Second, one may seek solution to the equation

$$\hat{L}_+ \varphi_{ll} = 0 \tag{9.52}$$

Once having found φ_{ll}, the remaining eigenfunctions of $\hat{L}^2$ and $\hat{L}_z$, corresponding to the orbital quantum number l,

$$\{\varphi_{lm}\} = (\varphi_{ll}, \varphi_{l,l-1}, \ldots, \varphi_{l,-l}) \tag{9.53}$$

are obtained by applying $\hat{L}_-$ to φ_{ll}. That is,

$$\begin{aligned} \varphi_{l,l-1} &= \hat{L}_- \varphi_{ll} \\ \varphi_{l,l-2} &= \hat{L}_- \varphi_{l,l-1} \end{aligned} \tag{9.54}$$

In either technique for obtaining the eigenfunctions φ_{lm}, it proves both convenient and practical to work in spherical coordinates (r, θ, ϕ) (see Fig. 1.6). These coordinates are related to the Cartesian coordinates (x, y, z) through the transformation equations

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \end{aligned} \tag{9.55}$$

With these equations, the Cartesian components of $\hat{\mathbf{L}}$, (9.3), are transformed to (see Problem 9.14)

$$\begin{aligned} \hat{L}_x &= i\hbar\left(\sin\phi \frac{\partial}{\partial\theta} + \cot\theta \cos\phi \frac{\partial}{\partial\phi}\right) \\ \hat{L}_y &= i\hbar\left(-\cos\phi \frac{\partial}{\partial\theta} + \cot\theta \sin\phi \frac{\partial}{\partial\phi}\right) \\ L_z &= -i\hbar \frac{\partial}{\partial\phi} \end{aligned} \tag{9.56}$$

Using expressions (9.56) we obtain first the ladder operators

$$\begin{aligned} \hat{L}_+ &= \hat{L}_x + i\hat{L}_y = \hbar e^{i\phi}\left(i \cot\theta \frac{\partial}{\partial\phi} + \frac{\partial}{\partial\theta}\right) \\ \hat{L}_- &= \hat{L}_x - i\hat{L}_y = \hbar e^{-i\phi}\left(i \cot\theta \frac{\partial}{\partial\phi} - \frac{\partial}{\partial\theta}\right) \end{aligned} \tag{9.57}$$

and second, the operator $\hat{L}^2$

$$\hat{L}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{9.58}$$

We are now prepared to seek solutions to (9.51). This is the first technique, as mentioned above, for finding the eigenstates φ_{lm}. These solutions are quite common