TABLE 9.1 The first few normalized spherical harmonics and corresponding associated Legendre polynomials[a]

$$Y_l^m(\theta, \phi) = \left[\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^m(\cos\theta)e^{im\phi}$$

$$\int_{-1}^{1} d\cos\theta \int_0^{2\pi} d\phi\; Y_l^m(Y_{l'}^{m'})^* = \delta_{mm'}\,\delta_{ll'}$$

$$P_0 = 1$$

$$P_1{}^1 = -\sin\theta$$

$$P_1{}^0 = \cos\theta$$

$$P_1{}^{-1} = \tfrac{1}{2}\sin\theta$$

$$P_2{}^2 = 3\sin^2\theta$$

$$P_2{}^1 = -3\sin\theta\cos\theta$$

$$P_2{}^0 = \tfrac{1}{2}(3\cos^2\theta - 1)$$

$$P_2{}^{-1} = \tfrac{1}{2}\sin\theta\cos\theta$$

$$P_2{}^{-2} = \tfrac{1}{8}\sin^2\theta$$

$$P_3{}^3 = -15\sin^3\theta$$

$$P_3{}^2 = 15\sin^2\theta\cos\theta$$

$$P_3{}^1 = -\tfrac{3}{2}\sin\theta(5\cos^2\theta - 1)$$

$$P_3{}^0 = \tfrac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

$$P_3{}^{-1} = \tfrac{1}{8}\sin\theta(5\cos^2\theta - 1)$$

$$P_3{}^{-2} = \tfrac{1}{8}\sin^2\theta\cos\theta$$

$$P_3{}^{-3} = \tfrac{1}{48}\sin^3\theta$$

$$Y_l^{-l} = \frac{1}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}}\sin^l\theta e^{-il\phi}$$

$$Y_l^0 = \sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)$$

$$\sum_{m=-l}^{l} |Y_l^m(\theta,\phi)|^2 = \frac{2l+1}{4\pi}$$

$$Y_l^{-m} = (-1)^m (Y_l^m)^*$$

$$Y_0{}^0 = \left(\frac{1}{4\pi}\right)^{1/2}$$

$$Y_1{}^1 = -\frac{1}{2}\left(\frac{3}{2\pi}\right)^{1/2}\sin\theta\, e^{i\phi}$$

$$Y_1{}^0 = \frac{1}{2}\left(\frac{3}{\pi}\right)^{1/2}\cos\theta$$

$$Y_1{}^{-1} = \frac{1}{2}\left(\frac{3}{2\pi}\right)^{1/2}\sin\theta e^{-i\phi}$$

$$Y_2{}^2 = \frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2}\sin^2\theta e^{2i\phi}$$

$$Y_2{}^1 = -\frac{1}{2}\left(\frac{15}{2\pi}\right)^{1/2}\sin\theta\cos\theta e^{i\phi}$$

$$Y_2{}^0 = \frac{1}{4}\left(\frac{5}{\pi}\right)^{1/2}(3\cos^2\theta - 1)$$

$$Y_2{}^{-1} = \frac{1}{2}\left(\frac{15}{2\pi}\right)^{1/2}\sin\theta\cos\theta e^{-i\phi}$$

$$Y_2{}^{-2} = \frac{1}{4}\left(\frac{15}{2\pi}\right)^{1/2}\sin^2\theta e^{-2i\phi}$$

$$Y_3{}^3 = -\frac{1}{8}\left(\frac{35}{\pi}\right)^{1/2}\sin^3\theta\, e^{3i\phi}$$

$$Y_3{}^2 = \frac{1}{4}\left(\frac{105}{2\pi}\right)^{1/2}\sin^2\theta\cos\theta e^{2i\phi}$$

$$Y_3{}^1 = -\frac{1}{8}\left(\frac{21}{\pi}\right)^{1/2}\sin\theta(5\cos^2\theta - 1)e^{i\phi}$$

$$Y_3{}^0 = \frac{1}{4}\left(\frac{7}{\pi}\right)^{1/2}(5\cos^3\theta - 3\cos\theta)$$

$$Y_3{}^{-1} = \frac{1}{8}\left(\frac{21}{\pi}\right)^{1/2}\sin\theta(5\cos^2\theta - 1)e^{-i\phi}$$

$$Y_3{}^{-2} = \frac{1}{4}\left(\frac{105}{2\pi}\right)^{1/2}\sin^2\theta\cos\theta e^{-2i\phi}$$

$$Y_3{}^{-3} = \frac{1}{8}\left(\frac{35}{\pi}\right)^{1/2}\sin^3\theta e^{-3i\phi}$$

[a] Defining relations for $P_l(\mu)$ and $P_l^{-m}(\mu)$ are given in Table 9.3. Comparison with other notations for the spherical harmonics and their related functions may be found in D. Park, *Introduction to the Quantum Theory*, 2nd ed., McGraw-Hill, New York, 1974.

TABLE 9.2 Properties of the Legendre polynomials

Generating Formulas

$$(1 - 2\mu s + s^2)^{-1/2} = \sum_{l=0}^{\infty} P_l(\mu) s^l$$

$$P_l(\mu) = \frac{1}{2^l l!} \frac{d^l}{d\mu^l} (\mu^2 - 1)^l \begin{cases} -1 \le \mu \le 1 \\ l = 0, 1, 2, 3, \ldots \end{cases}$$

Legendre's Equation

$$(1 - \mu^2) \frac{d^2 P_l(\mu)}{d\mu^2} - 2\mu \frac{dP_l(\mu)}{d\mu} + l(l + 1) P_l(\mu) = 0$$

Recurrence Relations

$$(l + 1) P_{l+1}(\mu) = (2l + 1)\mu P_l(\mu) - l P_{l-1}(\mu)$$

$$(1 - \mu^2) \frac{d}{d\mu} P_l(\mu) = -l\mu P_l(\mu) + l P_{l-1}(\mu)$$

Normalization and Orthogonality

$$\int_{-1}^{1} P_l(\mu) P_m(\mu)\, d\mu = \frac{2}{2l + 1} \qquad (l = m)$$

$$= 0 \qquad (l \neq m)$$

The First Few Polynomials

$P_0 = 1$	$P_2 = \frac{1}{2}(3\mu^2 - 1)$	$P_4 = \frac{1}{8}(35\mu^4 - 30\mu^2 + 3)$
$P_1 = \mu$	$P_3 = \frac{1}{2}(5\mu^3 - 3\mu)$	$P_5 = \frac{1}{8}(63\mu^5 - 70\mu^3 + 15\mu)$

Special Values

$$P_l(\mu) = (-1)^l P_l(-\mu) \qquad P_l(1) = 1$$

Polar plots of these functions for $l = 0, 1, 2$, and all accompanying m values, in any plane through the z axis, are sketched in Fig. 9.10.

The functions $Y_l^m(\theta, \phi)$ are a basis of the Hilbert space of square-integrable functions $\varphi(\theta, \phi)$ defined on the unit sphere. Such functions may be normalized as follows.

$$\|\varphi(\theta, \phi)\|^2 = \langle \varphi | \varphi \rangle = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta\, \varphi^* \varphi = 1 \tag{9.79}$$

The expansion of φ in spherical harmonics is given by

$$\varphi(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{|m| \le l} a_{lm} Y_l^m(\theta, \phi) \tag{9.80}$$

The coefficient of expansion a_{lm} is given by the inner product,

$$a_{lm} = \langle Y_l^m | \varphi \rangle = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta [Y_l^m(\theta, \phi)]^* \varphi(\theta, \phi) \tag{9.81}$$