Synthesis of the Planck and Bohr formulations of the correspondence principle

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Planck formulated the correspondence principle between quantum and classical mechanics as the limit in which the Planck constant h goes to zero. Bohr formulated the correspondence principle to be the limit of large quantum numbers. For three common quantum mechanical systems it is shown that in order for eigenvalues of quantum mechanical observables to have meaningful classical limits, it is necessary to take the double limit as both the Planck constant goes to zero and the quantum number goes to infinity, subject to the constraint that their product is equal to an appropriate classical action. This synthesis of the Bohr and Planck formulations of the correspondence principle is also used to show that the quantum mechanical transition frequency between adjacent levels approaches the corresponding classical frequency. The features these systems have in common in their classical limit are explained by general considerations of the classical limit of the Schrödinger equation.

I. INTRODUCTION

There are two different formulations of the correspondence principle between quantum and classical physics. One is Planck's formulation[1] in which the Planck constant h goes to zero.[2] Planck originally formulated this principle to show that his energy density for blackbody radiation approaches the correct classical Rayleigh–Jeans energy density.[1] The other is Bohr's formulation of the correspondence principle, in which the limit of large quantum numbers is used.[3,4] Bohr first enunciated this formulation[3] to show that in his model of the hydrogen atom the transition frequency between adjacent energy levels goes to the classical frequency of an electron in a circular orbit. In a series of papers, Liboff[5–8] emphasizes that these two formulations are not universally equivalent.

In this article, we show, for some simple quantum mechanical systems, that to obtain a meaningful classical limit of quantum mechanical eigenvalues, it is necessary to synthesize the Planck and Bohr formulations of the correspondence principle. Both formulations are used concurrently in the sense that the Planck constant goes to zero and the appropriate quantum number goes to infinity, subject to a constraint that their product be held fixed at the appropriate classical action. Meaningless results are obtained for the classical limit of quantum mechanical eigenvalues if one limit is taken without the other. The double limit as h goes to zero and the quantum number, say n, goes to infinity is subject to the constraint that $nh = J$, where J is the appropriate classical action. The correct value of the classical observable corresponding to the quantum mechanical eigenvalue is obtained in this constrained double limit.

The quantum mechanical systems we consider are the harmonic oscillator, the particle in a box, and the hydrogen atom. In these systems, we identify the basic quantum observables and give their eigenvalues. We then use concurrently the Planck and Bohr formulations of the correspondence principle subject to the constraint that the product of the quantum number and the Planck constant is equal to the appropriate classical action. The correct value of the classical observable is obtained. For the above systems, we show that in the constrained double limit the quantum mechanical transition frequency between adjacent levels becomes the frequency of the classical system.

The common features of these examples are explained by an analysis of the classical limit of the Schrödinger equation. The single-valuedness of the wavefunction leads to the quantization of the classical action $J = nh$ in the semiclassical case. The transition frequency between adjacent energy levels is a difference quotient that becomes the derivative of the energy with respect to the action $\partial E/\partial J$ in the classical limit. This derivative is the classical frequency.[9]

The synthesis of the Planck and Bohr formulations of the correspondence principle we develop here has been used previously in some contexts. For the energy of the harmonic oscillator, ter Haar[10] mentions the necessity of taking both the limits as the Planck constant goes to zero and the quantum number goes to infinity, keeping their product constant. Kubo[11] uses a similar limit to show that the Brillouin function approaches the Langevin function for magnetism. Qian and Huang[12] use the synthesized form of the correspondence principle to show that the quantum mechanical probability density goes over to a classical probability density, which they call the homogeneous ensemble. For the hydrogen atom, Messiah[13] mentions that both $h \to 0$ and $n \to \infty$ should be taken with their product held fixed. In the general case, ter Haar[14] states that for a given value of the action $J = nh$ the classical limit $h \to 0$ corresponds to the limit $n \to \infty$, which is another way of expressing the synthesized form of the correspondence principle. In a private conversation, Borowitz[15] mentioned the constrained double limit to Liboff.[6] An interesting discussion of the correspondence principle prior to the new quantum mechanics is given by several authors.[16]

In Sec. II, the harmonic oscillator is examined. The particle in a one-dimensional box is considered in Sec. III. The hydrogen atom is treated in Sec. IV. In Sec. V, it is shown that the common features in the three examples are in fact general manifestations of the classical limit of the Schrödinger equation. Finally, the conclusion is given in Sec. VI.

II. HARMONIC OSCILLATOR

The simplest quantum mechanical system to examine is the one-dimensional harmonic oscillator, for which the energy is the basic observable. For an oscillator of angular frequency ω the Schrödinger equation gives the energy eigenvalues

$$E_n = (n+\tfrac{1}{2})\hbar\omega, \quad n=0,1,2,3,..., \tag{1}$$

where $\hbar = h/2\pi$. In contrast, the energy of the classical harmonic oscillator E is a continuous variable from zero to arbitrarily large values. When only the Planck formulation of the correspondence principle in which the Planck constant h goes to zero is applied to Eq. (1), the result is zero for all n. On the other hand, when only the Bohr formulation of the correspondence principle in which n goes to infinity is applied to Eq. (1), the result is infinity. Therefore, neither the Planck nor the Bohr formulation of the correspondence principle alone gives the correct classical energy.

If, however, the Planck and Bohr formulations are applied concurrently, so that $h\to 0$ and $n\to\infty$, but with the constraint that their product nh is held fixed,[10] then E_n goes to its classical value E. The constant value at which the product nh is held fixed is the classical action J,[9]

$$nh = J = \pi m\omega A^2, \tag{2}$$

where A is the amplitude of the oscillator. Thus the constrained double limit of Eq. (1) gives

$$E_n \to E = \tfrac{1}{2}m\omega^2 A^2, \quad \text{as } n\to\infty, \quad h\to 0, \quad nh=J, \tag{3}$$

where E is the correct energy of a classical harmonic oscillator of angular frequency ω and amplitude A. The zero-point energy $\frac{1}{2}\hbar\omega$ in Eq. (1), which is obtained from the Schrödinger equation, goes to zero in the constrained double limit. The old quantum theory, in which Eq. (2) is used to quantize the system, does not give the zero-point energy. A general feature of the classical limit of quantum mechanics is that the part of the Schrödinger eigenvalue that is different from the eigenvalue of the old quantum theory goes to zero.

The transition frequency between adjacent states is

$$\omega_{n,n-1} = (E_n - E_{n-1})/\hbar = \omega \tag{4}$$

from Eq. (1), which is the classical frequency. That the transition frequency is the classical frequency is a special feature of the simple harmonic oscillator.[17]

The transition frequency in Eq. (4) can be written as

$$\omega_{n,n-1} = 2\pi(E_n - E_{n-1})/[nh-(n-1)h]$$
$$= 2\pi\frac{\Delta E}{\Delta J}, \tag{5}$$

where $J = nh$ is the classical action. In the constrained double limit, the difference quotient in Eq. (5) becomes[9,18]

$$\omega_{n,n-1} \to 2\pi\frac{\partial E}{\partial J} = \omega, \quad \text{as } h\to 0, \quad n\to\infty, \quad nh=J, \tag{6}$$

where ω is the classical angular frequency. The classical energy E in Eq. (3) can be written in terms of the action J in Eq. (2) as

$$E = \omega J/2\pi. \tag{7}$$

When Eq. (7) is used in Eq. (6), the classical angular frequency ω is indeed obtained. This behavior is a general feature of the synthesized form of the correspondence principle, which is discussed further in Sec. V.

III. PARTICLE IN A BOX

We now consider the classical limit of a quantum mechanical particle of mass m confined to a one-dimensional box of length d. The basic quantum mechanical observable of this system is the energy, for which a solution of the Schrödinger equation gives the eigenvalues

$$E_n = \hbar^2\pi^2 n^2/2md^2, \quad n = 1,2,3,... . \tag{8}$$

The Planck correspondence principle alone applied to Eq. (8) gives zero in the limit as $h\to 0$. The Bohr correspondence principle alone applied to Eq. (8) gives infinity in the limit as $n\to\infty$. In neither case is the correct classical value of the energy obtained. The synthesis of the Planck and Bohr formulations of the correspondence principle is the double limit $h\to 0$ and $n\to\infty$, with the constraint that nh is equal to the classical action J,

$$nh = J = p2d, \tag{9}$$

where p is the magnitude of the linear momentum. This constrained double limit of Eq. (8) gives

$$E_n \to E = p^2/2m, \quad \text{as } h\to 0, \quad n\to\infty, \quad nh=J, \tag{10}$$

where E is the classical energy.

The transition frequency between adjacent states is

$$\omega_{n,n-1} = (E_n - E_{n-1})/\hbar = \hbar\pi^2(2n-1)/2md^2 \tag{11}$$

from Eq. (8). The separate limits of $h\to 0$ and $n\to\infty$ give zero and infinity, respectively, for the transition frequency in Eq. (12). Therefore, the Planck and Bohr formulations of the correspondence principle applied alone do not give the correct classical angular frequency. When the synthesis of the Planck and Bohr formulations of the correspondence principle in which $nh = J$ is used, the classical action in Eqs. (9) and (11) gives the classical angular frequency

$$\omega_{n,n-1} \to \omega = 2\pi(p/m2d), \quad \text{as } n\to\infty, \quad h\to 0, \quad nh=J. \tag{12}$$

The classical angular frequency ω in this problem is 2π times the speed p/m of the particle divided by the distance $2d$ it travels in a complete cycle.

Equation (11) for the transition frequency is the difference quotient given in Eq. (5). In the constrained double limit, the transition frequency becomes[18]

$$\omega_{n,n-1} \to 2\pi\frac{\partial E}{\partial J} = \omega, \quad \text{as } h\to 0, \quad n\to\infty, \quad nh=J, \tag{13}$$

where ω is the classical angular frequency. The classical energy in Eq. (10) can be written in terms of the action in Eq. (9) as

$$E = J^2/8md^2. \tag{14}$$

When Eq. (14) is used in Eq. (13) and Eq. (9) is used for J, the angular frequency in Eq. (12) is obtained.

IV. HYDROGEN ATOM

Bohr first formulated his correspondence principle in connection with his model of the hydrogen atom to ensure that for large quantum numbers the transition frequency between two adjacent states equals the classical orbital frequency of the electron. In this section, we generalize Bohr's work by applying the synthesis of the Bohr and Planck formulations of the correspondence principle to the eigenvalues of the hydrogen atom obtained from the Schrödinger equation. The energy and angular momentum of elliptical orbits are obtained.

The basic quantum mechanical observables for the hydrogen atom are the energy, the square of the angular momentum $\mathbf{L}^2$ and the z component of the angular momentum L_z. The Schrödinger equation gives the eigenvalues of the energy operator as[13]

$$E_n = -\mu e^4/2(4\pi\epsilon_0)^2\hbar^2 n^2, \quad n = 1,2,3,..., \tag{15}$$

where n is the principal quantum number, μ is the reduced mass of the electron, $-e$ is the charge on the electron, and ϵ_0 is the permittivity of the vacuum. The eigenvalues of the square of the angular momentum operator are

$$L_l^2 = l(l+1)\hbar^2, \quad l = 0,1,2,..., n-1, \tag{16}$$

where l is the orbital angular momentum quantum number. The eigenvalues of the z component of the angular momentum are

$$L_{zm} = m\hbar, \quad m = 0, \ \pm 1, \ \pm 2,..., \ \pm l, \tag{17}$$

where m is the magnetic quantum number. If only the Planck constant goes to zero, the energy in Eq. (15) goes to infinity, while Eqs. (16) and (17) go to zero. If only the quantum numbers n, l, and $|m|$ go to infinity, the energy in Eq. (15) goes to zero, while Eqs. (16) and (17) go to infinity. In these cases, the limits do not correspond to the values of the classical variables. Therefore, neither the Planck nor the Bohr formulation of the correspondence principle alone gives the correct classical values. If a synthesis of the Planck and Bohr formulations of the correspondence principle is used, in which the Planck constant goes to zero and the quantum numbers n, l, and $|m|$ go to infinity, subject to the constraint that nh, lh, and mh are held fixed at the appropriate classical actions, then the limits of Eqs. (15)–(17) give the correct classical values of E, L^2, and L_z.

The constraints on the product of the quantum number and the Planck constant are obtained by setting the product equal to the appropriate classical action. The product of n and h is constrained to be the action[19]

$$nh = J_3 = (\pi\mu e^2 a/\epsilon_0)^{1/2}, \tag{18}$$

where a is the semimajor axis of the ellipse. The product of the angular momentum quantum number l and the Planck constant is constrained to be the action[19]

$$lh = J_2 = J_3(1-\epsilon^2)^{1/2}, \tag{19}$$

where $0 \leqslant \epsilon \leqslant 1$ is the eccentricity of the ellipse. The product of the magnetic quantum number m and the Planck constant is also equal to the appropriate classical action [19]

$$mh = J_1 = J_2 \cos\theta, \tag{20}$$

where θ is the polar angle of the angular momentum vector and the action J_2 is defined in Eq. (19).

The constrained double limit of the energy in Eq. (15) is

$$E_n \to E = -e^2/8\pi\epsilon_0 a, \quad \text{as } h\to 0, \ n\to\infty, \ nh = J_3, \tag{21}$$

where E is the classical energy. The constrained double limit of the square of the angular momentum in Eq. (16) is

$$L_l \to L = [\mu e^2 a(1-\epsilon^2)/4\pi\epsilon_0]^{1/2}, \quad \text{as } h\to 0, \ l\to\infty, \ lh = J_2, \tag{22}$$

where L is the classical angular momentum and J_2 is given in Eq. (19). Finally, the constrained double limit of Eq. (17) is

$$L_{zm} \to L\cos\theta, \quad \text{as } h\to 0, \ |m|\to\infty, \ mh = J_1 \tag{23}$$

from Eqs. (20) and (19).

The transition frequency between adjacent states is

$$\omega_{n,n-1} = (E_n - E_{n-1})/\hbar = \mu e^4(2n-1)/2(4\pi\epsilon_0)^2\hbar^3 n^2(n-1)^2 \tag{24}$$

from Eq. (15). The separate limits $h\to 0$ and $n\to\infty$ are infinity and zero, respectively, which do not correspond to the classical frequency. The constrained double limit, in contrast, gives the classical angular frequency ω,

$$\omega_{n,n-1} \to \omega = (e^2/4\pi\epsilon_0\mu a^3)^{1/2}, \quad \text{as } h\to 0, \ n\to\infty, \ nh = J_3. \tag{25}$$

Equation (25) implies Kepler's third law, which states that $\tau^2 \propto a^3$, where $\tau = 2\pi/\omega$ is the period and a is the semimajor axis of the ellipse.

The transition angular frequency in Eq. (25) is a difference quotient, as in Eq. (5). In the constrained double limit, the transition angular frequency becomes

$$\omega_{n,n-1} \to 2\pi\frac{\partial E}{\partial J_3} = \omega, \quad \text{as } h\to 0, \ n\to\infty, \ nh = J_3, \tag{26}$$

where ω is the classical angular frequency. The classical energy E in Eq. (21) can be expressed in terms of the action J_3 in Eq. (18) as[19]

$$E = -\mu e^4/8\epsilon_0^2 J_3^2. \tag{27}$$

When Eq. (27) is used in Eq. (26) and the action in Eq. (18) is used, Eq. (25) for the classical frequency is obtained.

V. GENERAL FORMULATION OF THE CLASSICAL LIMIT

A common theme exists in the previous examples. It is therefore appropriate to consider the general approach to the classical limit of energy eignevalues obtained from the Schrödinger equation.[20,21]

For the sake of simplicity, a system with only one degree of freedom x is considered. The stationary state Schrödinger equation for this system is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi, \tag{28}$$

where m is the mass of the particle and $V(x)$ is the potential energy.

The complex wavefunction $\psi(x)$ in Eq. (28) can be written in terms of a modulus $\rho^{1/2}$ and a phase $\phi/\hbar$ as

$$\psi = \rho^{1/2}\exp(i\phi/\hbar), \tag{29}$$

where $\rho = \psi^*\psi$ is the probability density of the particle. When Eq. (29) is substituted into the Schrödinger equation in Eq. (28), the result can be written as

$$p^2 = -\frac{\hbar^2}{2}\frac{d^2\ln\rho}{dx^2} - \frac{\hbar^2}{4}\left(\frac{d\ln\rho}{dx}\right)^2 + \left(\frac{d\phi}{dx}\right)^2 - i\hbar\frac{d\phi}{dx}\frac{d\ln\rho}{dx} - i\hbar\frac{d^2\phi}{dx^2}. \tag{30}$$

The magnitude of the momentum p is

$$p = [2m(E-V)]^{1/2}. \tag{31}$$

If the imaginary part of Eq. (30) is equated to zero, the result gives probability current conservation. If the real parts of each side of Eq. (30) are equated and the square root taken, the result is

$$\frac{d\phi}{dx} = p\left[1 + \frac{\hbar^2}{2p^2}\frac{d^2 \ln \rho}{dx^2} + \frac{\hbar^2}{2p^2}\left(\frac{d \ln \rho}{dx}\right)^2\right]^{1/2}. \quad (32)$$

If the system is periodic, Eq. (32) can be integrated over one cycle to give

$$\oint d\phi = \oint p\left[1 + \frac{\hbar^2}{2p^2}\frac{d^2 \ln \rho}{dx^2} + \frac{\hbar^2}{4p^2}\left(\frac{d \ln \rho}{dx}\right)^2\right]^{1/2} dx. \quad (33)$$

Since $\phi/\hbar$ is the phase of the wavefunction in Eq. (29), the change $\Delta\phi$ of ϕ over a complete cycle is

$$\Delta\phi = \oint d\phi = 2\pi n\hbar = nh, \quad (34)$$

where n is an integer. If the logarithm of the probability density ρ varies slowly over a de Broglie wavelength $\lambda = h/p$, then Eq. (33) becomes

$$\oint p\, dx = nh, \quad (35)$$

which is the Bohr–Sommerfeld condition.[21] The left-hand side of Eq. (35) is the classical action J. Thus it is not surprising that in Secs. II–IV the product nh had to be set equal to the value of the classical action. Furthermore, to ensure that the classical action does not collapse to zero as $h \to 0$ alone, or become infinite as $n \to \infty$ alone, it is necessary to take the combined limit

$$\oint p\, dx = J, \quad \text{as } h \to 0, \; n \to \infty, \; nh = J. \quad (36)$$

Since p is defined in Eq. (31) in terms of the energy E, Eq. (36) gives the action J as a function of the energy E. In Secs. II–IV the energy E as a function of J was obtained. A glance at Eqs. (7), (14), and (27) shows that to allow J to become zero or infinity would yield either a zero or infinite energy.

The classical frequency $\nu = \omega/2\pi$ of the system can be obtained from the period τ of the system. The period is

$$\tau = \oint dt = \nu^{-1}, \quad (37)$$

where the element of time $dt = dx/v$ and the speed $v = p/m$. If Eq. (36) is differentiated with respect to J and use is made of Eq. (31), the result is

$$\left(\oint dx\, mp^{-1}\right)\left(\frac{\partial E}{\partial J}\right) = 1. \quad (38)$$

The factor multiplying $\partial E/\partial J$ is the period τ, so

$$\frac{\partial E}{\partial J} = \nu, \quad (39)$$

which is equivalent to Eq. (6). Equation (39) is a general feature of classical mechanics[18] and shows why it is satisfied in Secs. II–IV.

VI. CONCLUSION

When Planck[1] obtained the classical Rayleigh–Jeans energy density for blackbody radiation by taking the limit as $h \to 0$ alone, he was applying the correspondence principle to a statistical average of the energy eigenvalues over a canonical probability distribution. There were, therefore, no quantum numbers to consider, since a sum over them had already been performed. When Bohr[3] applied his formulation of the correspondence principle to the transition frequency in his model of the hydrogen atom, he obtained the correct classical orbital frequency because he did not take the mathematical limit as the quantum number goes to infinity. Instead, Bohr considered that for large quantum numbers the quantum result was very nearly the classical result. Thus the quantum number n, say, was $n \gg 1$, but was not infinite. Otherwise, the transition frequencies in the hydrogen atom would collapse to zero, as Eq. (24) shows.

The correspondence principle that connects quantum mechanics to classical mechanics has often been thought to be imprecise because of Bohr's use of large, but not infinite, quantum numbers. However, when the Bohr and Planck formulations are synthesized, as shown in this article, the correspondence principle has a precise mathematical form.

An examination of a number of modern physics textbooks[22] shows that Bohr's formulation in which the quantum number is large, but finite, is used for the hydrogen atom. Planck's formulation is often used in the discussion of blackbody radiation. The synthesized formulation of the correspondence principle discussed in this article ends this dualistic approach. The synthesis of the Planck and Bohr formulations of the correspondence principle discussed here could be used in an undergraduate course to give students a deeper appreciation of the classical limit of quantum mechanics.

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[16]A. Sommerfeld, *Atomic Structure and Spectral Lines* (Methuen, London, 1923), 3rd ed., pp. 577–587; S. Tomonaga, *Quantum Mechanics* (North-Holland, Amsterdam, 1962), Vol. I, pp. 139–159; M. Born, *The Mechanics of the Atom* (Ungar, New York, 1960), pp. 60–71, 99–107.

[7]The transition frequency $\omega_{n,n-1}$ can be expressed as a function of E_n *for* $n \gg 1$ as $\omega_{n,n-1} = (2E_n/mA^2)^{1/2}$. This expression is functionally the same as the classical relation $\omega = (2E/mA^2)$.[1/2] In Refs. 5–8, this correspondence is called *form correspondence*. On the other hand, Refs. 5–8 define a system to have *frequency correspondence* when transition frequencies of the quantum spectrum approach arbitrarily close to the classical frequencies. In the synthesized Planck and Bohr formulations of the correspondence principle used in this article, both types of correspondence are satisfied.

[18]See, e.g., Ref. 9, p. 460, where the Hamiltonian $H(J)$ is used, which is the energy in this problem.

[19]See, e.g., Ref. 9, pp. 476–484.

[20]S. Borowitz, *Fundamentals of Quantum Mechanics* (Benjamin, New York, 1967), p. 159.

[21]M. Born, *Atomic Physics* (Hafner, New York, 1962), 7th ed., pp. 137 and 138.

[22]See, e.g., R. Eisberg and R. Resnick, *Quantum Physics* (Wiley, New York, 1974), pp. 128–129, 19; A. Beiser, *Concepts of Modern Physics* (McGraw-Hill, New York, 1981), 3rd ed., pp. 135, 312; P. A. Tipler, *Modern Physics* (Worth, New York, 1978), pp. 145, 150, 105.