$**$Problem 2.25 Check the uncertainty principle for the wave function in Equation 2.129. *Hint:* Calculating $\langle p^2 \rangle$ is tricky, because the derivative of ψ has a step discontinuity at $x = 0$. Use the result in Problem 2.24(b). *Partial answer:* $\langle p^2 \rangle = (m\alpha/\hbar)^2$.

Problem 2.25

$$\psi(x) = \frac{\sqrt{m\alpha}}{\hbar} e^{-m\alpha|x|/\hbar^2} = \frac{\sqrt{m\alpha}}{\hbar} \begin{cases} e^{-m\alpha x/\hbar^2}, & (x \geq 0), \\ e^{m\alpha x/\hbar^2}, & (x \leq 0). \end{cases}$$

$$\langle x \rangle = 0 \text{ (odd integrand).}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi|^2 dx = 2\frac{m\alpha}{\hbar^2} \int_0^{\infty} x^2 e^{-2m\alpha x/\hbar^2} dx = \frac{2m\alpha}{\hbar^2} 2 \left(\frac{\hbar^2}{2m\alpha}\right)^3 = \frac{\hbar^4}{2m^2\alpha^2}; \quad \sigma_x = \frac{\hbar^2}{\sqrt{2}m\alpha}.$$

$$\frac{d\psi}{dx} = \frac{\sqrt{m\alpha}}{\hbar} \left\{ \begin{array}{ll} -\frac{m\alpha}{\hbar^2} e^{-m\alpha x/\hbar^2}, & (x \geq 0) \\ \frac{m\alpha}{\hbar^2} e^{m\alpha x/\hbar^2}, & (x \leq 0) \end{array} \right\} = \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \left[-\theta(x) e^{-m\alpha x/\hbar^2} + \theta(-x) e^{m\alpha x/\hbar^2}\right].$$

$$\begin{aligned} \frac{d^2\psi}{dx^2} &= \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \left[-\delta(x) e^{-m\alpha x/\hbar^2} + \frac{m\alpha}{\hbar^2}\theta(x) e^{-m\alpha x/\hbar^2} - \delta(-x) e^{m\alpha x/\hbar^2} + \frac{m\alpha}{\hbar^2}\theta(-x) e^{m\alpha x/\hbar^2}\right] \\ &= \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \left[-2\delta(x) + \frac{m\alpha}{\hbar^2} e^{-m\alpha|x|/\hbar^2}\right]. \end{aligned}$$

In the last step I used the fact that $\delta(-x) = \delta(x)$ (Eq. 2.142), $f(x)\delta(x) = f(0)\delta(x)$ (Eq. 2.112), and $\theta(-x) + \theta(x) = 1$ (Eq. 2.143). Since $d\psi/dx$ is an odd function, $\langle p \rangle = 0$.

$$\begin{aligned} \langle p^2 \rangle &= -\hbar^2 \int_{-\infty}^{\infty} \psi \frac{d^2\psi}{dx^2} dx = -\hbar^2 \frac{\sqrt{m\alpha}}{\hbar} \left(\frac{\sqrt{m\alpha}}{\hbar}\right)^3 \int_{-\infty}^{\infty} e^{-m\alpha|x|/\hbar^2} \left[-2\delta(x) + \frac{m\alpha}{\hbar^2} e^{-m\alpha|x|/\hbar^2}\right] dx \\ &= \left(\frac{m\alpha}{\hbar}\right)^2 \left[2 - 2\frac{m\alpha}{\hbar^2} \int_0^{\infty} e^{-2m\alpha x/\hbar^2} dx\right] = 2\left(\frac{m\alpha}{\hbar}\right)^2 \left[1 - \frac{m\alpha}{\hbar^2}\frac{\hbar^2}{2m\alpha}\right] = \left(\frac{m\alpha}{\hbar}\right)^2. \end{aligned}$$

Evidently

$$\sigma_p = \frac{m\alpha}{\hbar}, \quad \text{so} \quad \sigma_x \sigma_p = \frac{\hbar^2}{\sqrt{2}m\alpha}\frac{m\alpha}{\hbar} = \sqrt{2}\frac{\hbar}{2} > \frac{\hbar}{2}. \ \checkmark$$