Let us suppose that at $t = 0$ a charge is concentrated within a small spherical region located somewhere in a conducting body. At every other point of the conductor the charge density is zero. The charge within the sphere now begins to fade away exponentially, but according to (21) no charge can reappear anywhere *within* the conductor. What becomes of it? Since the charge is conserved, the decay of charge within the spherical surface must be accompanied by an outward flow, or current. No charge can accumulate at any other interior point; hence the flow must be divergenceless. It will be arrested, however, on the outer surface of the conductor and it is here that we shall rediscover the charge that has been lost from the central sphere. This surface charge makes its appearance at the exact instant that the interior charge begins to decay, for the total charge is constant.

UNITS AND DIMENSIONS

1.8. The M.K.S. or Giorgi System.—An electromagnetic field thus far is no more than a complex of vectors subject to a postulated system of differential equations. To proceed further we must establish the physical dimensions of these vectors and agree on the units in which they are to be measured.

In the customary sense, an "absolute" system of units is one in which every quantity may be measured or expressed in terms of the three fundamental quantities mass, length, and time. Now in electromagnetic theory there is an essential arbitrariness in the matter of dimensions which is introduced with the factors ϵ_0 and μ_0 connecting **D** and **E**, **H** and **B** respectively in free space. No experiment has yet been imagined by means of which dimensions may be attributed to either ϵ_0 or μ_0 as an independent physical entity. On the other hand, it is a direct consequence of the field equations that the quantity

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} \tag{1}$$

shall have the dimensions of a velocity, and every arbitrary choice of ϵ_0 and μ_0 is subject to this restriction. The magnitude of this velocity cannot be calculated a priori, but by suitable experiment it may be measured. The value obtained by the method of Rosa and Dorsey of the Bureau of Standards and corrected by Curtis[1] in 1929 is

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.99790 \times 10^8 \qquad \text{meters/sec.,} \tag{2}$$

[1] Rosa and Dorsey, A New Determination of the Ratio of the Electrostatic Unit of Electricity, *Bur. Standards, Bull.* **3**, p. 433, 1907. Curtis, *Bur. Standards J. Research*, **3**, 63, 1929.

or for all practical purposes

$$c = 3 \times 10^8 \qquad \text{meters/sec.} \tag{3}$$

Throughout the early history of electromagnetic theory the absolute *electromagnetic system* of units was employed for all scientific investigations. In this system the centimeter was adopted as the unit of length, the gram as the unit of mass, the second as the unit of time, and as a fourth unit the factor μ_0 was placed arbitrarily equal to unity and considered dimensionless. The dimensions of ϵ_0 were then uniquely determined by (1) and it could be shown that the units and dimensions of every other quantity entering into the theory might be expressed in terms of centimeters, grams, seconds, and μ_0. Unfortunately, this absolute system failed to meet the needs of practice. The units of resistance and of electromotive force were, for example, far too small. To remedy this defect a *practical system* was adopted. Each unit of the practical system had the dimensions of the corresponding electromagnetic unit and differed from it in magnitude by a power of ten which, in the case of voltage and resistance at least, was wholly arbitrary. The practical units have the great advantage of convenient size and they are now universally employed for technical measurements and computations. Since they have been defined as arbitrary multiples of absolute units, they do not, however, constitute an absolute system. Now the quantities mass, length, and time are fundamental solely because the physicist has found it expedient to raise them to that rank. That there are other fundamental quantities is obvious from the fact that all electromagnetic quantities cannot be expressed in terms of these three alone. The restriction of the term "absolute" to systems based on mass, length, and time is, therefore, wholly unwarranted; one should ask only that such a system be self-consistent and that every quantity be defined in terms of a minimum number of basic, independent units. The antipathy of physicists in the past to the practical system of electrical units has been based not on any firm belief in the sanctity of mass, length, and time, but rather on the lack of self-consistency within that system.

Fortunately a most satisfactory solution has been found for this difficulty. In 1901 Giorgi,[1] pursuing an idea originally due to Maxwell, called attention to the fact that the practical system could be converted into an absolute system by an appropriate choice of fundamental units. It is indeed only necessary to choose for the unit of length the inter-

[1] GIORGI: Unità Razionali di Elettromagnetismo, *Atti dell' A.E.I.*, 1901. An historical review of the development of the practical system, including a report of the action taken at the 1935 meeting of the International Electrotechnical Commission and an extensive bibliography is given by Kennelly, *J. Inst. Elec. Engrs.*, **78**, 235–245, 1936. See also GLAZEBROOK, The M.K.S. System of Electrical Units, *J. Inst. Elec. Engrs.*, **78**, pp. 245–247.

national *meter*, for the unit of mass the *kilogram*, for the unit of time the *second*, and as a fourth unit any electrical quantity belonging to the practical system such as the coulomb, the ampere, or the ohm. From the field equations it is then possible to deduce the units and dimensions of every electromagnetic quantity in terms of these four fundamental units. Moreover the derived quantities will be related to each other exactly as in the practical system and may, therefore, be expressed in practical units. In particular it is found that the parameter μ_0 must have the value $4\pi \times 10^{-7}$, whence from (1) the value of ϵ_0 may be calculated. Inversely one might equally well *assume* this value of μ_0 as a fourth basic unit and then deduce the practical series from the field equations.

At a plenary session in June, 1935, the International Electrotechnical Commission adopted unanimously the m.k.s. system of Giorgi. Certain questions, however, still remain to be settled. No official agreement has as yet been reached as to the fourth fundamental unit. Giorgi himself recommended that the ohm, a material standard defined as the resistance of a specified column of mercury under specified conditions of pressure and temperature, be introduced as a basic quantity. If $\mu_0 = 4\pi \times 10^{-7}$ be chosen as the fourth unit and assumed dimensionless, all derived quantities may be expressed in terms of mass, length, and time alone, the dimensions of each being identical with those of the corresponding quantity in the absolute electromagnetic system and differing from them only in the size of the units. This assumption leads, however, to fractional exponents in the dimensions of many quantities, a direct consequence of our arbitrariness in clinging to mass, length, and time as the sole fundamental entities. In the absolute electromagnetic system, for example, the dimensions of charge are grams$^{\frac{1}{2}}$ · centimeters$^{\frac{1}{2}}$, an irrationality which can hardly be physically significant. These fractional exponents are entirely eliminated if we choose as a fourth unit the coulomb; for this reason, charge has been advocated at various times as a fundamental quantity quite apart from the question of its magnitude.[1] In the present volume we shall adhere exclusively to the meter-kilogram-second-coulomb system. A subsequent choice by the I.E.C. of some other electrical quantity as basic will in nowise affect the size of our units or the form of the equations.[2]

[1] See the discussion by WALLOT: *Elektrotechnische Zeitschrift*, Nos. 44–46, 1922. Also SOMMERFELD: "Ueber die Electromagnetischen Einheiten," pp. 157–165, Zeeman Verhandelingen, Martinus Nijhoff, The Hague, 1935; *Physik. Z.* **36**, 814–820, 1935.

[2] No ruling has been made as yet on the question of rationalization and opinion seems equally divided in favor and against. If one bases the theory on Maxwell's equations, it seems definitely advantageous to drop the factors 4π which in unrationalized systems stand before the charge and current densities. A rationalized system will be employed in this book.

To demonstrate that the proposed units do constitute a self-consistent system let us proceed as follows. The unit of current in the m.k.s. system is to be the absolute ampere and the unit of resistance is to be the absolute ohm. These quantities are to be such that the work expended per second by a current of 1 amp. passing through a resistance of 1 ohm is 1 joule (absolute). If R is the resistance of a section of conductor carrying a constant current of I amp., the work dissipated in heat in t sec. is

$$W = I^2Rt \qquad \text{joules.} \tag{4}$$

By means of a calorimeter the heat generated may be measured and thus one determines the relation of the unit of electrical energy to the unit quantity of heat. It is desired that the joule defined by (4) be identical with the joule defined as a unit of mechanical work, so that in the electrical as well as in the mechanical case

$$1 \text{ joule} = 0.2389 \qquad \text{gram-calorie (mean).} \tag{5}$$

Now we shall *define* the ampere on the basis of the equation of continuity (6), page 4, as the current which transports across any surface 1 coulomb in 1 sec. Then the ohm is a *derived* unit whose magnitude and dimensions are determined by (4):

$$1 \text{ ohm} = 1\,\frac{\text{watt}}{\text{ampere}^2} = 1\,\frac{\text{kilogram}\cdot\text{meter}^2}{\text{coulomb}^2\cdot\text{second}}, \tag{6}$$

since 1 watt is equal to 1 joule/sec. The resistivity of a medium is defined as the resistance measured between two parallel faces of a unit cube. The reciprocal of this quantity is the conductivity. The dimensions of σ follow from Eq. (17), page 15.

$$1 \text{ unit of conductivity} = \frac{1}{\text{ohm}\cdot\text{meter}} = 1\,\frac{\text{coulomb}^2\cdot\text{second}}{\text{kilogram}\cdot\text{meter}^3}. \tag{7}$$

In the United States the reciprocal ohm is usually called the mho, although the name *siemens* has been adopted officially by the I.E.C. The unit of conductivity is therefore 1 siemens/meter.

The volt will be defined simply as 1 watt/amp., or

$$1 \text{ volt} = 1\,\frac{\text{watt}}{\text{ampere}} = 1\,\frac{\text{kilogram}\cdot\text{meter}^2}{\text{coulomb}\cdot\text{second}^2}. \tag{8}$$

Since the unit of current density is 1 amp./meter^2, we deduce from the relation $\mathbf{J} = \sigma\mathbf{E}$ that

$$1 \text{ unit of } \mathbf{E} = 1\,\frac{\text{watt}}{\text{ampere}\cdot\text{meter}} = 1\,\frac{\text{volt}}{\text{meter}} = 1\,\frac{\text{kilogram}\cdot\text{meter}}{\text{coulomb}\cdot\text{second}^2}. \tag{9}$$

The power expended per unit volume by a current of density $\mathbf{J}$ is therefore $\mathbf{E} \cdot \mathbf{J}$ watts/meter³. It will be noted furthermore that the product of charge and electric field intensity $\mathbf{E}$ has the dimensions of force. Let a charge of 1 coulomb be placed in an electric field whose intensity is 1 volt/meter.

$$(10) \qquad 1 \text{ coulomb} \times 1 \frac{\text{volt}}{\text{meter}} = 1 \frac{\text{joule}}{\text{meter}} = 1 \frac{\text{kilogram} \cdot \text{meter}}{\text{second}^2}.$$

The unit of force in the m.k.s. system is called the *newton*, and is equivalent to 1 joule/meter, or 10^5 dynes.

The flux of the vector $\mathbf{B}$ shall be measured in *webers*,

$$(11) \qquad \Phi = \int_S \mathbf{B} \cdot \mathbf{n}\, da \qquad \text{webers},$$

and the intensity of the field $\mathbf{B}$, or flux density, may therefore be expressed in webers per square meter. According to (25), page 8,

$$(12) \qquad \int_C \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi}{dt} \qquad \frac{\text{webers}}{\text{second}}.$$

The line integral $\int_a^b \mathbf{E} \cdot d\mathbf{s}$ is measured in volts and is usually called the electromotive force (abbreviated e.m.f.) between the points a and b, although its value in a nonstationary field depends on the path of integration. The induced e.m.f. around any closed contour C is, therefore, equal to the rate of decrease of flux threading that contour, so that between the units there exists the relation

$$(13) \qquad 1 \text{ volt} = 1 \frac{\text{weber}}{\text{second}},$$

or

$$(14) \qquad 1 \text{ weber} = 1 \frac{\text{joule}}{\text{ampere}} = 1 \frac{\text{kilogram} \cdot \text{meter}^2}{\text{coulomb} \cdot \text{second}}.$$

It is important to note that the product of current and magnetic flux is an energy. Note also that the product of $\mathbf{B}$ and a velocity is measured in volts per meter, and is therefore a quantity of the same kind as $\mathbf{E}$.

$$(15) \qquad 1 \text{ unit of } \mathbf{B} = 1 \frac{\text{weber}}{\text{meter}^2} = 1 \frac{\text{kilogram}}{\text{coulomb} \cdot \text{second}}.$$

$$(16) \qquad 1 \text{ unit of } |\mathbf{B}|\,|\mathbf{v}| = 1 \frac{\text{weber}}{\text{meter}^2} \times 1 \frac{\text{meter}}{\text{second}} = 1 \frac{\text{volt}}{\text{meter}} = 1 \text{ unit of } |\mathbf{E}|.$$

The units which have been deduced thus far constitute an absolute system in the sense that each has been expressed in terms of the four basic quantities, mass, length, time, and charge. That this system is identical with the practical series may be verified by the substitutions

$$(17)\quad 1 \text{ kilogram} = 10^3 \text{ grams},\qquad 1 \text{ meter} = 10^2 \text{ centimeters},\qquad 1 \text{ coulomb} = \tfrac{1}{10} \text{ abcoulomb}.$$

The numerical factors which now appear in each relation are observed to be those that relate the practical units to the absolute electromagnetic units. For example, from (6),

$$(18)\quad 1 \text{ ohm} = 1\,\frac{\text{kilogram}\cdot\text{meter}^2}{\text{coulomb}^2\cdot\text{second}} = \frac{10^3 \text{ grams}\cdot 10^4 \text{ centimeters}^2}{10^{-2} \text{ abcoulomb}^2\cdot\text{seconds}} = 10^9 \quad \text{abohms};$$

and again from (8),

$$(19)\quad 1 \text{ volt} = 1\,\frac{\text{kilogram}\cdot\text{meter}^2}{\text{coulomb}\cdot\text{second}^2} = \frac{10^3 \text{ grams}\cdot 10^4 \text{ centimeters}^2}{10^{-1} \text{ abcoulomb}\cdot\text{second}^2} = 10^8 \quad \text{abvolts}.$$

The series must be completed by a determination of the units and dimensions of the vectors **D** and **H**. Since $\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$, it is necessary and sufficient that ϵ_0 and μ_0 be determined such as to satisfy Eq. (2) and such that the proper ratio of practical to absolute units be maintained. We shall represent mass, length, time, and charge by the letters M, L, T, and Q, respectively, and employ the customary symbol $[A]$ as meaning "the dimensions of A." Then from Eq. (31), page 9,

$$(20)\qquad \int_S \mathbf{D}\cdot\mathbf{n}\,da = q \qquad \text{coulombs}$$

and, hence,

$$(21)\qquad [\mathbf{D}] = \frac{\text{coulombs}}{\text{meter}^2} = \frac{Q}{L^2},$$

$$(22)\qquad [\epsilon_0] = \left[\frac{D}{\kappa_e E}\right] = \frac{\text{coulombs}}{\text{volt}\cdot\text{meter}} = \frac{Q^2T^2}{ML^3}.$$

The *farad,* a derived unit of capacity, is defined as the capacity of a conducting body whose potential will be raised 1 volt by a charge of 1 coulomb. It is equal, in other words, to 1 coulomb/volt. The parameter ϵ_0 in the m.k.s. system has dimensions, and may be measured in *farads per meter.*

By analogy with the electrical case, the line integral $\int_a^b \mathbf{H}\cdot d\mathbf{s}$ taken along a specified path is commonly called the magnetomotive force (abbreviated m.m.f.). In a stationary magnetic field

$$(23) \qquad \int_C \mathbf{H} \cdot d\mathbf{s} = I \qquad \text{amperes,}$$

where I is the current determined by the flow of charge through any surface spanning the closed contour C. If the field is variable, I must include the displacement current as in (28), page 9. According to (23) a magnetomotive force has the dimensions of current. In practice, however, the current is frequently carried by the turns of a coil or winding which is linked by the contour C. If there are n such turns carrying a current I, the total current threading C is nI ampere-turns and it is customary to express magnetomotive force in these terms, although dimensionally n is a numeric.

$$(24) \qquad [\text{m.m.f.}] = \text{ampere-turns,}$$

whence

$$(25) \qquad [\mathbf{H}] = \frac{\text{ampere-turns}}{\text{meter}} = \frac{Q}{LT}.$$

It will be observed that the dimensions of **D** and those of **H** divided by a velocity are identical. For the parameter μ_0 we find

$$(26) \qquad [\mu_0] = \left[\frac{B}{\kappa_m H}\right] = \frac{\text{volt} \cdot \text{second}}{\text{ampere} \cdot \text{meter}} = \frac{ML}{Q^2}.$$

As in the case of ϵ_0 it is convenient to express μ_0 in terms of a derived unit, in this case the *henry*, defined as 1 volt-second/amp. (The henry is that inductance in which an induced e.m.f. of 1 volt is generated when the inducing current is varying at the rate of 1 amp./sec.) The parameter μ_0 may, therefore, be measured in *henrys per meter.*

From (22) and (26) it follows now that

$$(27) \qquad \left[\frac{1}{\mu_0\epsilon_0}\right] = \frac{L^2}{T^2},$$

and hence that our system is indeed dimensionally consistent with Eq. (2). Since it is known that in the rationalized, absolute c.g.s. electromagnetic system μ_0 is equal in magnitude to 4π, Eq. (26) fixes also its magnitude in the m.k.s. system.

$$(28) \qquad \mu_0 = 4\pi \frac{\text{gram} \cdot \text{centimeters}}{\text{abcoulombs}^2} = 4\pi \frac{10^{-3}\ \text{kilogram} \cdot 10^{-2}\ \text{meter}}{10^2\ \text{coulombs}^2},$$

or

$$(29) \qquad \mu_0 = 4\pi \times 10^{-7} \frac{\text{kilogram} \cdot \text{meters}}{\text{coulombs}^2} = 1.257 \times 10^{-6} \frac{\text{henry}}{\text{meter}}.$$

The appropriate value of ϵ_0 is then determined from

$$(2) \qquad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.998 \times 10^8 \, \frac{\text{meters}}{\text{second}}$$

to be

$$(30) \qquad \epsilon_0 = 8.854 \times 10^{-12} \, \frac{\text{coulomb}^2 \cdot \text{seconds}^2}{\text{kilogram} \cdot \text{meter}^3} = 8.854 \times 10^{-12} \, \frac{\text{farad}}{\text{meter}}.$$

It is frequently convenient to know the reciprocal values of these factors.

$$(31) \qquad \frac{1}{\mu_0} = 0.7958 \times 10^6 \, \frac{\text{meters}}{\text{henry}}, \qquad \frac{1}{\epsilon_0} = 0.1129 \times 10^{12} \, \frac{\text{meters}}{\text{farad}},$$

and the quantities

$$(32) \qquad \sqrt{\frac{\mu_0}{\epsilon_0}} = 376.6 \text{ ohms}, \qquad \sqrt{\frac{\epsilon_0}{\mu_0}} = 2.655 \times 10^{-3} \text{ mho},$$

recur constantly throughout the investigation of wave propagation.

In Appendix I there will be found a summary of the units and dimensions of electromagnetic quantities in terms of mass, length, time, and charge.

THE ELECTROMAGNETIC POTENTIALS

1.9. Vector and Scalar Potentials.—The analysis of an electromagnetic field is often facilitated by the use of auxiliary functions known as potentials. At every ordinary point of space, the field vectors satisfy the system

$$\text{(I)} \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \qquad \text{(III)} \quad \nabla \cdot \mathbf{B} = 0,$$

$$\text{(II)} \quad \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}, \qquad \text{(IV)} \quad \nabla \cdot \mathbf{D} = \rho.$$

According to (III) the field of the vector $\mathbf{B}$ is always solenoidal. Consequently $\mathbf{B}$ can be represented as the curl of another vector $\mathbf{A}_0$.

$$(1) \qquad \mathbf{B} = \nabla \times \mathbf{A}_0.$$

However $\mathbf{A}_0$ is not uniquely defined by (1); for $\mathbf{B}$ is equal also to the curl of some vector $\mathbf{A}$,

$$(2) \qquad \mathbf{B} = \nabla \times \mathbf{A},$$

where

$$(3) \qquad \mathbf{A} = \mathbf{A}_0 - \nabla \psi,$$

and ψ is any arbitrary scalar function of position.