NLO corrections with the OPP method

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Outline

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2. OPP Reduction
   - Rational terms

3. Numerical Tests
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   - VVV production
The experimental programs of LHC require high precision predictions for multi-particle processes (also ILC of course)
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Introduction: LHC needs NLO

- The experimental programs of LHC require high precision predictions for multi-particle processes (also ILC of course)
- In the last years we have seen a remarkable progress in the theoretical description of multi-particle processes at tree-order, thanks to very efficient recursive algorithms
- The current need of precision goes beyond tree order. At LHC, most analyses require at least next-to-leading order calculations (NLO)
- As a result, a big effort has been devoted by several groups to the problem of an efficient computation of one-loop corrections for multi-particle processes!
### Wishlist Les Houches 2007

1. $pp \rightarrow VV + \text{jet}$
2. $pp \rightarrow t\bar{t}b\bar{b}$
3. $pp \rightarrow t\bar{t} + 2 \text{ jets}$
4. $pp \rightarrow WWW$
5. $pp \rightarrow VVb\bar{b}$
6. $pp \rightarrow VV + 2 \text{ jets}$
7. $pp \rightarrow V + 3 \text{ jets}$
8. $pp \rightarrow t\bar{t}b\bar{b}$
9. $pp \rightarrow 4 \text{ jets}$

Processes for which a NLO calculation is both desired and feasible

Will we “finish” in time for LHC?
What has been done? (2005-2007)

Some recent results → Cross Sections available

- $pp \rightarrow Z Z Z$ $pp \rightarrow t\bar{t}Z$ [Lazopoulos, Melnikov, Petriello]
- $pp \rightarrow H + 2$ jets [Campbell, et al., J. R. Andersen, et al.]
- $pp \rightarrow VV + 2$ jets via VBF [Bozzi, Jäger, Oleari, Zeppenfeld]
- $pp \rightarrow VV + 1$ jet [S. Dittmaier, S. Kallweit and P. Uwer]
- $pp \rightarrow t\bar{t} + 1$ jet [S. Dittmaier, P. Uwer and S. Weinzierl]

Mostly $2 \rightarrow 3$, very few $2 \rightarrow 4$ complete calculations.

- $e^+ e^- \rightarrow 4$ fermions [Denner, Dittmaier, Roth]
- $e^+ e^- \rightarrow HH\nu\bar{\nu}$ [GRACE group (Boudjema et al.)]

This is NOT a complete list
(A lot of work has been done at NLO → calculations & new methods)
Problems arising in NLO calculations

- Large **Number of Feynman diagrams**
- **Reduction to Scalar Integrals** (or sets of known integrals)
- **Numerical Instabilities** (inverse Gram determinants, spurious phase-space singularities)
- Extraction of **soft and collinear singularities** (we need virtual and real corrections)
- **Traditional** Method: Feynman Diagrams & Passarino-Veltman Reduction:
  - general applicability major achievements
  - but major problem: not designed @ amplitude level
- **Traditional** Method: Feynman Diagrams & Passarino-Veltman Reduction:

- **Semi-Numerical** Approach (Algebraic/Partly Numerical – Improved traditional) → Reduction to set of well-known integrals

- **Numerical** Approach (Numerical/Partly Algebraic) → Compute tensor integrals numerically

  - Ellis, Giele, Glover, Zanderighi;
  - Binoth, Guillet, Heinrich, Schubert;
  - Denner, Dittmaier; Del Aguila, Pittau;
  - Ferroglia, Passera, Passarino, Uccirati;
  - Nagy, Soper; van Hameren, Vollinga, Weinzierl;
Methods available

- **Traditional** Method: Feynman Diagrams & Passarino-Veltman Reduction:

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- **Analytic** Approach (Twistor-inspired)
→ extract information from lower-loop, lower-point amplitudes
→ determine scattering amplitudes by their poles and cuts
  - major advantage: designed to work @ amplitude level
  - *quadruple and triple cuts major simplifications*
  - Bern, Dixon, Dunbar, Kosower, Berger, Forde;
  - Anastasiou, Britto, Cachazo, Feng, Kunszt, Mastrolia;
- **Traditional** Method: Feynman Diagrams & Passarino-Veltman Reduction:

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★ **OPP Integrand-level reduction**

combine: reduction@integrand + n-particle cuts
Any \( m \)-point one-loop amplitude can be written, before integration, as

\[
A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}
\]

A bar denotes objects living in \( n = 4 + \epsilon \) dimensions

\[
\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2
\]

\[
\bar{q}^2 = q^2 + \bar{q}^2
\]

\[
\bar{D}_i = D_i + \bar{q}^2
\]

External momenta \( p_i \) are 4-dimensional objects
The old “master” formula

\[ \int A = \sum_{i_0 < i_1 < i_2 < i_3} d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3) \]

\[ + \sum_{i_0 < i_1 < i_2} c(i_0 i_1 i_2) C_0(i_0 i_1 i_2) \]

\[ + \sum_{i_0 < i_1} b(i_0 i_1) B_0(i_0 i_1) \]

\[ + \sum_{i_0} a(i_0) A_0(i_0) \]

+ rational terms
The old “master” formula

\[ A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \]

\[ N(q) \rightarrow q^{\mu_1} \cdots q^{\mu_m} \rightarrow g^{\mu_1 \mu_2} p_i^{\mu_3} \ldots \]

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The derivation of the reduction formula starts as in ref. [1] with the Schouten identity which is a relation between five Levi-Civita tensors:

\[ \epsilon^{p_1 p_2 p_3 p_4} Q_\mu = \epsilon^{\mu p_2 p_3 p_4} Q \cdot p_1 + \epsilon^{p_1 \mu p_3 p_4} Q \cdot p_2 + \epsilon^{p_1 p_2 \mu p_4} Q \cdot p_3 + \epsilon^{p_1 p_2 p_3 \mu} Q \cdot p_4. \]  

(6)
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which yields the final formula for the scalar one-loop five-point function:

\[ E_{01234}(w^2 - 4 \Delta_4 m_0^2) = D_{1234} [2 \Delta_4 - w \cdot (v_1 + v_2 + v_3 + v_4)] \]

\[ + D_{0234} v_1 \cdot w + D_{0134} v_2 \cdot w + D_{0124} v_3 \cdot w + D_{0123} v_4 \cdot w . \]

\[ (19) \]
The old "master" formula

\[ A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \]

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(19)

This method is completely different from the one used in ref. [3].
Started in 90’s, mainly QCD, amplitude level (analytical results)
Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower,
General expression for the 4-dim $N(q)$ at the integrand level in terms of $D_i$

\[
N(q) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
+ \sum_{i_0 < i_1 < i_2}^{m-1} \left[ c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
+ \sum_{i_0 < i_1}^{m-1} \left[ b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
+ \sum_{i_0}^{m-1} \left[ a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\]
\[ N(q) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \left[ d(i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i + \sum_{i_0 < i_1 < i_2}^{m-1} \left[ c(i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i + \sum_{i_0 < i_1}^{m-1} \left[ b(i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i + \sum_{i_0}^{m-1} \left[ a(i_0) \right] \prod_{i \neq i_0}^{m-1} D_i \]

The quantities \( d(i_0 i_1 i_2 i_3) \) are the coefficients of 4-point functions with denominators labeled by \( i_0, i_1, i_2, \) and \( i_3 \).

\( c(i_0 i_1 i_2), b(i_0 i_1), a(i_0) \) are the coefficients of all possible 3-point, 2-point and 1-point functions, respectively.
The quantities $\tilde{d}$, $\tilde{c}$, $\tilde{b}$, $\tilde{a}$ are the “spurious” terms

- They still depend on $q$ (integration momentum)
- They should vanish upon integration

What is the explicit expression of the spurious term?

Express any $q$ in $N(q)$ as

$$q^\mu = -p_0^\mu + \sum_{i=1}^{4} G_i \ell_i^\mu, \quad \ell_i^2 = 0$$

$$k_1 = \ell_1 + \alpha_1 \ell_2, \quad k_2 = \ell_2 + \alpha_2 \ell_1, \quad k_i = p_i - p_0$$

$$\ell_3^\mu = <\ell_1|\gamma^\mu|\ell_2>, \quad \ell_4^\mu = <\ell_2|\gamma^\mu|\ell_1>$$

The coefficients $G_i$ either reconstruct denominators $D_i$

$\rightarrow$ They give rise to $d$, $c$, $b$, $a$ coefficients
Spurious Terms - I


- Express any $q$ in $N(q)$ as

$$q'^\mu = -p'^\mu_0 + \sum_{i=1}^{4} G_i \ell^\mu_i, \quad \ell^2_i = 0$$

$$k_1 = \ell_1 + \alpha_1 \ell_2, \quad k_2 = \ell_2 + \alpha_2 \ell_1, \quad k_i = p_i - p_0$$

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- The coefficients $G_i$ either reconstruct denominators $D_i$ or vanish upon integration

  → They give rise to $d, c, b, a$ coefficients
  → They form the spurious $\tilde{d}, \tilde{c}, \tilde{b}, \tilde{a}$ coefficients
\begin{itemize}
  \item $\tilde{d}(q)$ term (only 1)
  \end{itemize}

\[
\tilde{d}(q) = \tilde{d} \quad T(q),
\]

where $\tilde{d}$ is a constant (does not depend on $q$).

\[
T(q) \equiv Tr[(\hat{\gamma} + \hat{\rho}_0)\hat{c}_1\hat{c}_2\hat{c}_3\gamma_5]
\]
\( \tilde{d}(q) \) term (only 1)

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\[
T(q) \equiv Tr\left[ (\not{q} + \not{p}_0) \ell_1 \ell_2 \ell_3 \gamma_5 \right]
\]

\( \tilde{c}(q) \) terms (they are 6)

\[
\tilde{c}(q) = \sum_{j=1}^{j_{\max}} \left\{ \tilde{c}_{1j}[(q + p_0) \cdot \ell_3]^j + \tilde{c}_{2j}[(q + p_0) \cdot \ell_4]^j \right\}
\]

In the renormalizable gauge, \( j_{\max} = 3 \)
\( \tilde{d}(q) \) term (only 1)

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\]

In the renormalizable gauge, \( j_{max} = 3 \)

\( \tilde{b}(q) \) and \( \tilde{a}(q) \) give rise to 8 and 4 terms, respectively
A simple example

\[ \int \frac{1}{D_0 D_1 D_2 D_3 D_4} \]
A simple example

\[
\int \frac{1}{D_0 D_1 D_2 D_3 D_4}
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\[
1 = \sum \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4}
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**A simple example**

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A SIMPLE EXAMPLE

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\int \frac{1}{D_0 D_1 D_2 D_3 D_4} 
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d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left( \frac{1}{D_{i_4}(q^+)} + \frac{1}{D_{i_4}(q^-)} \right)
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A simple example

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- Melrose, Nuovo Cim. 40 (1965) 181
A next to simple example

\[ \int \frac{1}{D_0 D_1 D_2 D_3 \ldots D_{m-1}} \]
A next to simple example

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\]
General strategy

Now we know the form of the spurious terms:

\[ N(q) = \sum_{i_0 < i_1 < i_2 < i_3} \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3} D_i + \sum_{i_0 < i_1 < i_2} \left[ c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2} D_i \]

\[ + \sum_{i_0 < i_1} \left[ b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1} D_i + \sum_{i_0} \left[ a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0} D_i \]

Our calculation is now reduced to an algebraic problem
General strategy

Now we know the form of the spurious terms:

\[ N(q) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i_i \neq i_0, i_1, i_2, i_3} D_i + \sum_{i_0 < i_1 < i_2}^{m-1} \left[ c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i_i \neq i_0, i_1, i_2} D_i + \sum_{i_0 < i_1}^{m-1} \left[ b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i_i \neq i_0, i_1} D_i + \sum_{i_0}^{m-1} \left[ a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i_i \neq i_0} D_i \]

Our calculation is now reduced to an algebraic problem

Extract all the coefficients by evaluating \( N(q) \) for a set of values of the integration momentum \( q \)
Now we know the form of the spurious terms:

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N(q) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \left[ d(i_0i_1i_2i_3) + \tilde{d}(q; i_0i_1i_2i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3} D_i + \sum_{i_0 < i_1 < i_2}^{m-1} \left[ c(i_0i_1i_2) + \tilde{c}(q; i_0i_1i_2) \right] \prod_{i \neq i_0, i_1, i_2} D_i \\
+ \sum_{i_0 < i_1}^{m-1} \left[ b(i_0i_1) + \tilde{b}(q; i_0i_1) \right] \prod_{i \neq i_0, i_1} D_i + \sum_{i_0}^{m-1} \left[ a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0} D_i
\]

Our calculation is now reduced to an algebraic problem

Extract all the coefficients by evaluating \( N(q) \) for a set of values of the integration momentum \( q \)

There is a very good set of such points: Use values of \( q \) for which a set of denominators \( D_i \) vanish → The system becomes “triangular”: solve first for 4-point functions, then 3-point functions and so on
We look for a $q$ of the form $q^\mu = -p_0^\mu + x_i \ell_i^\mu$ such that

$$D_0 = D_1 = D_2 = D_3 = 0$$

→ we get a system of equations in $x_i$ that has two solutions $q_0^\pm$
Our “master formula” for $q = q_0^\pm$ is:

$$N(q_0^\pm) = [d + \tilde{d} T(q_0^\pm)]$$

→ solve to extract the coefficients $d$ and $\tilde{d}$
Example

\[ N(q) - d - \tilde{d}(q) = \sum_{i=0}^{3} [c(i) + \tilde{c}(q; i)] D_i + \sum_{i_0 < i_1}^{3} \left[ b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] D_{i_0} D_{i_1} \]

\[ + \sum_{i_0=0}^{3} [a(i_0) + \tilde{a}(q; i_0)] D_{i \neq i_0} D_{j \neq i_0} D_{k \neq i_0} \]

Then we can move to the extraction of \( c \) coefficients using

\[ N'(q) = N(q) - d - \tilde{d} T(q) \]

and setting to zero three denominators (ex: \( D_1 = 0, D_2 = 0, D_3 = 0 \))
Example

\[ N(q) - d - \tilde{d}(q) = [c(0) + \tilde{c}(q; 0)] D_0 \]

We have infinite values of \( q \) for which

\[ D_1 = D_2 = D_3 = 0 \quad \text{and} \quad D_0 \neq 0 \]

→ Here we need 7 of them to determine \( c(0) \) and \( \tilde{c}(q; 0) \)
Let's go back to the integrand

\[ A(\bar{q}) = \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \]

Insert the expression for \( N(q) \) → we know all the coefficients

\[ N(q) = \sum_{i_0 < i_1 < i_2 < i_3} \sum_{i \neq i_0, i_1, i_2, i_3} \sum_{i_0 < i_1 < i_2} \sum_{i \neq i_0, i_1, i_2} \left[ d + \tilde{d}(q) \right] D_i + \left[ c + \tilde{c}(q) \right] \prod_{i \neq i_0, i_1, i_2} D_i + \cdots \]

Finally rewrite all denominators using

\[ \frac{D_i}{\bar{D}_i} = \bar{Z}_i , \quad \text{with} \quad \bar{Z}_i \equiv \left( 1 - \frac{\bar{q}^2}{D_i} \right) \]
RATIONAL TERMS - I

\[ A(\tilde{q}) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\tilde{D}_{i_0} \tilde{D}_{i_1} \tilde{D}_{i_2} \tilde{D}_{i_3}} \prod_{i \neq i_0, i_1, i_2, i_3} \tilde{Z}_i \]

\[ + \sum_{i_0 < i_1 < i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\tilde{D}_{i_0} \tilde{D}_{i_1} \tilde{D}_{i_2}} \prod_{i \neq i_0, i_1, i_2} \tilde{Z}_i \]

\[ + \sum_{i_0 < i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\tilde{D}_{i_0} \tilde{D}_{i_1}} \prod_{i \neq i_0, i_1} \tilde{Z}_i \]

\[ + \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\tilde{D}_{i_0}} \prod_{i \neq i_0} \tilde{Z}_i \]

The rational part is produced, after integrating over \( d^n q \), by the \( \tilde{q}^2 \) dependence in \( \tilde{Z}_i \)

\[ \tilde{Z}_i \equiv \left( 1 - \frac{\tilde{q}^2}{\tilde{D}_i} \right) \]
The "Extra Integrals" are of the form

\[ I_{s;\mu_1\cdots\mu_r}^{(n;2\ell)} \equiv \int d^n q \ \tilde{q}^{2\ell} \frac{q_{\mu_1} \cdots q_{\mu_r}}{\bar{D}(k_0) \cdots \bar{D}(k_s)}, \]

where

\[ \bar{D}(k_i) \equiv (\tilde{q} + k_i)^2 - m_i^2, \quad k_i = p_i - p_0 \]

These integrals:
- have dimensionality \( \mathcal{D} = 2(1 + \ell - s) + r \)
- contribute only when \( \mathcal{D} \geq 0 \), otherwise are of \( \mathcal{O}(\epsilon) \)
Expand in D-dimensions?

\[ \bar{D}_i = D_i + \tilde{q}^2 \]
Rational Terms - II

Expand in D-dimensions?

\[ N(q) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \left[ d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \tilde{D}_i + \sum_{i_0 < i_1 < i_2}^{m-1} \left[ c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \tilde{D}_i + \sum_{i_0 < i_1}^{m-1} \left[ b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1}^{m-1} \tilde{D}_i + \sum_{i_0}^{m-1} \left[ a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2) \right] \prod_{i \neq i_0}^{m-1} \tilde{D}_i + \tilde{P}(q) \prod_i^{m-1} \tilde{D}_i \]
Rational Terms - II

Expand in D-dimensions?

\[ N(q) = \sum_{i_0 < i_1 < i_2 < i_3}^{m-1} \left[ d(i_0 i_1 i_2; \bar{q}^2) + \bar{d}(q; i_0 i_1 i_2; \bar{q}^2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \]

\[ + \sum_{i_0 < i_1 < i_2}^{m-1} \left[ c(i_0 i_1; \bar{q}^2) + \bar{c}(q; i_0 i_1; \bar{q}^2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \]

\[ + \sum_{i_0 < i_1}^{m-1} \left[ b(i_0 i_1; \bar{q}^2) + \bar{b}(q; i_0 i_1; \bar{q}^2) \right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \]

\[ + \sum_{i_0}^{m-1} \left[ a(i_0; \bar{q}^2) + \bar{a}(q; i_0; \bar{q}^2) \right] \prod_{i \neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i \]

\[ m_i^2 \rightarrow m_i^2 + \bar{q}^2 \]
Polynomial dependence on $\tilde{q}^2$

\[
b(ij; \tilde{q}^2) = b(ij) + \tilde{q}^2 b^{(2)}(ij), \quad c(ijk; \tilde{q}^2) = c(ijk) + \tilde{q}^2 c^{(2)}(ijk).\]
Polynomial dependence on $\tilde{q}^2$

$$b(ij; \tilde{q}^2) = b(ij) + \tilde{q}^2 b^{(2)}(ij), \quad c(ijk; \tilde{q}^2) = c(ijk) + \tilde{q}^2 c^{(2)}(ijk).$$

$$\int d^n \bar{q} \frac{\tilde{q}^2}{\bar{D}_i \bar{D}_j} = -\frac{i \pi^2}{2} \left[ m_i^2 + m_j^2 - \frac{(p_i - p_j)^2}{3} \right] + \mathcal{O}(\epsilon),$$

$$\int d^n \bar{q} \frac{\tilde{q}^2}{\bar{D}_i \bar{D}_j \bar{D}_k} = -\frac{i \pi^2}{2} + \mathcal{O}(\epsilon), \quad \int d^n \bar{q} \frac{\tilde{q}^4}{\bar{D}_i \bar{D}_j \bar{D}_k \bar{D}_l} = -\frac{i \pi^2}{6} + \mathcal{O}(\epsilon).$$
Furthermore, by defining

\[ \mathcal{D}^{(m)}(q, \tilde{q}^2) \equiv \sum_{i_0 < i_1 < i_2 < i_3} \left[ d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2) \right] \prod_{i \neq i_0, i_1, i_2, i_3} \tilde{D}_i, \]

the following expansion holds

\[ \mathcal{D}^{(m)}(q, \tilde{q}^2) = \sum_{j=2}^{m} \tilde{q}^{(2j-4)} d^{(2j-4)}(q), \]

where the last coefficient is independent on \( q \)

\[ d^{(2m-4)}(q) = d^{(2m-4)}. \]
In practice, once the 4-dimensional coefficients have been determined, one can redo the fits for different values of $\tilde{q}^2$, in order to determine $b^{(2)}(ij)$, $c^{(2)}(ijk)$ and $d^{(2m-4)}$.

\[
R_1 = -\frac{i}{96\pi^2}d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2} c^{(2)}(i_0i_1i_2)
\]

\[
-\frac{i}{32\pi^2} \sum_{i_0<i_1} b^{(2)}(i_0i_1) \left(m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3}\right).
\]

RATIONAL TERMS - $R_2$

A different source of Rational Terms, called $R_2$, can also be generated from the $\epsilon$-dimensional part of $N(q)$

$$\bar{N}(\bar{q}) = N(q) + \bar{N}(\tilde{q}^2, \epsilon; q)$$

$$R_2 \equiv \frac{1}{(2\pi)^4} \int d^n \bar{q} \frac{\bar{N}(\tilde{q}^2, \epsilon; q)}{D_0 D_1 \cdots D_{m-1}} \equiv \frac{1}{(2\pi)^4} \int d^n \bar{q} R_2$$

$$\bar{q} = q + \tilde{q},$$

$$\bar{\gamma}_\mu = \gamma_\mu + \tilde{\gamma}_\mu,$$

$$\bar{g}^{\mu\nu} = g^{\mu\nu} + \tilde{g}^{\mu\nu}.$$

New vertices/particles or GKM-approach
RATIONAL TERMS - $R_2$

\[ N(\bar{q}) \equiv e^3 \left\{ \bar{\gamma}^\beta (\bar{Q}_1 + m_e) \gamma_\mu (\bar{Q}_2 + m_e) \bar{\gamma}^\beta \right\} \]
\[ = e^3 \left\{ \gamma_\beta (Q_1 + m_e) \gamma_\mu (Q_2 + m_e) \gamma^\beta \right\} \]
\[ - \epsilon (Q_1 - m_e) \gamma_\mu (Q_2 - m_e) + \epsilon \bar{q}^2 \gamma_\mu - \bar{q}^2 \gamma_\beta \gamma_\mu \gamma^\beta \]
RATIONAL TERMS - $R_2$

$$\int d^n \tilde{q} \frac{\tilde{q}^2}{\bar{D}_0 \bar{D}_1 \bar{D}_2} = -\frac{i\pi^2}{2} + \mathcal{O}(\epsilon),$$

$$\int d^n \tilde{q} \frac{q_\mu q_\nu}{\bar{D}_0 \bar{D}_1 \bar{D}_2} = -\frac{i\pi^2}{2\epsilon} g_{\mu\nu} + \mathcal{O}(1),$$

$$R_2 = -\frac{ie^3}{8\pi^2} \gamma_\mu + \mathcal{O}(\epsilon),$$

Diagram:

```
\mu
```

\[ = -\frac{ie^3}{8\pi^2} \gamma_\mu \]
Rational Terms - $R_2$

Rational counterterms

\[ \begin{align*}
\mu \quad \bullet \quad \nu &= -\frac{ie^2}{8\pi^2} g_{\mu\nu} \left(2m_e^2 - p^2/3\right) \\
\mu \quad \bullet \quad \nu &= \frac{ie^2}{16\pi^2} (-\rho + 2m_e) \\
\mu \quad \bullet \quad \nu &= \frac{ie^4}{12\pi^2} \left(g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}\right)
\end{align*} \]
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- We do not need to repeat this for all Feynman diagrams. We can group them and solve for (sub)amplitudes directly.
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**Evaluate scalar integrals**

- massive integrals \( \rightarrow \) FF [G. J. van Oldenborgh]
- massless + massive integrals \( \rightarrow \) OneLOop [A. van Hameren]
What we gain

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Cuttools

Properties of the master equation
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Properties of the master equation

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The $N \equiv N$ test

A tool to efficiently treat phase-space points with numerical instabilities
General expression for the 4-dim $N(q)$ at the integrand level in terms of $D_i$

\[
N(q) = \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[ d(i_0i_1i_2i_3) + \tilde{d}(q; i_0i_1i_2i_3) \right] \prod_{i \neq i_0,i_1,i_2,i_3}^{m-1} D_i \\
+ \sum_{i_0<i_1<i_2}^{m-1} \left[ c(i_0i_1i_2) + \tilde{c}(q; i_0i_1i_2) \right] \prod_{i \neq i_0,i_1,i_2}^{m-1} D_i \\
+ \sum_{i_0<i_1}^{m-1} \left[ b(i_0i_1) + \tilde{b}(q; i_0i_1) \right] \prod_{i \neq i_0,i_1}^{m-1} D_i \\
+ \sum_{i_0}^{m-1} \left[ a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\]
As an example we present 4-photon and 6-photon amplitudes (via fermionic loop of mass $m_f$)

Input parameters for the reduction:
- External momenta $p_i$
- Masses of propagators in the loop
- Polarization vectors
As an example we present 4-photon and 6-photon amplitudes (via fermionic loop of mass $m_f$)

Input parameters for the reduction:
- External momenta $p_i \to$ in this example massless, i.e. $p_i^2 = 0$
- Masses of propagators in the loop $\to$ all equal to $m_f$
- Polarization vectors $\to$ various helicity configurations
\[ \frac{F_{++++}^f}{\alpha^2 Q_f^4} = -8 \]
Four Photons – Comparison with Gounaris et al.

\[
\frac{F^f_{++++}}{\alpha^2 Q_f^4} = -8 + 8 \left( 1 + \frac{2\hat{u}}{\hat{s}} \right) B_0(\hat{u}) + 8 \left( 1 + \frac{2\hat{t}}{\hat{s}} \right) B_0(\hat{t}) \\
- 8 \left( \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2} \right) \left[ \hat{t} C_0(\hat{t}) + \hat{u} C_0(\hat{u}) \right] \\
- 4 \left[ \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2} \right] D_0(\hat{t}, \hat{u})
\]

Massless four-photon amplitudes
\[
\frac{F_{++++}^f}{\alpha^2 Q_f^4} = -8 + 8 \left( 1 + \frac{2\hat{u}}{\hat{s}} \right) B_0(\hat{u}) + 8 \left( 1 + \frac{2\hat{t}}{\hat{s}} \right) B_0(\hat{t}) \\
- 8 \left( \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2} - \frac{4m_f^2}{\hat{s}} \right) [\hat{t} C_0(\hat{t}) + \hat{u} C_0(\hat{u})] \\
- 4 \left[ 4m_f^4 - (2\hat{s} m_f^2 + \hat{t}\hat{u}) \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2} + \frac{4m_f^2 \hat{t}\hat{u}}{\hat{s}} \right] D_0(\hat{t}, \hat{u}) \\
+ 8m_f^2 (\hat{s} - 2m_f^2) [D_0(\hat{s}, \hat{t}) + D_0(\hat{s}, \hat{u})]
\]
\[
\frac{F^f_{++++}}{\alpha^2 Q_f^4} = -8 + 8 \left( 1 + \frac{2\hat{u}}{\hat{s}} \right) B_0(\hat{u}) + 8 \left( 1 + \frac{2\hat{t}}{\hat{s}} \right) B_0(\hat{t})
- 8 \left( \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2} - \frac{4m_f^2}{\hat{s}} \right) [\hat{t}C_0(\hat{t}) + \hat{u}C_0(\hat{u})]
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+ 8m_f^2(\hat{s} - 2m_f^2)[D_0(\hat{s}, \hat{t}) + D_0(\hat{s}, \hat{u})]
\]

Massive four-photon amplitudes

Results also checked for \( F^f_{++++} \) and \( F^f_{++--} \)
Massless case: $[+++---]$ and $[+---++]$

Plot presented by Nagy and Soper hep-ph/0610028
( also Binoth et al., hep-ph/0703311)
Six Photons – Comparison with Nagy-Soper and Mahlon

Massless case: \([+ + - - - - - -]\) and \([+ - - + + + +]\)

Analogous plot produced with OPP reduction

\(s |M|/\alpha^3\)

\(\theta\)
Massless case: \([++----] \) and \([++--++]\)

Same plot as before for a wider range of \(\theta\)
Massless case: $[+ + - - - -]$ and $[+ + - - + +]$

Same idea for a different set of external momenta
Massless result [Mahlon]
Massless result [Mahlon]

$m = 0.5 \text{ GeV}$
Six Photons with Massive Fermions

Massless result [Mahlon]

- $m = 0.5 \text{ GeV}$
- $m = 4.5 \text{ GeV}$
Massless result [Mahlon]

- $m = 0.5$ GeV
- $m = 4.5$ GeV
- $m = 12.0$ GeV
Six Photons with Massive Fermions

Massless result [Mahlon]

- $m = 0.5 \text{ GeV}$
- $m = 4.5 \text{ GeV}$
- $m = 12.0 \text{ GeV}$
- $m = 20.0 \text{ GeV}$
NLO corrections to tri-boson production

- \( pp \rightarrow ZZZ \)
- \( pp \rightarrow W^+ ZZ \)
- \( pp \rightarrow W^+ W^- Z \)
- \( pp \rightarrow W^+ W^- W^+ \)

$pp \rightarrow ZZZ$ VIRTUAL CORRECTIONS


\[
\sigma_{\text{NLO,virt}}^{\text{div}} = -C_F \frac{\alpha_s}{\pi} \frac{\Gamma(1 + \epsilon)}{(4\pi)^{-\epsilon}} (s_{12})^{-\epsilon} \left( \frac{1}{\epsilon^2} + \frac{3}{2\epsilon} \right) \sigma_{\text{LO}}
\]

POLES $1/\epsilon^2$ AND $1/\epsilon$
$pp \rightarrow WWZ$ VIRTUAL CORRECTIONS

A still naive implementation
A still naive implementation

- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices!
A still naive implementation

- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices!
- Calculate 4d and rational $R_1$ terms by CutTools
A still naive implementation

- Calculate the $N(q)$ by brute (numerical) force namely multiplying gamma matrices!
- Calculate 4d and rational $R_1$ terms by CutTools
- $R_2$ terms added by hand
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Comparison with LMP
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- Of course full agreement for the $1/\epsilon^2$ and $1/\epsilon$ terms
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Comparison with LMP

- Of course full agreement for the $1/\epsilon^2$ and $1/\epsilon$ terms
- An 'easy' agreement for all graphs with up to 4-point loop integrals
- A bit more work to uncover the differences in scalar function normalization that happen to show to order $\epsilon^2$ thus influence only 5-point loop integrals.
Typical precision:
Typical precision:

- LMP: 9.573(66) about 1% error
Typical precision:

- LMP: 9.573(66) about 1% error

- OPP:
  \[
  \begin{aligned}
  &-26.45706742815552 \\
  &-26.457067428165503661018557937723426
  \end{aligned}
  \]
Typical precision:

- LMP: 9.573(66) about 1% error

- OPP:
  \[-26.45706742815552\]
  \[-26.457067428165503661018557937723426\]

Typical time: $10^4$ times faster (for non-singular PS-points)
\( pp \rightarrow VVV \) REAL CORRECTIONS

\[
\sigma_{qq \bar{q}}^{NLO} = \int_{VVVg} \left[ d\sigma_R^{q\bar{q}} - d\sigma_A^{q\bar{q}} \right] + \int_{VVV} \left[ d\sigma_B^{q\bar{q}} + d\sigma^V_{q\bar{q}} + \int_g d\sigma_A^{q\bar{q}} + d\sigma_C^{q\bar{q}} \right]
\]
\[
\sigma_{q\bar{q}}^{NLO} = \int_{VVVg} \left[ d\sigma_{q\bar{q}}^R - d\sigma_{q\bar{q}}^A \right] + \int_{VVV} \left[ d\sigma_{q\bar{q}}^B + d\sigma_{q\bar{q}}^V + \int_g d\sigma_{q\bar{q}}^A + d\sigma_{q\bar{q}}^C \right]
\]

\[
D_{q_1 g_6, \bar{q}_2} = \frac{8\pi\alpha_s C_F}{2\tilde{x} p_1 \cdot p_6} \left( \frac{1 + \tilde{x}^2}{1 - \tilde{x}} \right) |M_{q\bar{q}}^{B}(\tilde{p}_{16}, p_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5)|^2
\]
\[ \sigma_{q\bar{q}}^{NLO} = \int_{VVVg} \left[ d\sigma_R^{q\bar{q}} - d\sigma_A^{q\bar{q}} \right] + \int_{VVV} \left[ d\sigma_B^{q\bar{q}} + d\sigma_V^{q\bar{q}} + \int_g d\sigma_A^{q\bar{q}} + d\sigma_C^{q\bar{q}} \right] \]

\[ D_{q_1 g_6; \bar{q}_2} = \frac{8\pi\alpha_s C_F}{2\tilde{x} p_1 \cdot p_6} \left( \frac{1 + \tilde{x}^2}{1 - \tilde{x}} \right) |M_{q\bar{q}}(\tilde{p}_{16}, p_2, \tilde{p}_3, \tilde{p}_4, \tilde{p}_5)|^2 \]

\[ \tilde{x} = \frac{p_1 \cdot p_2 - p_2 \cdot p_6 - p_1 \cdot p_6}{p_1 \cdot p_2} \]

\[ \tilde{p}_{16} = \tilde{x} p_1 \quad , \quad K = p_1 + p_2 - p_6 \quad , \quad \tilde{K} = \tilde{p}_{16} + p_2 \]

\[ \Lambda^{\mu\nu} = g^{\mu\nu} - \frac{2(K^\mu + \tilde{K}^\mu)(K^\nu + \tilde{K}^\nu)}{(K + \tilde{K})^2} + \frac{2\tilde{K}^\mu K^\nu}{K^2} \]

\[ \tilde{p}_j = \Lambda p_j \]
\[ d\sigma^R_{q\bar{q}} - d\sigma^A_{q\bar{q}} = \frac{C_S}{N} \frac{1}{2s_{12}} \left[ C_F |\mathcal{M}^R_{q\bar{q}}(\{p_j\}')|^2 - D_{q_1g_6,\bar{q}_2} - D_{\bar{q}_2g_6,q_1} \right] d\Phi_{VVVg} \]
\[
\begin{align*}
\sigma^R_{q\bar{q}} - \sigma^A_{q\bar{q}} &= \frac{C_S}{N} \frac{1}{2s_{12}} \left[ C_F |M^R_{q\bar{q}}(\{p_j\})|^2 - D_{q_1g_6,\bar{q}_2} - D_{\bar{q}_2g_6,q_1} \right] d\Phi_{VVVg} \\
\sigma^C_{q\bar{q}} + \int \sigma^A_{q\bar{q}} &= \frac{\alpha_s C_F}{2\pi} \frac{\Gamma(1 + \epsilon)}{(4\pi)^{-\epsilon}} \left( \frac{s_{12}}{\mu^2} \right)^{-\epsilon} \left[ \frac{2}{\epsilon^2} + \frac{3}{\epsilon} - \frac{2\pi^2}{3} \right] d\sigma^B_{q\bar{q}} \\
&+ \frac{\alpha_s C_F}{2\pi} \int_0^1 dx \ K^{q,q}(x) \sigma^B_{q\bar{q}}(xp_1, p_2) + \frac{\alpha_s C_F}{2\pi} \int_0^1 dx \ K^{\bar{q},\bar{q}}(x) \sigma^B_{q\bar{q}}(p_1, xp_2)
\end{align*}
\]

\[
K^{q,q}(x) = K^{\bar{q},\bar{q}}(x) = \left( \frac{1 + x^2}{1 - x} \right) \log \left( \frac{s_{12}}{\mu^2} \right) + \left( \frac{4 \log(1 - x)}{1 - x} \right) + (1 - x) - 2(1 + x) \log(1 - x)
\]
\[
\sigma_{gq}^{NLO} = \int_{VVV} \left[ \int_{q} d\sigma_{gq}^{A} + d\sigma_{gq}^{C} \right] + \int_{VVVq} \left[ d\sigma_{gq}^{R} - d\sigma_{gq}^{A} \right]
\]
\[
\sigma_{gq}^{NLO} = \int_{VVV} \left[ \int_{q} d\sigma_{gq}^{A} + d\sigma_{gq}^{C} \right] + \int_{VVVq} \left[ d\sigma_{gq}^{R} - d\sigma_{gq}^{A} \right]
\]

\[
d\sigma_{gq}^{R} - d\sigma_{gq}^{A} = \frac{C_{S}}{N} \frac{1}{2s_{12}} \left[ T_{R} |\mathcal{M}_{gq}^{R}|^{2} - \mathcal{D}^{g_{1}q_{6},q_{2}} \right] d\Phi_{VVVq}
\]
\[ \sigma^N_{gq} = \int_{VVV} \left[ \int_q d\sigma^A_{gq} + d\sigma^C_{gq} \right] + \int_{VVVq} \left[ d\sigma^R_{gq} - d\sigma^A_{gq} \right] \]

\[ d\sigma^R_{gq} - d\sigma^A_{gq} = \frac{C_S}{N} \frac{1}{2s_{12}} \left[ T_R |M^R_{gq}|^2 - D^{g_1q_6,q_2} \right] d\Phi_{VVVq} \]

\[ D^{g_1q_6,q_2} = \frac{8\pi\alpha_s}{\tilde{x}^2 p_1 \cdot p_6} \frac{T_R}{1 - 2 \tilde{x} (1 - \tilde{x})} |M^B_{q\bar{q}}(\tilde{p}_j)|^2 \]
\[
d\sigma^C_{gq} + \int_q d\sigma^A_{gq} = \frac{\alpha_s T_R}{2\pi} \int_0^1 dx \, \mathcal{K}^{g,q}(x) \, d\sigma_{q\bar{q}}(xp_1, p_2)
\]

\[
\mathcal{K}^{g,q}(x) = [x^2 + (1 - x)^2] \log \left( \frac{s_{12}}{\mu_F^2} \right) + 2x(1 - x) + 2[x^2 + (1 - x)^2] \log(1 - x)
\]
\[ d\sigma^C_{gq} + \int d\sigma^A_{gq} = \frac{\alpha_s T_R}{2\pi} \int_0^1 dx \, K^{g,q}(x) \, d\sigma^B_{q\bar{q}}(xp_1, p_2) \]

\[ K^{g,q}(x) = [x^2 + (1 - x)^2] \log \left( \frac{s_{12}}{\mu_F^2} \right) + 2x(1 - x) + 2[x^2 + (1 - x)^2] \log(1 - x) \]

\[ d\sigma(P_1, P_2) = \sum_{ab} \int dz_1 dz_2 f_a(z_1, \mu_F) f_b(z_2, \mu_F) d\sigma_{ab}(z_1 P_1, z_2 P_2) \]

\( q\bar{q}, \bar{q}q, gq, qg, g\bar{q}, \bar{q}g \)
$pp \rightarrow VVV$ REAL CORRECTIONS

\[ d\sigma^C_{gq} + \int_q d\sigma^A_{gq} = \frac{\alpha_s T_R}{2\pi} \int_0^1 dx \mathcal{K}^{g,q}(x) d\sigma_{q\bar{q}}(xp_1, p_2) \]

\[ \mathcal{K}^{g,q}(x) = [x^2 + (1 - x)^2] \log \left( \frac{s_{12}}{\mu_F^2} \right) + 2x(1 - x) + 2[x^2 + (1 - x)^2] \log(1 - x) \]

\[ d\sigma(P_1, P_2) = \sum_{ab} \int dz_1 dz_2 f_a(z_1, \mu_F) f_b(z_2, \mu_F) d\sigma_{ab}(z_1 P_1, z_2 P_2) \]

$q\bar{q}, \bar{q}q, gq, qg, g\bar{q}, \bar{q}g$

- check also with phase-space slicing method
Virtual contributions obtained with Cuttools

$O(100\,ms)$ per "event" $\rightarrow$ factor $O(10 - 10^2)$
Virtual contributions obtained with Cuttools

$O(100 \text{ms})$ per "event" $\rightarrow$ factor $O(10 - 10^2)$

Real contributions obtained with Helac

Positive/negative (un)weighted events
Virtual contributions obtained with Cuttools

\[ O(100 \, ms) \text{ per "event" } \rightarrow \text{ factor } O(10 - 10^2) \]

Real contributions obtained with Helac

Positive/negative (un)weighted events

<table>
<thead>
<tr>
<th>Process</th>
<th>scale ( \mu )</th>
<th>Born cross section [fb]</th>
<th>NLO cross section [fb]</th>
</tr>
</thead>
<tbody>
<tr>
<td>ZZZ</td>
<td>( 3M_Z )</td>
<td>9.7(1)</td>
<td>15.3(1)</td>
</tr>
<tr>
<td>WZZ</td>
<td>( 2M_Z + M_W )</td>
<td>20.2(1)</td>
<td>40.4(2)</td>
</tr>
<tr>
<td>WWZ</td>
<td>( M_Z + 2M_W )</td>
<td>96.8(6)</td>
<td>185.5(8)</td>
</tr>
<tr>
<td>WWW</td>
<td>( 3M_W )</td>
<td>82.5(5)</td>
<td>146.2(6)</td>
</tr>
</tbody>
</table>
\[ pp \rightarrow VVV \text{ NLO} \]


\[ ZZW^+ \]

<table>
<thead>
<tr>
<th>Scale</th>
<th>program</th>
<th>( \sigma^{\text{LO}} ) [fb]</th>
<th>( \sigma^{\text{NLO}} ) [fb]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} \times (3 m_Z) )</td>
<td>VBFNLO</td>
<td>20.42 ± 0.03</td>
<td>43.02 ± 0.08</td>
</tr>
<tr>
<td></td>
<td>Ref. [3]</td>
<td>20.2 ± 0.1</td>
<td>43.0 ± 0.2</td>
</tr>
<tr>
<td>( 2 m_Z + m_W )</td>
<td>VBFNLO</td>
<td>20.30 ± 0.03</td>
<td>39.87 ± 0.08</td>
</tr>
<tr>
<td></td>
<td>Ref. [3]</td>
<td>20.2 ± 0.1</td>
<td>40.4 ± 0.2</td>
</tr>
<tr>
<td>( (3 m_Z) )</td>
<td>VBFNLO</td>
<td>20.24 ± 0.03</td>
<td>39.86 ± 0.07</td>
</tr>
<tr>
<td></td>
<td>Ref. [3]</td>
<td>20.0 ± 0.1</td>
<td>39.7 ± 0.2</td>
</tr>
<tr>
<td>( 2 \times (3 m_Z) )</td>
<td>VBFNLO</td>
<td>20.03 ± 0.03</td>
<td>37.39 ± 0.07</td>
</tr>
<tr>
<td></td>
<td>Ref. [3]</td>
<td>19.7 ± 0.1</td>
<td>37.8 ± 0.2</td>
</tr>
</tbody>
</table>

\[ W^+ W^- W^+ \]

<table>
<thead>
<tr>
<th>Scale</th>
<th>program</th>
<th>( \sigma^{\text{LO}} ) [fb]</th>
<th>( \sigma^{\text{NLO}} ) [fb]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{2} \times (3 m_Z) )</td>
<td>VBFNLO</td>
<td>82.7 ± 0.1</td>
<td>152.5 ± 0.3</td>
</tr>
<tr>
<td></td>
<td>Ref. [3]</td>
<td>82.7 ± 0.5</td>
<td>153.2 ± 0.6</td>
</tr>
<tr>
<td>( 3m_W )</td>
<td>VBFNLO</td>
<td>82.8 ± 0.1</td>
<td>145.2 ± 0.3</td>
</tr>
<tr>
<td></td>
<td>Ref. [3]</td>
<td>82.5 ± 0.5</td>
<td>146.2 ± 0.6</td>
</tr>
<tr>
<td>( (3 m_Z) )</td>
<td>VBFNLO</td>
<td>82.8 ± 0.1</td>
<td>143.8 ± 0.3</td>
</tr>
<tr>
<td></td>
<td>Ref. [3]</td>
<td>81.4 ± 0.5</td>
<td>144.5 ± 0.6</td>
</tr>
<tr>
<td>( 2 \times (3 m_Z) )</td>
<td>VBFNLO</td>
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<td>136.8 ± 0.3</td>
</tr>
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<td>139.1 ± 0.6</td>
</tr>
</tbody>
</table>
\( pp \rightarrow VVV \) NLO

- **ZZZ**
- **W^+ ZZ**
- **W^+ W^- Z**
- **W^+ W^- W^+**
$pp \rightarrow VVV$ NLO

**Graphs:**
- ZZZ
- $W^+ ZZ$
- $W^+ W^- Z$
- $W^+ W^- W^+$
$pp \rightarrow VVV$ NLO

$\log(\frac{d\sigma}{dy})$

$K$–factor

$y$

$y$
\[ pp \rightarrow VVV \text{ NLO} \]

<table>
<thead>
<tr>
<th>scale</th>
<th>( \sigma_B )</th>
<th>( \sigma_{NLO} )</th>
<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu = \frac{M}{2} )</td>
<td>82.7(5)</td>
<td>153.2(6)</td>
<td>1.85</td>
</tr>
<tr>
<td>( \mu = M )</td>
<td>81.4(5)</td>
<td>144.5(6)</td>
<td>1.77</td>
</tr>
<tr>
<td>( \mu = 2M )</td>
<td>81.8(5)</td>
<td>139.1(6)</td>
<td>1.70</td>
</tr>
</tbody>
</table>

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<th>( K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu = \frac{M}{2} )</td>
<td>20.2(1)</td>
<td>43.0(2)</td>
<td>2.12</td>
</tr>
<tr>
<td>( \mu = M )</td>
<td>20.0(1)</td>
<td>39.7(2)</td>
<td>1.99</td>
</tr>
<tr>
<td>( \mu = 2M )</td>
<td>19.7(1)</td>
<td>37.8(2)</td>
<td>1.91</td>
</tr>
</tbody>
</table>
Existing Tools 2008

**BlackHat**


**Rocket**


**CutTools**

Still using Feynman Graphs, but a new (OPP) reduction approach
One Loop Amplitude Calculation

- Still using Feynman Graphs, but a new (OPP) reduction approach
- Unitarity-like approach
One Loop Amplitude Calculation

- Still using Feynman Graphs, but a new (OPP) reduction approach
- Unitarity-like approach
- Dyson-Schwinger recursion
Reliable cross section computation and event generation for multiparticle processes, with $\sim 10^{-12}$ particles in the final state.


Amplitude calculation-I

Web page

http://www.cern.ch/helac-phegas
Amplitude calculation - I

Colour Configuration - EWK⊕QCD

- Ordinary approach $SU(N)$-type

$$A^{a_1\ldots a_n} = \sum Tr(T^{a_1\sigma_1} \ldots T^{a_n\sigma_n}) \ A(\sigma_1 \ldots \sigma_n)$$

$$C_{ij} = \sum Tr(T^{a_1\sigma_1} \ldots T^{a_n\sigma_n}) Tr(T^{a'_1\sigma'_1} \ldots T^{a'_n\sigma'_n})$$

Quarks and gluons treated differently
Amplitude calculation - I

Colour Configuration - EWK$\oplus$QCD

- New approach $U(N)$-type

Each color-configuration amplitude is proportional to

$$D_i = \delta_{1,\sigma_i(1)}\delta_{2,\sigma_i(2)}\cdots\delta_{n,\sigma_i(n)}$$

where $\sigma_i$ represents the $i$-th permutation of the set $1, 2, \ldots, n$.

- quarks $1\ldots n$
- antiquarks $\sigma_i(1\ldots n)$ and
- gluons $= q\bar{q}$

$$C_{ij} = \sum D_i D_j = N_c^\alpha, \quad \alpha = \langle \sigma_1, \sigma_2 \rangle$$
Amplitude calculation-I

\[ \sum f^{abc} t^{a}_{AB} t^{b}_{CD} t^{c}_{EF} = -\frac{i}{4} (\delta_{AD}\delta_{CF}\delta_{EB} - \delta_{AF}\delta_{CB}\delta_{ED}) \]

\[ \delta_{1\sigma_{2}} \delta_{2\sigma_{3}} \delta_{3\sigma_{1}} \]
Amplitude calculation-I

\[ \sum t_A^a t_C^b = \frac{1}{2} (\delta_{AD} \delta_{CB} - \frac{1}{N_c} \delta_{AB} \delta_{AC}) \]
\[ \sum t_{AB}^a t_{CD}^b = \frac{1}{2} (\delta_{AD} \delta_{CB} - \frac{1}{N_c} \delta_{AB} \delta_{AC}) \]
\begin{align*}
\delta_1 \sigma_3 \delta_3 \sigma_2 \delta_2 \sigma_4 \delta_4 \sigma_1
\end{align*}
Amplitude calculation-I
Amplitude calculation-I
Amplitude calculation-I

\[
\text{Diagram with nodes and arrows representing amplitude calculation.}
\]
Amplitude calculation-II

\[
\begin{align*}
&= + \\
&+ + \\
&+ + \\
&+ + \\
&+ + \\
&+ \ldots \\
\end{align*}
\]
Amplitude calculation-II
INFO =============================================
INFO COLOR 1 out of 1
INFO number of nums 25
INFO NUM 1 of 25
INFO 4 24 35 7 1 1 8 35 4 16 35 5 0 0 0 0 1 1
INFO 4 28 35 8 1 1 4 35 3 24 35 7 0 0 0 0 1 1
INFO 4 30 35 9 1 1 2 35 2 28 35 8 0 0 0 0 1 1
INFO 4 62 35 10 1 1 32 35 6 30 35 9 0 0 0 0 1 2
INFO 4 8 4 2 1 35 35 35 35 0 0 0 0 0 0 4 3
INFOYY 3
INFO NUM 2 of 25
INFO 18 24 13 7 1 1 8 35 4 16 13 5 0 0 0 0 1 1
INFO 18 28 13 8 1 1 4 35 3 24 13 7 0 0 0 0 1 1
INFO 18 30 13 9 1 1 2 35 2 28 13 8 0 0 0 0 1 1
INFO 17 62 35 10 1 1 32 -13 6 30 13 9 0 0 0 0 -1 2
INFO 4 8 4 2 1 13 13 13 13 0 0 0 0 0 0 4 -3
INFOYY -3
INFO NUM 3 of 25
INFO 4 12 35 7 1 1 4 35 3 8 35 4 0 0 0 0 1 1
INFO 4 28 35 8 1 1 4 35 3 24 35 7 0 0 0 0 1 2
INFO 5 28 35 8 2 2 4 35 3 8 35 4 16 35 5 0 1 5
INFO 4 30 35 9 1 1 2 35 2 28 35 8 0 0 0 0 1 1
INFO 4 62 35 10 1 1 32 35 6 30 35 9 0 0 0 0 1 2
INFO 3 12 2 1 35 35 35 0 0 0 0 0 0 3 3
INFOYY 3
INFO NUM 4 of 25
INFO 4 12 35 7 1 1 4 35 3 8 35 4 0 0 0 0 1 1
INFO 18 28 13 8 1 1 12 35 7 16 13 5 0 0 0 0 1 1
INFO 18 30 13 9 1 1 2 35 2 28 13 8 0 0 0 0 1 1
INFO 17 62 35 10 1 1 32 -13 6 30 13 9 0 0 0 0 -1 2
INFO 3 12 2 1 13 13 13 13 0 0 0 0 0 0 3 -3
INFOYY -3

Costas G. Papadopoulos (Athens) OPP Reduction HP2 74 / 79
Rational Terms - $R_2$

\[
\frac{1}{2} p^2 g_{\mu\nu} - \frac{1}{3} p_\mu p_\nu
\]

\[-\frac{11}{6} V_{\mu\nu\rho}\]

\[-\frac{16}{3} g_{\mu\rho} g_{\nu\sigma} + \frac{7}{3} g_{\mu\nu} g_{\rho\sigma} + \frac{7}{3} g_{\rho\nu} g_{\mu\sigma}\]
Amplitude calculation-II

INFO NUM 19 of 25
INFO 22 14 35 5 1 1 2 35 2 4 35 3 8 35 4 0 1 5
INFO 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
INFOYY 1
INFO NUM 20 of 25
INFO 4 12 35 5 1 1 4 35 3 8 35 4 0 0 0 0 1 1
INFO 21 14 35 6 1 1 2 35 2 12 35 5 0 0 0 0 1 1
INFO 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
INFOYY 1
INFO NUM 21 of 25
INFO 21 6 35 5 1 1 2 35 2 4 35 3 0 0 0 0 1 1
INFO 4 14 35 6 1 1 8 35 4 6 35 5 0 0 0 0 1 2
INFO 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
INFOYY 1
INFO NUM 22 of 25
INFO 21 12 35 5 1 1 4 35 3 8 35 4 2 0 0 0 1 1
INFO 4 14 35 6 1 1 2 35 2 12 35 5 0 0 0 0 1 1
INFO 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
INFOYY 1
INFO NUM 23 of 25
INFO 21 12 35 5 1 1 4 35 3 8 35 4 0 0 0 0 1 1
INFO 4 14 35 6 1 1 2 35 2 12 35 5 0 0 0 0 1 1
INFO 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
INFOYY 1
INFO NUM 24 of 25
INFO 4 12 35 5 1 2 4 35 3 8 35 4 0 0 0 0 1 1
INFO 20 12 35 5 2 2 12 35 5 0 0 0 0 0 0 0 1 0
INFO 4 14 35 6 1 1 2 35 2 12 35 5 0 0 0 0 1 1
INFO 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
INFOYY 1
INFO NUM 25 of 25
INFO 4 6 35 5 1 2 2 35 2 4 35 3 0 0 0 0 1 1
INFO 20 6 35 5 2 2 6 35 5 0 0 0 0 0 0 0 1 0
INFO 4 14 35 6 1 1 8 35 4 6 35 5 0 0 0 0 1 2
INFO 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
Rational Terms - $R_2$
Rational Terms - $R_2$

- WI tested for $n = 4, 5, 6$ gluonic amplitude
W1 tested for $n = 4, 5, 6$ gluonic amplitude

Any type and number of particles
The "unitarity" method
Reduction at the integrand level
Reduction at the integrand level

- changes the computational approach at one loop
Outlook

Reduction at the integrand level

- changes the computational approach at one loop
- Numerical but still algebraic: speed and precision not a problem
Outlook

Reduction at the integrand level

- changes the computational approach at one loop
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Achieved
Outlook

Reduction at the integrand level

- changes the computational approach at one loop
- Numerical but still algebraic: speed and precision not a problem

Achieved

- Understand potential stability problems
Outlook

Reduction at the integrand level

- changes the computational approach at one loop
- Numerical but still algebraic: speed and precision not a problem

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A generic NLO calculator seems feasible