$

thus, the efficiency of a Diesel engine is less than the efficiency of an Otto engine or a Joule engine with the same compression ratio. The reason is clear: in the Diesel engine the expansion ratio is smaller than in Otto or Joule engines.

The entropy change of the universe after one cycle of the Diesel engine is

$$\begin{aligned}\Delta S^{\text{universe}}_{\text{Diesel cycle},\,\circlearrowright} &= -\frac{Q_h}{T_h} - \frac{Q_c}{T_c} = -\frac{\frac{nRT_1\gamma}{\gamma-1}\,(r_c - 1)\,r^{\gamma-1}}{T_3} + \frac{\frac{nRT_1}{\gamma-1}\left(r_c^\gamma - 1\right)}{T_1} \\ &= \frac{nR}{\gamma - 1}\left[r_c^\gamma - 1 - \gamma\frac{(r_c - 1)\,r^{\gamma-1}}{r_c r^{\gamma-1}}\right] = \frac{nR}{\gamma - 1}\left[r_c^\gamma - 1 - \gamma\left(1 - \frac{1}{r_c}\right)\right].\end{aligned} \tag{55}$$

This depends only on the cut-off ratio and on the nature of the gas; it is easy to verify[18] that $\Delta S^{\text{universe}}_{\text{Diesel cycle},\,\circlearrowright} > 0$, so the clockwise Diesel cycle is also irreversible.

In order to transform this heat engine into a heat pump/refrigerator, we proceed as in Otto and Joule cycles: we invert the cycle, following the counterclockwise direction, and we use two new reservoirs, at temperatures T_4 and T_2. In process $1 \to 4$, the gas is in contact with a reservoir at temperature T_4; in processes $4 \to 3$ and $2 \to 1$, the gas is thermally isolated; in process $3 \to 2$, the gas is in contact with a reservoir at temperature T_2. We obtain

$$Q_{14} = -Q_{41} = \frac{nRT_1}{\gamma - 1}\left(r_c^\gamma - 1\right);$$

$$Q_{43} = 0\,;\; Q_{32} = -Q_{23} = -\frac{nRT_1\gamma}{\gamma - 1}\,(r_c - 1)\,r^{\gamma-1}\,;\; Q_{21} = 0.$$

[16] Since $r > r_c$, we can write $\gamma\,(r_c - 1)\,r^{\gamma-1} - \left(r_c^\gamma - 1\right) > \gamma\,(r_c - 1)\,r_c^{\gamma-1} - \left(r_c^\gamma - 1\right)$. So we must study the function $f_1(x) = \gamma\,(x - 1)\,x^{\gamma-1} - (x^\gamma - 1)$ for $x > 1$, with $\gamma > 1$.

[17] The inequality $\frac{r_c^\gamma - 1}{\gamma(r_c - 1)} > 1$ is proved by studying the function $f_2(x) = x^\gamma - 1 - \gamma(x - 1)$ for $x > 1$, with $\gamma > 1$.

[18] We must study the function $f_3(x) = x^\gamma - 1 - \gamma\left(1 - \frac{1}{x}\right)$ for $x > 1$, with $\gamma > 1$.

Imposing now the condition $T'_{\rm c} = T_4 < T_2 = T'_{\rm h}$, we will have

$$Q'_{\rm h} = Q_{32} = -\frac{nRT_1\gamma}{\gamma-1}\,(r_{\rm c}-1)\,r^{\gamma-1} < 0\,, \qquad Q'_{\rm c} = Q_{14} = \frac{nRT_1}{\gamma-1}\left(r_{\rm c}^{\gamma}-1\right) > 0\,,$$

$$Q'_{\rm cycle} = Q'_{\rm h} + Q'_{\rm c} = -\frac{nRT_1}{\gamma-1}\left[\gamma\,(r_{\rm c}-1)\,r^{\gamma-1} - \left(r_{\rm c}^{\gamma}-1\right)\right] < 0\,,$$

$$W'_{\rm cycle} = -Q'_{\rm cycle} = \frac{nRT_1}{\gamma-1}\left[\gamma\,(r_{\rm c}-1)\,r^{\gamma-1} - \left(r_{\rm c}^{\gamma}-1\right)\right] > 0\,.$$

It should be noted that condition $T_4 < T_2$ is equivalent to $r_{\rm c}^{\gamma} < r^{\gamma-1}$; once we fix the values of $r_{\rm c}$ and γ, this condition establishes a minimum value for the compression ratio r.

The coefficients of performance of this inverted Diesel cycle are

$$\text{COP}^{\rm HP}_{\text{Diesel cycle}} = \frac{|Q'_{\rm h}|}{|W'_{\rm cycle}|} = \frac{\frac{nRT_1\gamma}{\gamma-1}\,(r_{\rm c}-1)\,r^{\gamma-1}}{\frac{nRT_1}{\gamma-1}\left[\gamma\,(r_{\rm c}-1)\,r^{\gamma-1} - \left(r_{\rm c}^{\gamma}-1\right)\right]} = \frac{1}{1-\frac{r_{\rm c}^{\gamma}-1}{\gamma(r_{\rm c}-1)r^{\gamma-1}}} > 1\,, \tag{56}$$

$$\text{COP}^{\rm R}_{\text{Diesel cycle}} = \frac{|Q'_{\rm c}|}{|W'_{\rm cycle}|} = \frac{\frac{nRT_1}{\gamma-1}\left(r_{\rm c}^{\gamma}-1\right)}{\frac{nRT_1}{\gamma-1}\left[\gamma\,(r_{\rm c}-1)\,r^{\gamma-1} - \left(r_{\rm c}^{\gamma}-1\right)\right]} = \frac{\frac{r_{\rm c}^{\gamma}-1}{\gamma(r_{\rm c}-1)r^{\gamma-1}}}{1-\frac{r_{\rm c}^{\gamma}-1}{\gamma(r_{\rm c}-1)r^{\gamma-1}}} > 0\,. \tag{57}$$

These two coefficients verify the general relation (9) and

$$\text{COP}^{\rm HP}_{\text{Diesel cycle}} = \frac{1}{\epsilon_{\text{Diesel cycle}}}\,. \tag{58}$$

The entropy change of the universe after one counterclockwise Diesel cycle is

$$\begin{aligned}\Delta S^{\rm universe}_{\text{Diesel cycle},\,\circlearrowleft} &= -\frac{Q'_{\rm h}}{T'_{\rm h}} - \frac{Q'_{\rm c}}{T'_{\rm c}} = \frac{\frac{nRT_1\gamma}{\gamma-1}\,(r_{\rm c}-1)\,r^{\gamma-1}}{T_2} - \frac{\frac{nRT_1}{\gamma-1}\left(r_{\rm c}^{\gamma}-1\right)}{T_4}\\ &= \frac{nR}{\gamma-1}\left[\frac{\gamma\,(r_{\rm c}-1)\,r^{\gamma-1}}{r^{\gamma-1}} - \frac{r_{\rm c}^{\gamma}-1}{r_{\rm c}^{\gamma}}\right] = \frac{nR}{\gamma-1}\left[\gamma\,(r_{\rm c}-1) - \left(1-\frac{1}{r_{\rm c}^{\gamma}}\right)\right]\end{aligned} \tag{59}$$

and is *not equal* to the result obtained for the clockwise cycle. This non-equality could be related to the fact, observed at the beginning of the analysis of this cycle, that the two non-adiabatic processes are of different kinds (one isobaric, the other isochoric).

Anyway, $\Delta S^{\rm universe}_{\text{Diesel cycle},\,\circlearrowleft}$ depends only on the cut-off ratio and on the nature of the gas, and it is easy to verify[19] that $\Delta S^{\rm universe}_{\text{Diesel cycle},\,\circlearrowleft} > 0$, so the counterclockwise Diesel cycle is also irreversible.

Let us calculate the difference between expressions (55) and (59). We have

$$\begin{aligned}\Delta S^{\rm universe}_{\text{Diesel cycle},\,\circlearrowright} - \Delta S^{\rm universe}_{\text{Diesel cycle},\,\circlearrowleft} &= \frac{nR}{\gamma-1}\left[r_{\rm c}^{\gamma} - 1 - \gamma + \frac{\gamma}{r_{\rm c}} - \gamma r_{\rm c} + \gamma + 1 - \frac{1}{r_{\rm c}^{\gamma}}\right]\\ &= \frac{nR}{\gamma-1}\left[r_{\rm c}^{\gamma} - \frac{1}{r_{\rm c}^{\gamma}} - \gamma\left(r_{\rm c} - \frac{1}{r_{\rm c}}\right)\right]\end{aligned} \tag{60}$$

and it can be proved[20] that this difference is always positive, so

$$\Delta S^{\rm universe}_{\text{Diesel cycle},\,\circlearrowright} > \Delta S^{\rm universe}_{\text{Diesel cycle},\,\circlearrowleft}\,, \tag{61}$$

that is, the clockwise cycle (representing Diesel engine) generates more entropy than the counterclockwise cycle (representing Diesel heat pump/refrigerator).

[19] Analysing the function $f_4(x) = \gamma(x-1) - \left(1-\frac{1}{x^{\gamma}}\right)$ for $x > 1$, with $\gamma > 1$.
[20] Studying the function $f_5(x) = x^{\gamma} - \frac{1}{x^{\gamma}} - \gamma\left(x - \frac{1}{x}\right)$ for $x > 1$, with $\gamma > 1$.

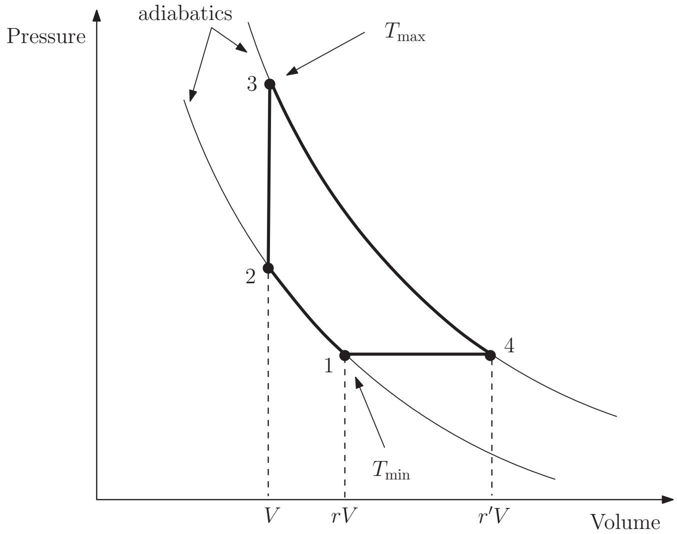

Figure 10. Volume–pressure diagram of the Atkinson cycle.

Finally, comparing the expressions of W'_{cycle} and $T'_{\text{h}}\Delta S^{\text{universe}}_{\text{Diesel cycle, }\circlearrowleft}$, we realize that condition $W'_{\text{cycle}} > T'_{\text{h}}\Delta S^{\text{universe}}_{\text{Diesel cycle, }\circlearrowleft}$ is equivalent to $r_{\text{c}}^{\gamma} < r^{\gamma-1}$, which, as we saw, is the same as $T_4 < T_2$, the necessary condition to transform a Diesel engine, by inversion, into a heat pump/refrigerator. Thus, as in all previous cycles, we can interpret the value of $T'_{\text{h}}\Delta S^{\text{universe}}_{\text{Diesel cycle, }\circlearrowleft}$ as the minimum work we have to do on the gas to obtain a Diesel heat pump/refrigerator.

9. Atkinson cycle

Let us consider the Atkinson cycle shown in figure 10, with two adiabatic processes, one isochoric process and one isobaric process, where $r' > r > 1$; clearly, r is the compression ratio and r' is the expansion ratio[21]. Depending on the source[22], this cycle is also known as the Sargent cycle or Humphrey cycle. Similar to the Diesel cycle, this cycle contains three different kinds of processes instead of two. The points of the cycle with the highest and lowest temperatures are also shown.

Since $p_1 = p_4$, we can write

$$\frac{T_1}{V_1} = \frac{T_4}{V_4} \implies \frac{T_4}{T_1} = \frac{V_4}{V_1} = \frac{r'}{r} \,. \tag{62}$$

[21] As in the Diesel cycle, we also have here two pressure ratios: $r_{\text{p}} = \frac{p_2}{p_1} = r^{\gamma}$ and $r'_{\text{p}} = \frac{p_3}{p_4} = r'^{\gamma}$.

[22] The designation Atkinson cycle [5] has a historical explanation, because the cycle was used by James Atkinson (1846–1914) in his internal combustion engine, in 1882. The name Sargent cycle [6, 7] is in honour of Charles Elliotte Sargent (1862–?), who used it in various gas engines he patented from 1905 on. The designation Humphrey cycle [7] is based on a pump, patented by Herbert Albert Humphrey (1868–1951) in 1906, that implemented this cycle with a gaseous mixture and where the work was used to pump water; this designation is also very common in the literature concerning pulse detonation engines [8, 9].

Since adiabatics can be described by $TV^{\gamma-1} = \text{constant}$, we have

$$\begin{cases} T_2V_2^{\gamma-1} = T_1V_1^{\gamma-1} \quad \Longrightarrow \quad \dfrac{T_2}{T_1} = \left(\dfrac{V_1}{V_2}\right)^{\gamma-1} = r^{\gamma-1} \\ T_3V_3^{\gamma-1} = T_4V_4^{\gamma-1} \quad \Longrightarrow \quad \dfrac{T_3}{T_4} = \left(\dfrac{V_4}{V_3}\right)^{\gamma-1} = \left(r'\right)^{\gamma-1} \quad \Longrightarrow \quad \dfrac{T_3}{T_1} = \dfrac{T_3}{T_4}\dfrac{T_4}{T_1} = \dfrac{r'^{\gamma}}{r}, \end{cases} \tag{63}$$

so

$$\begin{cases} T_3 - T_2 = T_1\left(\dfrac{T_3}{T_1} - \dfrac{T_2}{T_1}\right) = T_1\left(\dfrac{r'^{\gamma}}{r} - r^{\gamma-1}\right) = T_1\dfrac{r'^{\gamma} - r^{\gamma}}{r} \\ T_4 - T_1 = T_1\left(\dfrac{T_4}{T_1} - 1\right) = T_1\left(\dfrac{r'}{r} - 1\right) = T_1\dfrac{r' - r}{r} \end{cases} . \tag{64}$$

Let us assume that the cycle is performed in the clockwise direction. In processes $1 \to 2$ and $3 \to 4$, the gas is thermally isolated; in process $2 \to 3$, the gas is in contact with a thermal reservoir at temperature $T_\text{h} = T_3$; and in process $4 \to 1$, the gas is in contact with a thermal reservoir at temperature $T_\text{c} = T_1$. We have

$$Q_{12} = 0\,; \quad Q_{23} = nc_V\Delta T_{23} = n\frac{R}{\gamma-1}(T_3 - T_2) = \frac{nRT_1}{\gamma-1}\frac{r'^{\gamma} - r^{\gamma}}{r}\,;$$

$$Q_{34} = 0\,; \quad Q_{41} = nc_p\Delta T_{41} = -n\frac{\gamma R}{\gamma-1}(T_4 - T_1) = -\frac{nRT_1\gamma}{\gamma-1}\frac{r' - r}{r}\,;$$

$$Q_\text{h} = Q_{23} = \frac{nRT_1}{\gamma-1}\frac{r'^{\gamma} - r^{\gamma}}{r} > 0\,, \quad Q_\text{c} = Q_{41} = -\frac{nRT_1\gamma}{\gamma-1}\frac{r' - r}{r} < 0\,,$$

$$Q_\text{cycle} = Q_\text{h} + Q_\text{c} = \frac{nRT_1}{\gamma-1}\frac{r'^{\gamma} - r^{\gamma} - \gamma(r' - r)}{r} > 0\,,$$

$$W_\text{cycle} = -Q_\text{cycle} = -\frac{nRT_1}{\gamma-1}\frac{r'^{\gamma} - r^{\gamma} - \gamma(r' - r)}{r} < 0\,.$$

Since the gas is absorbing heat from the reservoir at higher temperature, is rejecting heat to the reservoir at lower temperature and is doing work, this represents a heat engine. The efficiency of Atkinson engine is given by

$$\epsilon_\text{Atkinson cycle} = 1 - \frac{|Q_\text{c}|}{|Q_\text{h}|} = 1 - \frac{\frac{nRT_1\gamma}{\gamma-1}\frac{r'-r}{r}}{\frac{nRT_1}{\gamma-1}\frac{r'^{\gamma}-r^{\gamma}}{r}} = 1 - \frac{\gamma(r' - r)}{r'^{\gamma} - r^{\gamma}}\,, \tag{65}$$

and can be written in the form[23]

$$\epsilon_\text{Atkinson cycle} = 1 - \frac{\gamma r\left(\frac{r'}{r} - 1\right)}{r'^{\gamma}\left[1 - \left(\frac{r}{r'}\right)^{\gamma}\right]} = 1 - \frac{r}{r'^{\gamma}}\left[\frac{\gamma\left(\frac{r'}{r} - 1\right)}{1 - \left(\frac{r}{r'}\right)^{\gamma}}\right] < 1 - \frac{r}{r'^{\gamma}} = \epsilon_\text{Carnot cycle}$$

or in the form[24]

$$\epsilon_\text{Atkinson cycle} = 1 - \frac{\gamma r\left(\frac{r'}{r} - 1\right)}{r^{\gamma}\left[\left(\frac{r'}{r}\right)^{\gamma} - 1\right]} = 1 - \frac{1}{r^{\gamma-1}}\left[\frac{\gamma\left(\frac{r'}{r} - 1\right)}{\left(\frac{r'}{r}\right)^{\gamma} - 1}\right] > 1 - \frac{1}{r^{\gamma-1}} = \epsilon_\text{Otto cycle}\,.$$

[23] The inequality results from $\frac{\gamma\left(\frac{r'}{r}-1\right)}{1-\left(\frac{r}{r'}\right)^{\gamma}} > 1$, easily proved through the function $f_4(x)$ of a previous footnote for $x > 1$, with $\gamma > 1$.

[24] The inequality results from $\frac{\gamma\left(\frac{r'}{r}-1\right)}{\left(\frac{r'}{r}\right)^{\gamma}-1} < 1$, which is proved through the function $f_2(x)$ of a previous footnote for $x > 1$, with $\gamma > 1$.

Thus, an Atkinson engine is less efficient than a Carnot engine functioning between two reservoirs at maximum and minimum temperatures. Nevertheless, it is more efficient than a Otto engine or a Joule engine with the same compression ratio. Why? Because the expansion ratio is greater in Atkinson engine than in Otto or Joule engines. Combining these results with those obtained in the previous sections, we obtain

$$0 < \epsilon_{\text{Diesel cycle}} < \epsilon_{\text{Otto cycle}} = \epsilon_{\text{Joule cycle}} < \epsilon_{\text{Atkinson cycle}} < \epsilon_{\text{Carnot cycle}} < 1\,.$$

The entropy change of the universe after one cycle of the Atkinson engine is

$$\begin{aligned}\Delta S^{\text{universe}}_{\text{Atkinson cycle}, \circlearrowright} &= -\frac{Q_{\text{h}}}{T_{\text{h}}} - \frac{Q_{\text{c}}}{T_{\text{c}}} = -\frac{\frac{nRT_1}{\gamma-1}\frac{r'^{\gamma}-r^{\gamma}}{r}}{T_3} + \frac{\frac{nRT_1\gamma}{\gamma-1}\frac{r'-r}{r}}{T_1} \\ &= \frac{nR}{\gamma-1}\left[\gamma\left(\frac{r'}{r}-1\right) - \left(1-\frac{r^{\gamma}}{r'^{\gamma}}\right)\right], \end{aligned} \tag{66}$$

an expression that depends only on the ratio $\frac{r'}{r}$ and on the nature of the gas; it can be shown that $\Delta S^{\text{universe}}_{\text{Atkinson cycle}, \circlearrowright} > 0$, so the clockwise Atkinson cycle is also irreversible.

In order to transform this heat engine into a heat pump/refrigerator, we proceed as in Otto, Joule and Diesel cycles: we invert the cycle to the counterclockwise direction, and we use two new reservoirs, at temperatures T_4 and T_2. In process $1 \to 4$, the gas is in contact with a reservoir at temperature T_4; in processes $4 \to 3$ and $2 \to 1$, the gas is thermally isolated; in process $3 \to 2$, the gas is in contact with a reservoir at temperature T_2. We obtain

$$Q_{14} = -Q_{41} = \frac{nRT_1\gamma}{\gamma-1}\frac{r'-r}{r}\,;\; Q_{43} = 0\,;\; Q_{32} = -Q_{23} = -\frac{nRT_1}{\gamma-1}\frac{r'^{\gamma}-r^{\gamma}}{r}\,;\; Q_{21} = 0\,.$$

Imposing the condition $T'_{\text{c}} = T_4 < T_2 = T'_{\text{h}}$, we will obtain the following formulae:

$$\begin{aligned} Q'_{\text{h}} &= Q_{32} = -\frac{nRT_1}{\gamma-1}\frac{r'^{\gamma}-r^{\gamma}}{r} < 0\,, \quad Q'_{\text{c}} = Q_{14} = \frac{nRT_1\gamma}{\gamma-1}\frac{r'-r}{r} > 0\,, \\ Q'_{\text{cycle}} &= Q'_{\text{h}} + Q'_{\text{c}} = -\frac{nRT_1}{\gamma-1}\frac{r'^{\gamma}-r^{\gamma}-\gamma(r'-r)}{r} < 0\,, \\ W'_{\text{cycle}} &= -Q'_{\text{cycle}} = \frac{nRT_1}{\gamma-1}\frac{r'^{\gamma}-r^{\gamma}-\gamma(r'-r)}{r} > 0\,. \end{aligned}$$

Similar to what occurred in the previous section, condition $T_4 < T_2$ is equivalent to $r^{\gamma} > r'$; once we fix the values of $\frac{r'}{r}$ and γ, this condition establishes a minimum value for the compression ratio r.

The coefficients of performance of this inverted Atkinson cycle are

$$\text{COP}^{\text{HP}}_{\text{Atkinson cycle}} = \frac{|Q'_{\text{h}}|}{|W'_{\text{cycle}}|} = \frac{\frac{nRT_1}{\gamma-1}\frac{r'^{\gamma}-r^{\gamma}}{r}}{\frac{nRT_1}{\gamma-1}\frac{r'^{\gamma}-r^{\gamma}-\gamma(r'-r)}{r}} = \frac{1}{1-\frac{\gamma(r'-r)}{r'^{\gamma}-r^{\gamma}}} > 1\,, \tag{67}$$

$$\text{COP}^{\text{R}}_{\text{Atkinson cycle}} = \frac{|Q'_{\text{c}}|}{|W'_{\text{cycle}}|} = \frac{\frac{nRT_1\gamma}{\gamma-1}\frac{r'-r}{r}}{\frac{nRT_1}{\gamma-1}\frac{r'^{\gamma}-r^{\gamma}-\gamma(r'-r)}{r}} = \frac{\frac{\gamma(r'-r)}{r'^{\gamma}-r^{\gamma}}}{1-\frac{\gamma(r'-r)}{r'^{\gamma}-r^{\gamma}}} > 0\,, \tag{68}$$

verifying the general relation (9) and

$$\text{COP}^{\text{HP}}_{\text{Atkinson cycle}} = \frac{1}{\epsilon_{\text{Atkinson cycle}}}\,. \tag{69}$$

Taking into account the previous comparison between the efficiencies of the various cycles, and combining relations (17), (40), (49), (58) and (69), we would obtain the following result:

$$1 < \text{COP}^{\text{HP}}_{\text{Carnot cycle}} < \text{COP}^{\text{HP}}_{\text{Atkinson cycle}} < \text{COP}^{\text{HP}}_{\text{Otto cycle}} = \text{COP}^{\text{HP}}_{\text{Joule cycle}} < \text{COP}^{\text{HP}}_{\text{Diesel cycle}}\,.$$

The second inequality seems to violate the Carnot theorem: apparently, we would have heat pumps with a coefficient of performance greater than the coefficient of performance of a Carnot heat pump. But there is no violation at all: we should note that Atkinson, Otto, Joule and Diesel heat pumps, appearing in the previous expression, function between reservoirs at temperatures $T'_\mathrm{c} = T_4 < T_2 = T'_\mathrm{h}$, whereas the Carnot heat pump, with which they are being compared, functions between reservoirs at temperatures $T_\mathrm{c} = T_1 < T_3 = T_\mathrm{h}$.

If we compare $\mathrm{COP}^{\mathrm{HP}}_{\text{Diesel cycle}}$ with the coefficient of performance of a Carnot heat pump functioning between reservoirs at temperatures T_4 and T_2, we verify that the latter is greater, according to Carnot theorem, and we obtain (we leave the details to the reader)

$$1 < \mathrm{COP}^{\mathrm{HP}}_{\text{Atkinson cycle}} < \mathrm{COP}^{\mathrm{HP}}_{\text{Otto cycle}} = \mathrm{COP}^{\mathrm{HP}}_{\text{Joule cycle}} < \mathrm{COP}^{\mathrm{HP}}_{\text{Diesel cycle}} < \mathrm{COP}^{\mathrm{HP}}_{\text{Carnot cycle}} .$$

The entropy change of the universe after one counterclockwise Atkinson cycle is

$$\begin{aligned}\Delta S^{\text{universe}}_{\text{Atkinson cycle, }\circlearrowleft} &= -\frac{Q'_\mathrm{h}}{T'_\mathrm{h}} - \frac{Q'_\mathrm{c}}{T'_\mathrm{c}} = \frac{\frac{nRT_1}{\gamma-1}\frac{r'^\gamma - r^\gamma}{r}}{T_2} - \frac{\frac{nRT_1\gamma}{\gamma-1}\frac{r'-r}{r}}{T_4} \\ &= \frac{nR}{\gamma-1}\left[\frac{r'^\gamma - r^\gamma}{r^\gamma} - \frac{\gamma(r'-r)}{r'}\right] = \frac{nR}{\gamma-1}\left[\frac{r'^\gamma}{r^\gamma} - 1 - \gamma\left(1 - \frac{r}{r'}\right)\right] \quad (70)\end{aligned}$$

and is *not equal* to the result obtained for the clockwise cycle. Once again, this non-equality should be related to the fact that the two non-adiabatic processes are of different kinds (one isobaric, the other isochoric). Anyway, $\Delta S^{\text{universe}}_{\text{Atkinson cycle, }\circlearrowleft}$ depends only on the ratio $\frac{r'}{r}$ and on the nature of the gas, and it is easy to verify[25] that $\Delta S^{\text{universe}}_{\text{Atkinson cycle, }\circlearrowleft} > 0$, which shows that the counterclockwise Atkinson cycle is also irreversible.

Let us calculate the difference between expressions (66) and (70). We have

$$\begin{aligned}\Delta S^{\text{universe}}_{\text{Atkinson cycle, }\circlearrowright} - \Delta S^{\text{universe}}_{\text{Atkinson cycle, }\circlearrowleft} &= \frac{nR}{\gamma-1}\left[\frac{\gamma r'}{r} - \gamma - 1 + \frac{r^\gamma}{r'^\gamma} - \frac{r'^\gamma}{r^\gamma} + 1 + \gamma - \frac{\gamma r}{r'}\right] \\ &= -\frac{nR}{\gamma-1}\left[\frac{r'^\gamma}{r^\gamma} - \frac{r^\gamma}{r'^\gamma} - \gamma\left(\frac{r'}{r} - \frac{r}{r'}\right)\right] \quad (71)\end{aligned}$$

and it can be shown[26] that this difference is always negative, so

$$\Delta S^{\text{universe}}_{\text{Atkinson cycle, }\circlearrowright} < \Delta S^{\text{universe}}_{\text{Atkinson cycle, }\circlearrowleft} , \quad (72)$$

that is, the clockwise cycle (representing Atkinson engine) generates less entropy than the counterclockwise cycle (representing Atkinson heat pump/refrigerator).

To conclude this section, if we compare expressions of W'_{cycle} and $T'_\mathrm{h}\Delta S^{\text{universe}}_{\text{Atkinson cycle, }\circlearrowleft}$, we realize that condition $W'_{\text{cycle}} > T'_\mathrm{h}\Delta S^{\text{universe}}_{\text{Atkinson cycle, }\circlearrowleft}$ is equivalent to $r^\gamma > r'$, which, as we saw, is the same as $T_4 < T_2$, the necessary condition to transform an Atkinson engine, by inversion, into a heat pump/refrigerator. Thus, as in all previous cases, we can interpret the quantity $T'_\mathrm{h}\Delta S^{\text{universe}}_{\text{Atkinson cycle, }\circlearrowleft}$ as the minimum work we have to do on the gas to implement an Atkinson heat pump/refrigerator.

In this and all previous cycles, we saw that the inversion of a heat engine cycle, made under certain conditions, generated the cycle of a heat pump/refrigerator; conditions which allow that transformation were always expressed in terms of one or more parameters, and it was always possible to interpret them as the minimum work we have to bring to the system.

Thus it would seem reasonable to think that the transformation of a heat engine cycle into a heat pump/refrigerator cycle is always possible under specified conditions. But that idea is wrong, as we will show in the following section.

[25] Analysing the function $f_3(x)$ of a previous footnote for $x > 1$, with $\gamma > 1$.
[26] Analysing the function $f_5(x)$ of a previous footnote for $x > 1$, with $\gamma > 1$.

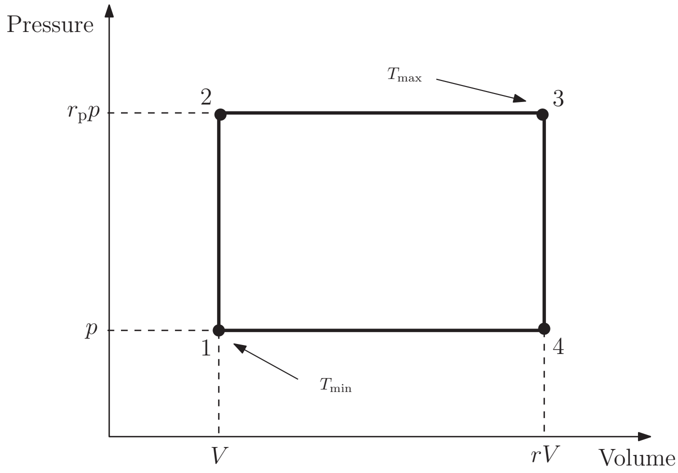

Figure 11. Volume–pressure diagram of the rectangular cycle.

10. Rectangular cycle

Let us now consider the rectangular cycle shown in figure 11, with two isochoric processes and two isobaric processes, where $r, r_p > 1$ (r could be named compression or expansion ratio, and r_p could be named pressure ratio). Minimum and maximum temperatures occur at points 1 and 3, respectively. Although this cycle shows similarities with some of the previous cycles, we should note that it does not include any isothermal or adiabatic process.

Let us assume that the cycle follows the clockwise direction. In processes $1 \to 2 \to 3$, the gas is in contact with a thermal reservoir at temperature $T_h = T_3$, and in processes $3 \to 4 \to 1$, the gas is in contact with a thermal reservoir at temperature $T_c = T_1$. We have

$$Q_{12} = nc_V \Delta T_{12} = n\frac{R}{\gamma - 1}(T_2 - T_1) = \frac{r_p pV - pV}{\gamma - 1} = \frac{r_p - 1}{\gamma - 1} pV\,;$$
$$Q_{23} = nc_p \Delta T_{23} = n\frac{\gamma R}{\gamma - 1}(T_3 - T_2) = \frac{\gamma(r r_p pV - r_p pV)}{\gamma - 1} = \frac{\gamma r_p (r - 1)}{\gamma - 1} pV\,;$$
$$Q_{34} = nc_V \Delta T_{34} = n\frac{R}{\gamma - 1}(T_4 - T_3) = \frac{rpV - r r_p pV}{\gamma - 1} = -\frac{r(r_p - 1)}{\gamma - 1} pV\,;$$
$$Q_{41} = nc_p \Delta T_{41} = n\frac{\gamma R}{\gamma - 1}(T_1 - T_4) = \frac{\gamma(pV - rpV)}{\gamma - 1} = -\frac{\gamma(r - 1)}{\gamma - 1} pV\,.$$

So

$$Q_h = Q_{12} + Q_{23} = \frac{(r_p - 1) + \gamma r_p (r - 1)}{\gamma - 1} pV > 0\,,$$
$$Q_c = Q_{34} + Q_{41} = -\frac{r(r_p - 1) + \gamma(r - 1)}{\gamma - 1} pV < 0\,,$$
$$Q_{cycle} = Q_h + Q_c = (r - 1)(r_p - 1) pV > 0, \quad W_{cycle} = -Q_{cycle} = -(r - 1)(r_p - 1) pV < 0.$$

Thus, the gas is absorbing heat from the reservoir at higher temperature, is rejecting heat to the reservoir at lower temperature and is doing work; this clearly represents a heat engine. The efficiency of this heat engine is given by

$$\epsilon_{\text{rectangular cycle}} = \frac{|W_{\text{cycle}}|}{|Q_{\text{h}}|} = \frac{(r-1)(r_{\text{p}}-1)pV}{\frac{(r_{\text{p}}-1)+\gamma r_{\text{p}}(r-1)}{\gamma-1}pV} = \frac{(\gamma-1)(r-1)(r_{\text{p}}-1)}{(r_{\text{p}}-1)+\gamma r_{\text{p}}(r-1)} \tag{73}$$

and if we compare it with the efficiency of a Carnot engine functioning between the same reservoirs,

$$\epsilon_{\text{Carnot cycle}} = 1 - \frac{T_1}{T_3} = 1 - \frac{nRT_1}{nRT_3} = 1 - \frac{pV}{rr_{\text{p}}pV} = 1 - \frac{1}{rr_{\text{p}}} = \frac{rr_{\text{p}}-1}{rr_{\text{p}}}\,, \tag{74}$$

we find that $\epsilon_{\text{rectangular cycle}} < \epsilon_{\text{Carnot cycle}}$ because

$$\epsilon_{\text{Carnot cycle}} - \epsilon_{\text{rectangular cycle}} = \frac{(r_{\text{p}}-1)(r^2 r_{\text{p}}-1) + \gamma r_{\text{p}}(r-1)^2}{rr_{\text{p}}\left[(r_{\text{p}}-1)+\gamma r_{\text{p}}(r-1)\right]} > 0\,.$$

The entropy change of the universe after one cycle of this heat engine is

$$\begin{aligned}
\Delta S^{\text{universe}}_{\text{rectangular cycle},\,\circlearrowleft} &= -\frac{Q_{\text{h}}}{T_{\text{h}}} - \frac{Q_{\text{c}}}{T_{\text{c}}} = -\frac{\frac{(r_{\text{p}}-1)+\gamma r_{\text{p}}(r-1)}{\gamma-1}pV}{\frac{rr_{\text{p}}pV}{nR}} + \frac{\frac{r(r_{\text{p}}-1)+\gamma(r-1)}{\gamma-1}pV}{\frac{pV}{nR}} \\
&= \frac{nR}{\gamma-1}\left[r(r_{\text{p}}-1)+\gamma(r-1) - \frac{(r_{\text{p}}-1)+\gamma r_{\text{p}}(r-1)}{rr_{\text{p}}}\right] \\
&= \frac{nR}{(\gamma-1)}\frac{(r_{\text{p}}-1)(r^2 r_{\text{p}}-1)+\gamma r_{\text{p}}(r-1)^2}{rr_{\text{p}}} > 0\,,
\end{aligned} \tag{75}$$

showing the irreversible character of this cycle.

Let us try, as in the examples of previous sections, to transform this heat engine into a heat pump/refrigerator. In order to do so, we invert the rectangular cycle, performing it in the counterclockwise direction. In processes $1 \to 4 \to 3$, the gas is in contact with a thermal reservoir at temperature $T'_{\text{h}} = T_{\text{h}} = T_3$, and in processes $3 \to 2 \to 1$, the gas is in contact with a thermal reservoir at temperature $T'_{\text{c}} = T_{\text{c}} = T_1$. We have

$$\begin{aligned}
&Q_{14} = -Q_{41} = \frac{\gamma(r-1)}{\gamma-1}pV\,; \quad Q_{43} = -Q_{34} = \frac{r(r_{\text{p}}-1)}{\gamma-1}pV\,; \\
&Q_{32} = -Q_{23} = -\frac{\gamma r_{\text{p}}(r-1)}{\gamma-1}pV\,; \quad Q_{21} = -Q_{12} = -\frac{r_{\text{p}}-1}{\gamma-1}pV\,, \quad \text{so} \\
&Q'_{\text{h}} = Q_{14} + Q_{43} = \frac{r(r_{\text{p}}-1)+\gamma(r-1)}{\gamma-1}pV > 0\,, \\
&Q'_{\text{c}} = Q_{32} + Q_{21} = -\frac{(r_{\text{p}}-1)+\gamma r_{\text{p}}(r-1)}{\gamma-1}pV < 0\,, \\
&Q'_{\text{cycle}} = Q'_{\text{h}} + Q'_{\text{c}} = -(r-1)(r_{\text{p}}-1)pV < 0\,, \quad W'_{\text{cycle}} = -Q'_{\text{cycle}} = (r-1)(r_{\text{p}}-1)pV > 0\,.
\end{aligned}$$

Thus, the gas is still absorbing heat from the reservoir at higher temperature, it is still rejecting heat to the reservoir at lower temperature, but now work is being done on the gas; this situation does not correspond to a heat engine, nor a heat pump/refrigerator. Its schematic representation is shown in figure 4(a).

So it makes no sense here to calculate the efficiency or the coefficients of performance.

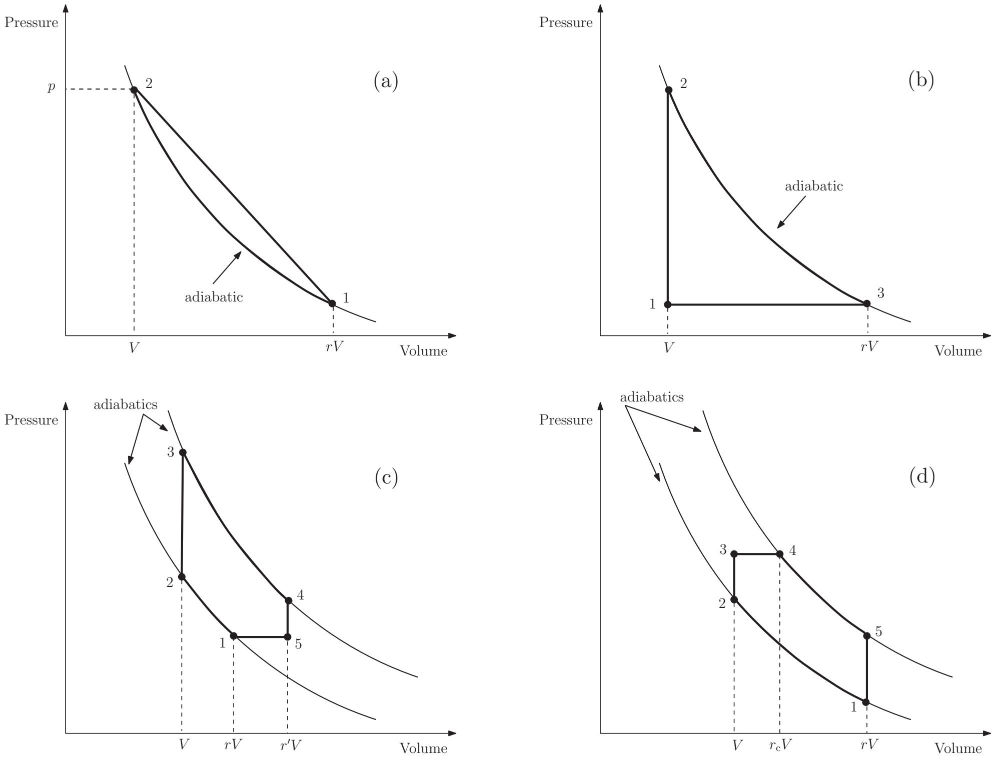

Figure 12. Volume–pressure diagram of (a) 'Sadly Cannot' cycle [10, 11], (b) Lenoir cycle[27], (c) Miller cycle[28] and (d) Trinkler cycle[29].

The entropy change of the universe during this counterclockwise rectangular cycle is

$$\Delta S^{\text{universe}}_{\text{rectangular cycle, } \circlearrowleft} = -\frac{Q'_{\text{h}}}{T_{\text{h}}} - \frac{Q'_{\text{c}}}{T_{\text{c}}} = -\frac{\frac{r(r_{\text{p}}-1)+\gamma(r-1)}{\gamma-1}pV}{\frac{rr_{\text{p}}pV}{nR}} + \frac{\frac{(r_{\text{p}}-1)+\gamma r_{\text{p}}(r-1)}{\gamma-1}pV}{\frac{pV}{nR}}$$
$$= \frac{nR}{\gamma-1}\left[(r_{\text{p}}-1)+\gamma r_{\text{p}}(r-1) - \frac{r(r_{\text{p}}-1)+\gamma(r-1)}{rr_{\text{p}}}\right]$$
$$= \frac{nR}{(\gamma-1)}\frac{r(r_{\text{p}}-1)^2+\gamma(r-1)(rr_{\text{p}}^2-1)}{rr_{\text{p}}} > 0\,, \tag{76}$$

so this cycle is also irreversible; additionally, it is easy to prove that

$$\Delta S^{\text{universe}}_{\text{rectangular cycle, } \circlearrowright} - \Delta S^{\text{universe}}_{\text{rectangular cycle, } \circlearrowleft} = -nR\frac{(r-1)(r_{\text{p}}-1)(rr_{\text{p}}+1)}{rr_{\text{p}}} < 0\,, \quad \text{so} \tag{77}$$

$$\Delta S^{\text{universe}}_{\text{rectangular cycle, } \circlearrowright} < \Delta S^{\text{universe}}_{\text{rectangular cycle, } \circlearrowleft}\,, \tag{78}$$

that is, the clockwise cycle (representing the rectangular heat engine) generates less entropy than the counterclockwise cycle.

We leave to the reader, as an exercise, to verify that, independent of the values of γ, r or r_{p}, the work W'_{cycle} is always less than $T_1\Delta S^{\text{universe}}_{\text{rectangular cycle, } \circlearrowleft}$; since T_1 is the minimum temperature during the cycle, that work will be less than $T_j\Delta S^{\text{universe}}_{\text{rectangular cycle, } \circlearrowleft}$ $(j = 1, 2, 3, 4)$. In particular, $W'_{\text{cycle}} < T'_{\text{h}}\Delta S^{\text{universe}}_{\text{rectangular cycle, } \circlearrowleft}$. Thus the necessary condition to implement a heat pump/refrigerator with this rectangular cycle does not verify in any circumstances. So we have here an example of a heat engine whose cycle cannot be transformed, by inversion, into a heat pump/refrigerator.

11. Some other cycles

In this work, we have restricted the study to four-process cycles; naturally, we could also study two-process or three-process cycles, or even cycles with five or more processes. Some interesting cycles are shown in figure 12, and we invite the reader to extend our analysis to those cycles. In particular, the reader can calculate the efficiencies of those clockwise cycles, compare them with the results obtained in this work, calculate and compare the entropy changes of the universe during the clockwise and counterclockwise cycles, study the necessary conditions for the counterclockwise cycle to represent a heat pump/refrigerator and calculate the corresponding coefficients of performance whenever possible.

12. Conclusions

We have studied in detail some thermodynamic cycles usually referred to in textbooks. We began studying the clockwise version of those cycles (corresponding to heat engines), and calculated (and compared) their efficiencies. We have obtained the results

$$0 < \epsilon_{\text{Ericsson cycle}} < \epsilon_{\text{Stirling cycle}} < \epsilon_{\text{Carnot cycle}} < 1$$

for cycles with two isothermal processes, and the results

$$0 < \epsilon_{\text{Diesel cycle}} < \epsilon_{\text{Otto cycle}} = \epsilon_{\text{Joule cycle}} < \epsilon_{\text{Atkinson cycle}} < \epsilon_{\text{Carnot cycle}} < 1$$

for cycles with two adiabatic processes.

We can also compare the efficiencies of the first group of cycles with those of the second group if, in expressions (19) and (27), we identify the two compression ratios (concerning isothermal processes in the first group cycles, and adiabatic processes in the second group cycles) and replace ΔT by $T_{\max} - T_{\min}$ (that is, by $T_3 - T_1$) and T_2 by $T_{\max}$ (that is, by T_3). It is not difficult to show that $\epsilon_{\text{Stirling cycle}} < \epsilon_{\text{Otto cycle}}$, but there is no general relation between $\epsilon_{\text{Ericsson cycle}}$ and $\epsilon_{\text{Diesel cycle}}$, nor between $\epsilon_{\text{Stirling cycle}}$ and $\epsilon_{\text{Diesel cycle}}$, as we can see through the following numerical examples, computed for a diatomic gas ($\gamma = 7/5$):

$$\epsilon_{\text{Diesel cycle}} = 0,25 < \epsilon_{\text{Ericsson cycle}} = 0,27 < \epsilon_{\text{Stirling cycle}} = 0,33 \quad \text{for } r_c = 3 \text{ and } r = 4;$$

$$\epsilon_{\text{Ericsson cycle}} = 0,32 < \epsilon_{\text{Diesel cycle}} = 0,36 < \epsilon_{\text{Stirling cycle}} = 0,39 \quad \text{for } r_c = 3 \text{ and } r = 6;$$

$$\epsilon_{\text{Ericsson cycle}} = 0,35 < \epsilon_{\text{Stirling cycle}} = 0,42 < \epsilon_{\text{Diesel cycle}} = 0,43 \quad \text{for } r_c = 3 \text{ and } r = 8.$$

We have also calculated the entropy change of the universe for each heat engine; it was clear that the unique reversible heat engine functioning between two thermal reservoirs uses a Carnot cycle. Although we have derived some interesting expressions for these entropy changes, e.g., (37) and (46), those formulae are only valid for certain ‘symmetric’ cycles.

Then we have proceeded to the inversion of the heat engine cycles. We have verified that, when it is inverted, the unique cycle that automatically generates a heat pump/refrigerator cycle in all circumstances is the Carnot cycle. For the other cycles the inversion *can* represent a heat pump/refrigerator, but only under certain conditions, and sometimes it is required to employ thermal reservoirs different from those used in the heat engines. Whenever possible, we have calculated and compared the corresponding coefficients of performance. If the appropriate condition is satisfied, we have obtained

$$1 < \mathrm{COP}^{\mathrm{HP}}_{\text{Ericsson cycle}} < \mathrm{COP}^{\mathrm{HP}}_{\text{Stirling cycle}} < \mathrm{COP}^{\mathrm{HP}}_{\text{Carnot cycle}}$$

$$1 < \mathrm{COP}^{\mathrm{HP}}_{\text{Atkinson cycle}} < \mathrm{COP}^{\mathrm{HP}}_{\text{Otto cycle}} = \mathrm{COP}^{\mathrm{HP}}_{\text{Joule cycle}} < \mathrm{COP}^{\mathrm{HP}}_{\text{Diesel cycle}} < \mathrm{COP}^{\mathrm{HP}}_{\text{Carnot cycle}}.$$

We have also computed the corresponding entropy change of the universe for each heat pump/refrigerator, and we have established a relation between its value and the necessary condition to obtain a heat pump/refrigerator in terms of the work we must supply: $W'_{\text{cycle}} > T'_{\mathrm{h}} \Delta S^{\text{universe}}_{\text{cycle},\circlearrowleft}$. Typically, this condition translates into a compression ratio large enough. Concerning the relation between the entropy change of the universe in clockwise and counterclockwise cycles, we have seen that the two values are equal in some cycles (Carnot, Stirling, Ericsson, Otto and Joule); from the other cycles we have studied, we conclude that anything can happen: for instance, $\Delta S^{\text{universe}}_{\text{cycle},\circlearrowright} > \Delta S^{\text{universe}}_{\text{cycle},\circlearrowleft}$ in the Diesel cycle, and $\Delta S^{\text{universe}}_{\text{cycle},\circlearrowright} < \Delta S^{\text{universe}}_{\text{cycle},\circlearrowleft}$ in the Atkinson cycle.

Finally, we have shown an example of a cycle (the rectangular cycle) we could use to implement a heat engine but not a heat pump/refrigerator, independent of the compression ratio or the temperatures of the reservoirs.

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