

An economic analogy to thermodynamics

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We develop analogies between economic systems and thermodynamics, and show how economic quantities can characterize the state of an economic system in equilibrium. We argue that just as a physical system in thermodynamic equilibrium requires a nonmechanical variable (the temperature T) to specify its state, so does an economic system. In addition, both systems must have a corresponding conjugate quantity, the entropy S . We also develop economic analogies to the free energy, Maxwell relations, and the Gibbs–Duhem relationship. Assuming that economic utility can be measured, we develop an operational definition of an economic temperature scale. We also develop an analogy to statistical mechanics, which leads to Gaussian fluctuations. © 1999 American Association of Physics Teachers.

I. INTRODUCTION

We propose a theory for the phenomenological description of economic systems, based on the analogous description of thermodynamic systems. Like thermodynamics, the present theory has predictive power only to the extent that it can relate different sets of measurements. Within this context, it makes many statements that extend beyond conventional economic theory. Fundamental to our analysis is the assumption that the economic system be in equilibrium; however, it is not our purpose to determine when a system is in equilibrium.

Several years ago, Kagel and Battalio¹ performed a set of experiments on the economic behavior of rats. Price was determined by the number of bar-presses per payoff from a fluid dispenser. The rats were given two desirable commodities whose prices were then varied. These authors found rat behavior to be both deterministic (because of the well-defined average behavior), and probabilistic (because of fluctuations about the average behavior). The observed fluctuations suggest that there is an economic equivalent to the temperature T and its conjugate variable S .

Relatively little of the economics literature discusses fluctuations of a single economic system, a notable exception being the work by Theil.² However, there is a long tradition in economics of considering analogies between economic and physical systems.³ In particular, there have been studies in economics that develop various implications of the concept of entropy.^{4–7} However, Refs. 4–7 do not use entropy in the same way as we do here—as a quantity that defines the internal state of an economic system.

In Sec. II we give the basic elements of our analogy between economics and thermodynamics. To provide background for the details of the analogy, we review the formalism and history of thermodynamics in Sec. III. In Sec. IV we discuss the economic analogies to thermodynamics, and observe that the present formalism can be used to give a more precise meaning to Marshall's concept of surplus. Specifically, the formalism implies that an increase in the Marshallian surplus is due to either increased leisure (Veblenian surplus) or increased efficiency (Smithian surplus). In Sec. V we discuss thermometry (and "entrometry") in thermodynamics, and in Sec. VI we discuss thermometry and entrometry in economics. Section VII discusses an analogy to statistical mechanics, which supports our thermodynamic analogy by an alternate argument based on a wealth-

maximization principle that leads to Gaussian fluctuations. Section VIII concludes with a brief summary and discussion.

II. ECONOMIC ANALOGIES

Examples of economic systems of interest are an individual consumer or a small country, each of which is embedded within a larger economic system. Consider an individual consumer. A fundamental assumption in economics is that the consumer employs a *utility function* U to choose to purchase one good over another. For many purposes, it is sufficient for the utility to be an ordinal quantity (that is, it specifies only relative ordering). However, to make the full analogy to thermodynamics, we must take the utility U to be a real number. We assume U to be given in a convenient set of units, such as 1998 dollars, and we also assume that U is measurable.⁸ The formalism we develop is falsifiable, and can be overdetermined by a proper set of measurements, thus providing constraints on its consistency. To explain the analogy, we begin by discussing certain fundamental relations in economics.

First consider the measurable economic quantity known as wealth,

$$W = \lambda M + pN \quad (\text{economics}), \quad (1)$$

where λ and M represent the value and amount of money, and p and N represent vectors of prices and numbers of goods. (In principle, λ and M can be considered vectors if the consumer has holdings in more than one type of currency. Or, we can lump both goods and money into the p and N vectors.) Because W is conserved in transactions, as is the total energy E of a thermodynamic system, it is tempting to consider them to be analogous. From the viewpoint of maximization, a more natural analogy is between $-E$ and W , and this analogy is pursued in Sec. VII, which discusses an analogy to statistical mechanics. However, from the viewpoint of making an analogy to the dependence on the extensive thermodynamic variables, we will see in Sec. IV that the most appropriate analogy is between E and U .

Economics assumes that the value of an individual consumer's money and goods is summarized by the value of U , which typically exceeds W . The excess is known as the surplus (a term due to Alfred Marshall), for which we introduce the notation Ψ (for "psurplus").^{9–11} Thus

Table I. Summary of the suggested analogies between thermodynamic and economic systems. Not listed are the direct analogies between S and T .

Thermodynamics	$-F$	$-E$	TS	μ	N
Economics	$W(\text{wealth})$	$U(\text{utility})$	$\Psi(\text{surplus})$	$p(\text{price})$	$N(\# \text{ of goods})$

$$\Psi = U - W \quad (\text{economics}). \quad (2)$$

In a primitive or very poor economy, there is no surplus, so $\Psi = 0$. In this case, every individual performs the same economic function at the same efficiency, and there is no benefit from specialization and trade. The surplus Ψ cannot be negative; for typical economic systems $\Psi > 0$. Although Eq. (2) appears only to define another unknown quantity, Ψ , in terms of U , this economic relationship is useful because it has a thermodynamic analogue.

The Helmholtz free energy of a system with N identical particles is defined as

$$F = -PV + \mu N \quad (\text{thermodynamics}), \quad (3)$$

where P is the pressure, V is the volume, and μ is the chemical potential of the particles. We may think of $-PV$ as analogous to λM . (There may be a deeper relationship, wherein the ‘‘economic engine’’ uses money λdM rather than work $-PdV$, but we do not pursue this relationship here.) The quantity in thermodynamics analogous to the price p is the chemical potential μ . The energy E is related to F in terms of the temperature T and entropy S via

$$TS = E - F \quad (\text{thermodynamics}). \quad (4)$$

Note that, according to the third law of thermodynamics, $S = 0$ for a system at $T = 0$.

A comparison of Eqs. (2) and (4) suggests another analogy, that of Ψ and TS . By taking a system with zero surplus (and thus zero economic temperature) to have zero economic entropy, we assume the economic analogue of the third law of thermodynamics.

Because the surplus is zero for an undeveloped economy, we make the tentative common language identification of economic temperature T with the *level of economic development*. This usage seems consistent with the idea that T is an intensive quantity. We propose no common language definition of economic entropy, although we expect it to be related to economic variety, which in turn may be a measure of the economic value of leisure. We summarize the thermodynamic and economic analogies in Table I.

III. THERMODYNAMICS, ENTROPY, AND EQUILIBRIUM

The concepts of temperature and entropy are well developed in thermodynamics. However, when thermodynamics was in its nascent state, these concepts were obscure. To provide background for our development of the economic analogy to thermodynamics, we give a brief review of certain key ideas in thermodynamics.

Thermodynamics deals with the transformation of heat into mechanical work, and dates from the work of Carnot (1824), who established the principle limiting the amount of work obtainable from heat under given conditions. The rea-

soning by which Carnot established his principle ‘‘... is justly regarded as one of the most remarkable triumphs of the deductive method.’’¹²

Carnot was concerned with the efficiency of heat engines. He determined that for a heat engine operating between two nearby temperatures, the ideal efficiency (useful work divided by the heat provided by the hotter reservoir) is proportional to the temperature difference times an unknown but universal function of the average temperature. Carnot’s thinking is the basis of the second law of thermodynamics. It is remarkable that it was established well before the experimental studies (1845) by James Joule that established the first law of thermodynamics that heat is a form of energy.

According to Joule, a system with a fixed number of particles can change its energy in either of two ways: heat dQ can enter the system or work dW can be done on the system. Energy conservation is written as

$$dE = dQ + dW. \quad (5)$$

From Carnot’s studies we can show that, in equilibrium, $dQ = TdS$, where T is the temperature and the entropy S is a function of the state of the system. Hence $dS = dQ/T$ is a true differential. It took 30 years to conceive and then to establish this simple but nontrivial statement. Because the work done on the system is $dW = -PdV$, energy conservation takes the form

$$dE = TdS - PdV. \quad (6)$$

Although the energy of a thermodynamic system is uniquely defined, its heat content (the integral of dQ) and its work content (the integral of $-PdV$) are not. A system can go from one energy state to another via an infinite number of processes where the contributions of heat and work differ. Thus neither dQ nor dW is an exact differential. However, $dW/P = -dV$ and $dQ/T = dS$ are exact differentials.

Finally, if particles can enter or leave the system by an amount dN , there is an energy change μdN . The fundamental relation of thermodynamics combines these changes of energy to yield

$$dE = TdS - PdV + \mu dN. \quad (7)$$

Thermodynamics involves two important classes of variables. We define them by an example. Consider two chambers of identical gases at the same temperature, pressure, and chemical potential. On connecting the chambers, the energy E , volume V , and number of particles N of the combined system will be the sum of the energies, volumes, and numbers of particles for the individual systems. Such variables are called *extensive*. The entropy S also is an extensive variable. On the other hand, the temperature T , pressure P , and chemical potential μ of the combined system will be unchanged. Such variables are called *intensive*.¹³

Another way to write Eq. (7) is to consider the energy as an extensive variable and a function of the three extensive variables S , V , and N :

$$E = E(S, V, N). \quad (8)$$

One goal of thermodynamics is to provide a theoretical framework so that experimental measurements can determine this functional dependence for a given system. A knowledge of $E(S, V, N)$ completely characterizes the thermodynamic state of the system. For this reason $E(S, V, N)$ is called the *state function* for the system.

Because changes in the energy E are characterized by changes in S , V , and N , we have from Eq. (8):

$$dE = \frac{\partial E}{\partial S} dS + \frac{\partial E}{\partial V} dV + \frac{\partial E}{\partial N} dN. \quad (9)$$

A comparison of Eqs. (7) and (9) leads to the identifications

$$T \equiv \left(\frac{\partial E}{\partial S} \right)_{V, N}, \quad P \equiv - \left(\frac{\partial E}{\partial V} \right)_{S, N}, \quad \mu \equiv \left(\frac{\partial E}{\partial N} \right)_{S, V}. \quad (10)$$

The Maxwell relations are a consequence of the fact that the order of the cross derivatives of $E(S, V, N)$ does not matter. Hence, from Eqs. (8) to (10), we have

$$\begin{aligned} \frac{\partial^2 E}{\partial S \partial V} &= \frac{\partial^2 E}{\partial V \partial S} \quad \text{or} \quad - \left(\frac{\partial P}{\partial S} \right)_{V, N} = \left(\frac{\partial T}{\partial V} \right)_{S, N}, \\ \frac{\partial^2 E}{\partial S \partial N} &= \frac{\partial^2 E}{\partial N \partial S} \quad \text{or} \quad \left(\frac{\partial \mu}{\partial S} \right)_{V, N} = \left(\frac{\partial T}{\partial N} \right)_{S, V}, \\ \frac{\partial^2 E}{\partial N \partial V} &= \frac{\partial^2 E}{\partial V \partial N} \quad \text{or} \quad - \left(\frac{\partial P}{\partial N} \right)_{S, V} = \left(\frac{\partial \mu}{\partial V} \right)_{S, N}. \end{aligned} \quad (11)$$

These relations guarantee that integrals over dE in S , V , N space are path independent.

Because the energy is an extensive quantity, it satisfies¹³

$$E(\alpha S, \alpha V, \alpha N) = \alpha E(S, V, N), \quad (12)$$

for arbitrary $\alpha > 0$. That is, scaling the extensive quantities S , V , and N also scales the extensive quantity E , which depends upon S , V , and N . Note that T , P , and μ are unchanged because this scaling corresponds to a system that is α times larger. Differentiating the left-hand side of Eq. (12) with respect to α , and then using both Eqs. (12) and (10), gives

$$\begin{aligned} \frac{dE(\alpha S, \alpha V, \alpha N)}{d\alpha} &= \frac{\partial E(\alpha S, \alpha V, \alpha N)}{\partial(\alpha S)} \frac{\partial(\alpha S)}{\partial\alpha} \\ &\quad + \frac{\partial E(\alpha S, \alpha V, \alpha N)}{\partial(\alpha V)} \frac{\partial(\alpha V)}{\partial\alpha} \\ &\quad + \frac{\partial E(\alpha S, \alpha V, \alpha N)}{\partial(\alpha N)} \frac{\partial(\alpha N)}{\partial\alpha} \\ &= \frac{\partial E(S, V, N)}{\partial S} S + \frac{\partial E(S, V, N)}{\partial V} V \\ &\quad + \frac{\partial E(S, V, N)}{\partial N} N = TS - PV + \mu N. \end{aligned} \quad (13)$$

Differentiating the right-hand side of Eq. (12) gives

$$\frac{dE(\alpha S, \alpha V, \alpha N)}{d\alpha} = E(S, V, N). \quad (14)$$

Equating Eqs. (13) and (14) yields the fundamental relationship

$$E(S, V, N) = TS - PV + \mu N. \quad (15)$$

Subtracting Eq. (9) from the differential of Eq. (15) yields the Gibbs–Duhem relation

$$0 = SdT - VdP + Nd\mu. \quad (16)$$

For some purposes, the set (T, V, N) rather than (S, V, N) is a more natural set of variables. In this case, the appropriate thermodynamic potential is the Helmholtz free energy F . Combining Eqs. (3) and (15) yields

$$F = E - TS. \quad (17)$$

Using Eq. (10) and Eq. (17), dF satisfies

$$dF = dE - TdS - SdT = -SdT - PdV + \mu dN, \quad (18)$$

which is consistent with the natural set of variables (T, V, N) . Like $E(S, V, N)$, $F(T, V, N)$ is a state function. Note that $dF = 0$ for a system at fixed V and N in contact with a thermal reservoir that fixes T . Further, at fixed T and N , the work $dW = -PdV$ done on the system equals dF . This relation is the origin of the terminology *free energy*.

Equation (17) is the *Legendre transformation* of the energy and enables us to go from a function with natural variables (S, V, N) to one with natural variables (T, V, N) . The variables S and T are said to be *dual* to one another, as are the variables V and P and N and μ . If a system has two possible states with the same T , V , and N , the state with the lower free energy $F(T, V, N)$ is thermodynamically stable.

Using its natural variables, Eq. (3) for F may be written as

$$F(T, V, N) = -PV + \mu N. \quad (19)$$

By taking two cross-derivatives with respect to its variables, we can derive three new Maxwell relations from the free energy $F(T, V, N)$.

IV. RELATING THERMODYNAMICS AND ECONOMICS

Our goal in making an analogy between economics and thermodynamics is to provide a theoretical framework so that economics measurements can determine the functional dependence of the utility U on the economic parameters that specify the state of an economic system. A knowledge of the state function as a function of the appropriate economic parameters completely characterizes the economic system.

From the economic relations introduced in Sec. I, we have

$$\Psi = TS, \quad (20)$$

and

$$U = TS + W = TS + \lambda M + pN. \quad (21)$$

A comparison of Eq. (21) for U to Eq. (15) for E suggests that, from the point of view of its natural set of variables, we have

$$U = U(S, M, N). \quad (22)$$

Relation (22) is our fundamental assumption.

The economic equivalent of Eq. (7) is

$$dU = TdS + \lambda dM + pdN, \quad (23)$$

where

$$T \equiv \left(\frac{\partial U}{\partial S} \right)_{M,N}, \quad \lambda \equiv \left(\frac{\partial U}{\partial M} \right)_{S,N}, \quad p \equiv \left(\frac{\partial U}{\partial N} \right)_{S,M}. \quad (24)$$

Let us now apply this theoretical structure.

In the *The Wealth of Nations*, Adam Smith distinguishes between two measures of utility.¹⁴ One measure is the “value in exchange.” In economics it is conventional to identify the value in exchange with the price p . From (24), we take this measure to be the *marginal utility per good* dU/dN at fixed S and M . (This last statement is deceptive, because we have not yet given S an operational definition.) Another measure is the “value in use,” which is less readily identified.

We will identify “value in use” with the *marginal utility per good* dU/dN for another set of fixed variables. For simplicity, we will take M to be fixed, but we cannot be explicit about the second variable that is to be held fixed, and will simply denote it as x . (Perhaps the second variable is T ; our uncertainty about which variable is to be held fixed is an echo of conventional economics, where the imprecise phrase “all other quantities held constant” is commonly invoked.) From Eq. (23) we then have

$$\left(\frac{\partial U}{\partial N} \right)_{x,M} = T \left(\frac{\partial S}{\partial N} \right)_{x,M} + p. \quad (25)$$

One of the great triumphs of nineteenth century “margin-*alist*” economic theory is the following statement: A consumer will purchase goods subject to the condition that the ratio of the value of any good in use to its price p takes on a common value.⁸ This statement follows on requiring that, for fixed market values of goods, U be maximized for each good. Fixed market value means that the goods 1 and 2 are exchanged in the marketplace subject to the condition

$$0 = p_1 dN_1 + p_2 dN_2. \quad (26)$$

The maximization of U requires that

$$0 = \frac{\partial U}{\partial N_1} dN_1 + \frac{\partial U}{\partial N_2} dN_2. \quad (27)$$

Combining Eqs. (26) and (27) then yields

$$\frac{1}{p} \frac{\partial U}{\partial N} = \text{constant} \quad (28)$$

for each good. Thus, as desired, the ratio of value in use to price is a constant.

Using Eq. (25), Eq. (28) can be expressed as

$$\frac{1}{p} \left(\frac{\partial U}{\partial N} \right)_{x,M} = \frac{T}{p} \left(\frac{\partial S}{\partial N} \right)_{x,M} + 1 = \text{constant}. \quad (29)$$

From Eq. (29), the constancy of this ratio for all goods does not depend on whether the price is included in the computation of utility. The present formalism helps us focus on the issue of “what is held constant.” Let m represent the value in use (marginal utility per good, at fixed x and M):

$$m \equiv \left(\frac{\partial U}{\partial N} \right)_{x,M}. \quad (30)$$

Note that m is specified in monetary units. From Eqs. (28) and (29), the ratio m/p has the same dimensionless value for all goods.

The ratio m/p can be generalized to include the value of currency, thus permitting the study of saving. Specifically, define

$$m_\lambda \equiv \left(\frac{\partial U}{\partial M} \right)_{x,N}. \quad (31)$$

Then the ratio of value in use to value in exchange for money, m_λ/λ , takes on the same value as m/p for goods.

If we use Eq. (21) to relate W and U , the analogy to the development associated with F leads to

$$dW = -SdT + \lambda dM + p dN, \quad (32)$$

where

$$S \equiv -\frac{\partial W}{\partial T}, \quad \lambda \equiv \frac{\partial W}{\partial M}, \quad p \equiv \frac{\partial W}{\partial N}. \quad (33)$$

It is implicit that two of the three quantities (T , M , N) are held constant in the partial derivatives. From Eq. (32) we may write the functional dependence

$$W = W(T, M, N). \quad (34)$$

Another standard economic relationship states that when an individual consumer interacts with the market, the price (marginal cost per good) is determined by the market. We can obtain this result by assuming that, in equilibrium, the total wealth of the consumer and of the market is maximized at fixed temperature and money. Considering the market to be a reservoir r , we have from Eq. (32)

$$dW_r = -S_r dT_r + \lambda_r dM_r + p_r dN_r. \quad (35)$$

Subject to the conditions $dT = dT_r = 0$, conservation of money ($dM + dM_r = 0$), and conservation of goods ($dN + dN_r = 0$), we find by adding Eqs. (32) and (35) that

$$dW + dW_r = (\lambda - \lambda_r) dM + (p - p_r) dN. \quad (36)$$

The right-hand side of Eq. (36) is zero for arbitrary variations dM and dN only if the value of money to the consumer is the same as the value of money to the market: $\lambda = \lambda_r$. Similarly, for the value of a good, we have $p = p_r$.

Note that Eq. (1) gives the differential

$$dW = \lambda dM + p dN + M d\lambda + N dp. \quad (37)$$

The consistency of Eqs. (37) and (32) requires that

$$0 = SdT + M d\lambda + N dp. \quad (38)$$

Equation (38) is the analogue of Eq. (16), the Gibbs–Duhem relation. Among other things, it implies that a decrease in the price of money or goods (as when the state of economic development increases) is accompanied by an increase in the economic temperature. This qualitative behavior is expected from conventional economic reasoning. Specifically, if all prices and currency values are increased by a common factor, then the system does not really change: By Eq. (38) the temperature is simply increased by the common factor. However, Eq. (38) holds for more general variations. Note that scaling by a factor of 2 means calling a one-dollar bill a two-dollar bill, etc. To really change the value of the dollar would require printing more dollars, which is a real cost that cannot be scaled away.

Equation (38) has an important application. We can write the change in the Marshallian surplus, $\Psi = TS$, as

$$d\psi = TdS + SdT = TdS - Md\lambda - Ndp. \quad (39)$$

We interpret the term TdS as the change in the economic value of leisure. We shall call TdS the *Veblenian surplus* (for the economist Thorstein Veblen, known for his studies of leisure). As we will show shortly (and as is well-known to economists), the term $-Ndp$ is the change in the consumer's surplus of goods. We shall call $-Ndp$ the *Smithian surplus* (for the economist Adam Smith, who argued that efficiencies produced by specialization lead to such surpluses). Likewise, the term $-Md\lambda$ may be interpreted as a currency surplus, due to efficiencies produced by specialization.¹⁵ We shall also call this currency surplus a *Smithian surplus*.

To see that $-Ndp$ is a surplus, note that the cost of incrementally purchasing goods (where the first goods are scarce and, hence, costly) is $\int_0^N p dN$, where $p(N)$, the price of the N th good, decreases as N increases (that is, $dp/dN < 0$.) However, when purchased all at once, the actual cost to the consumer is Np , the number of goods times the latest cost per good. The difference is

$$\int_0^N p dN - Np = - \int_{p(N)}^{p(0)} N dp. \quad (40)$$

The difference is positive, because for the limits of integration in Eq. (40), dp is negative. Hence $-Ndp$ is the change in the consumer's surplus of goods.

Equation (39) shows that there are two types of surplus: the *Veblenian surplus* of leisure and the *Smithian surplus* from efficiency due to specialization. These ideas are present in economics, but we are unaware of any previous formal statement that relates the *Marshallian*, *Veblenian*, and *Smithian* surpluses. Note that the statement of the constancy of m/p for all goods purchased by a given consumer is the same as the statement that the ratio of *Veblenian surplus* per good to the price per good is a constant for all goods purchased by a given consumer.

The following "Maxwell relations" are an immediate consequence of the fact that the order of the cross derivatives of $U = U(S, M, N)$ does not matter. Thus

$$\frac{\partial^2 U}{\partial S \partial N} = \frac{\partial^2 U}{\partial N \partial S} \quad \text{or} \quad \left(\frac{\partial p}{\partial S} \right)_{M, N} = \left(\frac{\partial T}{\partial N} \right)_{S, M}, \quad (41)$$

$$\frac{\partial^2 U}{\partial S \partial M} = \frac{\partial^2 U}{\partial M \partial S} \quad \text{or} \quad \left(\frac{\partial \lambda}{\partial S} \right)_{M, N} = \left(\frac{\partial T}{\partial M} \right)_{S, N}, \quad (42)$$

$$\frac{\partial^2 U}{\partial M \partial N} = \frac{\partial^2 U}{\partial N \partial M} \quad \text{or} \quad \left(\frac{\partial p}{\partial N} \right)_{S, V} = \left(\frac{\partial \lambda}{\partial M} \right)_{S, N}. \quad (43)$$

In economics, these relations are known as *Slutsky conditions*.^{16,17} They guarantee that integrals over dU in S , N , M space are path independent. Similar Slutsky conditions can be derived from W , for which the natural variables are T , M , and N . Equations (41) and (42) are new; Eq. (43) is already known in a form where the dependence on entropy is not made explicit. The addition of the variables T and S helps make more precise the meaning of the phrase "all other quantities held constant."

V. THERMOMETRY IN THERMODYNAMICS

We have so far assumed that economic "temperature" is a well-defined quantity. However, in thermodynamics it took

hundreds of years before the qualitative thermometers based upon the height of a column of fluid were properly calibrated against an absolute temperature scale T . The ideal gas law makes the determination of absolute temperature a relatively simple one. When it is applicable to real gases at low density N/V and high absolute temperature T , the ideal gas law, $PV = Nk_B T$, makes thermometry relatively easy (k_B is Boltzmann's constant). Hence, a measurement of P and N/V gives T .

It will be difficult to find an economic analogue to the ideal gas thermometer. One difference is that the speed of the particles of an ideal gas has no upper bound and a bounded lower limit on its energy, whereas an economic agent has bounded values for its wealth and utility. However, magnetic salts behave, for a limited range of temperatures, as if they have only a finite number of energy levels.

Common thermometry involves reading the height of a column of fluid or the position of a pointer attached to a coil of wire. Both of these quantities depend upon the thermal expansion coefficient. Other forms of thermometry depend upon other temperature-dependent variables. At room temperature, these can be calibrated against another thermometer based on the ideal gas law. At lower temperature it is convenient to use the carbon resistor thermometer; its electrical resistance is a measure of the temperature. In the millikelvin range, calibrations are done with thermometers that use certain magnetic salts. At very low and very high temperatures, it is difficult to perform any thermometry at all. We therefore consider the general problem of how we can calibrate a measurement of a measured quantity (for example, electrical resistance), which we will call τ , against an absolute thermodynamic temperature T .¹⁸

Recall that $T=0$ for all thermodynamic temperature scales. However, there is no absolute scale for temperature. By setting (approximately) 273 deg to be the freezing point of water, or using the temperature of the triple point of a pure material, we determine the Kelvin scale of temperature T_K . On another planet, the thermodynamic temperature scale would be different, but only by a scale factor. If the inhabitants of the other planet called the freezing point of water 546 deg, we would know that their temperatures are all twice as high as on the kelvin scale.

Determining the temperature scale. In general, the thermometer property τ depends on T , P , and N . For this thermometer, we must determine the thermodynamic temperature T as a function of τ , V , and N . Our discussion is an extension of Landau and Lifshitz,¹⁸ who consider a thermometer that measures a quantity τ depending only upon the temperature T : $\tau = \tau(T)$.

Consider a measurable quantity, the heat gain $dQ = TdS$, and how it varies with a change in pressure at fixed temperature T and fixed N . We will find it helpful to employ the following Maxwell relation [based on the Gibbs free energy $G(T, P, N) = E - TS + PV$]:

$$-\left(\frac{\partial S}{\partial P} \right)_{T, N} = \left(\frac{\partial V}{\partial T} \right)_{P, N}. \quad (44)$$

From Eq. (44) we can write

$$\begin{aligned} \left(\frac{\partial Q}{\partial P}\right)_{T,N} &= T \left(\frac{\partial S}{\partial P}\right)_{T,N} = -T \left(\frac{\partial V}{\partial T}\right)_{P,N} \\ &= -T \left(\frac{\partial V}{\partial \tau}\right)_{P,N} \left(\frac{\partial \tau}{\partial T}\right)_{P,N}, \end{aligned} \quad (45)$$

which can be re-arranged to read

$$\frac{1}{T} \left(\frac{\partial T}{\partial \tau}\right)_{P,N} = - \frac{(\partial V/\partial \tau)_{P,N}}{(\partial Q/\partial P)_{T,N}} \equiv f(\tau, P, N). \quad (46)$$

The left-hand side of Eq. (46) is at fixed P and N , thus explaining the notation $f(\tau, P, N)$. Equation (46) tells us that the fractional change in temperature can be expressed in terms of the measurable quantities $f(\tau, P, N)$ and $d\tau$. We can integrate Eq. (46) to obtain

$$\ln \frac{T}{T_0} = \int_{\tau_0}^{\tau} f(\tau, P, N) d\tau, \quad f(\tau, P, N) = - \frac{(\partial V/\partial \tau)_{P,N}}{(\partial Q/\partial P)_{T,N}}, \quad (47)$$

where $dQ \equiv TdS$. In this way we can obtain the temperature scale T as $T = T(\tau, P, N)$ in terms of measurable quantities.

By using fixed V and varying N , we can obtain $T(\tau, V, \mu)$ via the relations

$$\ln \frac{T}{T_0} = \int_{\tau_0}^{\tau} g(\tau, V, \mu) d\tau, \quad g(\tau, V, \mu) \equiv \frac{(\partial N/\partial \tau)_{\mu, V}}{(\partial Q/\partial \mu)_{T, V}}. \quad (48)$$

Thus there is more than one way to obtain a temperature scale. Equations (47) and (48) must be consistent with one another.

The above discussion suggests at least eight ways to perform thermometry. At fixed T , we measure the dependence of dQ on either dV or dP , and at either fixed N or fixed μ , we measure the dependence of dQ on either dN or $d\mu$ at either fixed V or fixed P . Two pairs of these eight measurements lead to τ as a function of the same fixed variables.

VI. THERMOMETRY IN ECONOMICS

Having described some of the complexity of thermometry in physics, we now indicate how to perform thermometry in economics. To do so, it is essential that we be able to measure $dQ \equiv TdS$, the Veblenian surplus, and that we have a quantity that can serve as a thermometer. By Eq. (23), we have

$$TdS = dU - \lambda dM - p dN. \quad (49)$$

Hence, if we can measure dU , λ , dM , p , and dN , then we can measure $dQ \equiv TdS$.

We will assume that we have some economic indicator τ that depends upon T , λ , and N . To employ the relation $\tau(T, \lambda, N)$, we will require a new thermal Slutsky condition, obtained using the economic analogue of the Gibbs free energy. We write

$$\mathcal{V}(T, p, M) = U - TS - \lambda M, \quad (50)$$

where $\mathcal{V} = pN$ is obtained by using Eq. (21) and is the monetary value of goods. Its differential satisfies

$$d\mathcal{V} = -SdT - Md\lambda + p dN, \quad (51)$$

which follows on substituting Eq. (23) in the differential of Eq. (50). Then

$$\left(\frac{\partial^2 \mathcal{V}}{\partial \lambda \partial T}\right)_N = \left(\frac{\partial^2 \mathcal{V}}{\partial T \partial \lambda}\right)_N, \quad (52)$$

which leads to

$$\left(\frac{\partial S}{\partial \lambda}\right)_{\lambda, N} = \left(\frac{\partial M}{\partial T}\right)_{\lambda, N}. \quad (53)$$

Using Eq. (53) and $dQ \equiv TdS$, we have

$$\begin{aligned} \left(\frac{\partial Q}{\partial \lambda}\right)_{T, N} &= T \left(\frac{\partial S}{\partial \lambda}\right)_{T, N} = -T \left(\frac{\partial M}{\partial T}\right)_{\lambda, N} \\ &= -T \left(\frac{\partial M}{\partial \tau}\right)_{\lambda, N} \left(\frac{\partial \tau}{\partial T}\right)_{\lambda, N}, \end{aligned} \quad (54)$$

which can be re-arranged to read

$$\frac{1}{T} \left(\frac{\partial T}{\partial \tau}\right)_{\lambda, N} = \frac{(\partial M/\partial \tau)_{\lambda, N}}{(\partial Q/\partial \lambda)_{T, N}} \equiv f(\tau, \lambda, N). \quad (55)$$

The left-hand side of Eq. (55) is at fixed λ and N , so that the right-hand side must be written in terms of the variables τ , λ , and N , thus explaining the notation $f(\tau, \lambda, N)$. We can then integrate from T_0 and the corresponding τ_0 to obtain

$$\ln \frac{T}{T_0} = \int_{\tau_0}^{\tau} f(\tau, \lambda, N) d\tau, \quad f(\tau, \lambda, N) = \frac{(\partial M/\partial \tau)_{\lambda, N}}{(\partial Q/\partial \lambda)_{T, N}}, \quad (56)$$

where $dQ \equiv TdS$ as usual. From Eq. (56) we can obtain, in principle, the temperature scale $T(\tau, \lambda, N)$ in terms of the measurable quantities τ , λ , N . We need to be able to measure the quantities appearing in the expression for $f(\tau, \lambda, N)$. Especially difficult to measure is the change in the Veblenian surplus, $dQ \equiv TdS$. By Eq. (24), this quantity is also the change in utility at fixed goods and money, and in that form its measurability is the subject of debate among economists.¹⁹ The present work is predicated upon the assumption of the measurability of utility. Other forms of economic thermometry that are perhaps more practical to implement are possible by analogy to thermometry in thermodynamics.

VII. AN ANALOGY TO STATISTICAL MECHANICS

In statistical mechanics we weight each microstate s with energy E_s of a system in equilibrium with a heat reservoir by the factor $\exp(-E_s/T)$. This weight has the following desirable properties: the probabilities for independent systems are multiplicative; and the energy of independent systems is both additive and conserved in interactions between the system and the reservoir. In this way we obtain the partition function

$$Z = \sum_{\text{states}} e^{-E_s/T}. \quad (57)$$

From Eq. (57), the free energy F is defined via

$$Z = e^{-F/T}, \quad (58)$$

so that

$$F = -T \ln Z. \quad (59)$$

A knowledge of F yields the thermodynamic properties of the physical system.

To make the analogy to economics, it is important to note that the number of economic states is bounded from above, just as for certain magnetic salts. Moreover, an economic agent will seek to maximize utility or wealth (according to what variables are held constant), whereas a physical system will seek to minimize energy or free energy. Four possible weightings of economic states suggest themselves: (1) $e^{-U/T}$; (2) $e^{U/T}$; (3) $e^{-W/T}$; and (4) $e^{W/T}$. If the relationship $U = W + TS$ is to be maintained, then $e^{U/T}$ and $e^{-W/T}$ lead to undesirable definitions whereby entropy is negative. Of the remaining two weightings, $e^{-U/T}$ would appear to be the most natural, because of the thermodynamic analogy. However, $e^{-U/T}$ is undesirable because it favors states of low utility (a minimization principle) and employs a quantity that is not conserved in economic exchanges. On the other hand, weighting by $e^{W/T}$ favors states of higher wealth (a maximization principle) and employs a quantity that is conserved in economic exchanges. This weight also is consistent with the idea that T is a measure of economic development. Low temperature (less developed) economies strive for high W states whereas high temperature (highly developed) economies prefer variety, that is, economic states are equally weighted.

Consider an economics microstate s with wealth W_s . We define the partition function

$$Z = \sum_{\text{states}} e^{W_s/T}, \quad (60)$$

and the utility U by

$$Z = e^{U/T}, \quad (61)$$

so that

$$U = T \ln Z. \quad (62)$$

Making the usual assumption that the sum is dominated by the most probable states (at the most probable wealth), Eq. (62) becomes

$$Z \approx \Gamma e^{W/T}, \quad \Gamma = \sum_{\text{states}} 1, \quad (63)$$

where the sum is over economic states with wealth $W_s = W$. Combining Eqs. (60)–(63) yields

$$U = W + T \ln \Gamma. \quad (64)$$

Hence, to agree with (2) and $\Psi = TS$, we identify

$$S = \ln \Gamma. \quad (65)$$

Equation (65) is the economic analogue of the famous relationship due to Boltzmann. It relates economic entropy to economic variety.

We will not explore how to count economic states, which is a complex and difficult subject, involving considerations as complex as the determination of the “phase space” asso-

ciated with leisure. (Recall the difficulties associated with state counting in physics before quantum mechanics defined the unit of phase space.)

From $F = E - TS$, low temperature favors E minimization, whereas high temperature favors S maximization. Correspondingly, from $U = W + TS$, low economic temperature favors W maximization, whereas high economic temperature favors S maximization. From the viewpoint of maximization, $-F$ and U are analogous. This analogy is in contrast to our earlier discussion, where E and U are analogous from the viewpoint of natural variables. Note that U calculated from Eq. (62) will be written in terms of T , which is not its natural variable. By use of $W = U - TS$, where $S = \int_0^T (dU/dT)|_{M,N}$, the function $W(T, M, N)$ can be obtained, from which the economics can be calculated.

Equation (65) can be made the basis of a theory of fluctuations about economic equilibrium in analogy to the theory of fluctuations about thermodynamic equilibrium. For example, the fluctuations δN in the number of particles N satisfy¹⁸

$$\overline{(\delta N)^2} = T \frac{\partial N}{\partial \mu}, \quad (66)$$

where T has the same units (energy) as μ . Analogously, the fluctuations δN in the number of goods N have a mean square average which satisfies

$$\overline{(\delta N)^2} = -T \frac{\partial N}{\partial p}, \quad (67)$$

where T has the same units (for example, 1998 dollars) as p . The derivative in Eq. (67), which appeared in our earlier discussion of Eq. (40), is proportional to the *price-elasticity* $(p/N) \partial N / \partial p$. Theil² assumed that, for a given economic agent, each good will satisfy a relationship similar to Eq. (67). Although he did not give the coefficient of proportionality, his economic intuition led him to suggest that the coefficient of proportionality should be the same for all goods. (He did not include the concept of temperature in this context.) Equation (67) goes beyond Thiel in predicting that the fluctuations increase for fixed T , if $\partial N / \partial p$ (akin to a susceptibility or compressibility) increases, and that the fluctuations increase as T increases, for fixed $\partial N / \partial p$. Note that if $\overline{(\delta N)^2}$ and $\partial N / \partial p$ can be measured for an equilibrium system with Gaussian fluctuations, then Eq. (67) could be used to determine the economic temperature.

VIII. SUMMARY AND DISCUSSION

Assuming that the utility U can be given a dollar value, so that economic thermometry can be performed (in principle), we have developed an analogy between the economic quantities surplus Ψ , utility U , and wealth W , and the thermodynamic quantities TS , energy E , and Helmholtz free energy F . In Sec. II we tentatively made a common language identification of T with the *level of economic development*, but we have avoided a common language identification of S . The quantities T and S correspond, at least in part, to psychological variables, which often are invoked in economic discussions of utility.

This analogy has led to a number of apparently new results: an economic Gibbs–Duhem relation; the equating of Marshallian surplus Ψ to TS ; the use of the economic Gibbs–Duhem relation to recast the Smithian surplus $-Ndp - Md\lambda$ as SdT ; the equating of TdS to increase in utility at fixed goods N and money M ; the interpretation of TdS as Veblenian surplus; new and more precise Slutsky relations; and the relationship between measurements of utility and the establishment of an economic temperature scale. We also developed an analogy to the relation between statistical physics and thermodynamics. Moreover, Eq. (65) can, in principle, be used to study near-equilibrium fluctuations.

In the late nineteenth century, Irving Fisher³ developed a mechanical analogy between economics and physics, invoking force and distance to be analogous to price and number of goods, respectively. In the present work we have made a thermodynamic analogy, invoking the chemical potential and number of goods, and especially the temperature and entropy. Such a viewpoint would have been recognized immediately by Fisher’s advisor—J. Willard Gibbs himself.

Neither we nor professional economists know if the assumption of economic equilibrium is correct. Furthermore, we do not know if econometric measurements are accurate enough to test the theory, even if a system is found that is thought to be in equilibrium. Nevertheless, even if no economic system is found to be in true equilibrium, the analogies we have proposed here may have value for systems that are slightly out of equilibrium.

A surprisingly large number of apparently new results have been obtained using the macroscopic reasoning associated with thermodynamics. Of course, it is possible that a better analogy can be developed. We assume economic systems that are permanent and are in equilibrium and that wealth is conserved under exchange with fixed prices. These assumptions cannot be literally true. Moreover, although the formalism allows for an internal variable (the temperature), this value has been thought of as a societally determined quantity (the state of economic development), rather than a physiological-psychological variable associated with the individual. Probably the best way to test the economic-thermodynamic analogy is by applying it to economic measurements on systems that are thought to be in, or nearly in, economic equilibrium. Note that economic individuality will appear in the micro-aspects of the economic-statistical-mechanics analogy of Sec. VII. Probably the best way to test this analogy is by developing microscopic models for the behavior of specific economic systems. Even if the analogies require modification, they suggest directions for future research.

Suggested problems: The interested reader should consult a basic economics text, such as Ref. 8, and a graduate text such as Ref. 10. Reference 3 discusses various analogies that have been made to physics.

- (1) Develop a theory of near-equilibrium fluctuations. Note that economic fluctuations are often non-Gaussian.²⁰
- (2) Develop an analogy to heat engines operating at different temperatures. To what extent can λdM be considered the economic equivalent of work $-PdV$?
- (3) Develop an analogy to a thermodynamic system with a surface and surface tension. In the economic system, land becomes a commodity with a certain value that is analogous to surface area.

- (4) Develop an alternate form of economic thermometry, where $\tau = \tau(T, M, N)$, in contrast to $\tau = \tau(T, \lambda, N)$ of Eq. (55).
- (5) Develop a statistical mechanical model for the economic behavior of a rat with a fixed daily number of bar presses and a choice between two fluids with the same nutritional content but different flavors. Consider the fluctuations in the number of bar presses. How does price effect the fluctuations? How does temperature effect the fluctuations?

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¹J. H. Kagel, R. C. Battalio, L. Green, and H. Rachlin, “Consumer demand theory applied to choice behavior of rats,” in *Limits to Action: The Allocation of Individual Behavior*, edited by J. E. R. Staddon (Academic, New York, 1980). A summary of this work and others may be found in J. H. Kagel, R. C. Battalio, and L. Green, *Economic Choice Theory: An Experimental Analysis of Animal Behavior* (Cambridge U.P., Cambridge, 1995).

²H. Theil, *Theory and Measurement of Consumer Demand* (American Elsevier, New York, 1975), Vol. 1.

³See the critical discussion of this neoclassical approach in P. Mirowski, *More Heat Than Light* (Cambridge U.P., Cambridge, 1989).

⁴N. Georgescu-Roegen, *The Entropy Law and the Economic Process* (Harvard U.P., Cambridge, MA, 1971).

⁵J. Hermiter, *An Entropy Model of Brand Purchase Behavior* (Marketing Science Institute, Cambridge, MA, 1972).

⁶D. F. Batten, *Spatial Analysis of Interacting Economies: The Role of Entropy and Information Theory in Spatial-Input Modelling* (Kluwer-Nijhoff, Boston, 1983).

⁷D. K. Foley, “A statistical equilibrium theory of markets,” *J. Econ. Theory* **62**, 321–345 (1994).

⁸In discussing utility, the expression “psychological utility” is employed by the classic text by P. Samuelson, *Economics* (McGraw–Hill, New York, 1970), 8th ed., p. 410. It is not obvious how to measure utility, but it is believed that *relative* measures of utility have meaning. For some purposes economists take a cautious position on utility, treating it as being subject only to priority-ordering (ordinal values). For other purposes economists make the leap of faith that utility can be given a dollar value, so that more quantitative theories can be constructed.

⁹See p. 417 of Ref. 8.

¹⁰E. Silberberg, *The Structure of Economics: A Mathematical Analysis* (McGraw–Hill, New York, 1990), p. 396.

¹¹E. K. Browning and J. M. Browning, *Microeconomic Theory and Applications* (HarperCollins, New York, 1992), 4th ed., p. 87.

¹²W. Benton, *Encyclopedia Britannica* (Chicago, 1966), Vol. 22, p. 92.

¹³H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics* (Wiley, New York, 1985).

¹⁴Adam Smith, *An Inquiry into the Nature and Causes of the Wealth of Nations*, published in Dublin, printed for Messrs. Whitestone and others (1776). This work can be found at <http://www.bibliomania.com/NonFiction/Smith/Wealth>. See Chapter IV, “Of the Origin and Use of Money.”

¹⁵Some economies employ large rocks as currency; others employ cows; others gold; others paper. Clearly, the cost of money is not an invariant.

Moreover, gold had a tendency to become debased. Complex printed notes are subject to the debasement known as counterfeiting. One can imagine that there has been debasement associated with cows and large rocks. Clearly, the formalism, as in Eq. (39), must permit the value of currency λ to vary.

¹⁶W. E. Diewert, "Generalized Slutsky conditions for aggregate consumer demand functions," *J. Econ. Theory* **15**, 353–362 (1977).

¹⁷A. Barten, "Evidence on the Slutsky conditions for demand equations," *Rev. Econ. Statistics* **49**, 77–84 (1967).

¹⁸L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison–Wesley, Reading, MA, 1969), 2nd ed.

¹⁹See p. 301 of Ref. 10.

²⁰B. B. Mandelbrot, *Fractals and Scaling in Finance: Discontinuity, Concentration, Risk* (Springer-Verlag, New York, 1997).

PHYSICS ENVY

Other scientists at the meeting said they believed that population genetics is a robust science, but that scientists studying molecular evolution must accept the nature of their research. "We are really about documenting patterns and trying to explain those patterns. A lot of biology is inherently descriptive," Dr. Staton said.

"We all have physics envy," said Rollin Richmond, a former student of Dobzhansky's and provost at the State University of New York at Stony Brook. "We can't prove something the way a physicist proves an electron exists. You have to have a tolerance for ambiguity."

David L. Wheeler, *The Chronicle of Higher Education*, 14 February 1997.