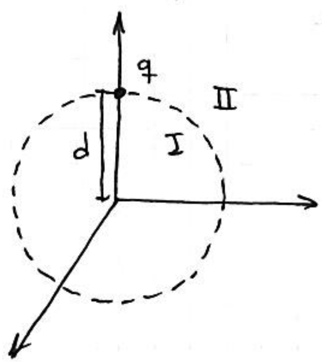
Ejemplo:



$$\varphi(\underline{r}) = \frac{q}{|\underline{r} - d\hat{z}|}$$

Dividimos el espacio en dos zonas $\nexists$ p $\nabla^2\varphi = 0$

En I: $\varphi_I = \sum_{lm} A_{lm} r^l Y_{lm}$

En II: $\varphi_{II} = \sum_{lm} \frac{B_{lm}}{r^{l+1}} Y_{lm}$

y φ debe ser continuo en la interface $r=d$

$$\Rightarrow \sum_{lm} A_{lm} d^l Y_{lm}(\theta\phi) = \sum_{lm} \frac{B_{lm}}{d^{l+1}} Y_{lm}(\theta\phi) \qquad (1)$$

Además

$$-\left.\frac{\partial\varphi_{II}}{\partial r}\right|_{r=d} + \left.\frac{\partial\varphi_I}{\partial r}\right|_{r=d} = 4\pi\,\sigma(\theta\phi)$$

con $\sigma = N\,\delta(\theta)$ $\nexists$ p $\int \sigma\, dS = q$

$$\int \sigma\, dS = \int N\,\delta(\theta)\, d^2 \operatorname{sen}\theta\, d\theta\, d\phi = q$$

$$\Rightarrow \quad \sigma(\theta,\phi) = \frac{q\,\delta(\theta)}{2\pi d^2 \operatorname{sen}\theta}$$

luego

$$\sum_{lm}\left[(l+1)\frac{B_{lm}}{d^{l+2}} + l A_{lm} d^{l-1}\right] Y_{lm}(\theta\phi) = 4\cancel{\pi}\,\frac{q\,\delta(\theta)}{2\cancel{\pi} d^2 \operatorname{sen}\theta} \qquad (2)$$

De (1)

$$A_{lm} d^{l} = \frac{B_{lm}}{d^{l+1}} \Longrightarrow B_{lm} = A_{lm} d^{2l+1}$$

y de (2), usando ortogonalidad

$$\int Y^{*}_{l'm'} \sum_{lm}\left[(l+1)\frac{B_{lm}}{d^{l+2}} + l A_{lm} d^{l-1}\right] Y_{lm}\, d\Omega = \int Y^{*}_{l'm'}\, \frac{2q\,\delta(\theta)}{d^2 \operatorname{sen}\theta}\, d\Omega$$

$$\Longrightarrow \frac{l'+1}{d^{l'+2}} B_{l'm'} + l' d^{l'-1} A_{l'm'} = \frac{4\pi q}{d^2}\sqrt{\frac{2l'+1}{4\pi}\frac{(l'-m')!}{(l'+m')!}}\, P_{l}^{m'}(1)\,\delta_{m',0} \qquad (3)$$

luego si $m \neq 0$, $A_{lm} = B_{lm} = 0$ (axisimetría)

y $\left\{ \begin{array}{l} \varphi_{I} = \sum_{l} A'_{l} r^{l} P_{l}(\cos\theta) \\ \varphi_{II} = \sum_{l} \frac{B'_{l}}{r^{l+1}} P_{l}(\cos\theta) \end{array} \right.$ en término de los $P_l(\cos\theta)$ en lugar de los $Y_{lm}(\theta,\varphi)$ (requiere cuidado en la normalización)

Usando $B_{lm} = A_{lm} d^{2l+1}$ en (3)

$$A_{l0}\, d^{l-1}(2l+1) = \frac{4\pi q}{d^2}\sqrt{\frac{2l+1}{4\pi}}\, P_l(1)$$

coef. de los Y_{lm}

$$\Longrightarrow A_{l0} = \frac{q}{d^{l+1}}\sqrt{\frac{4\pi}{2l+1}}\, \underbrace{P_l(1)}_{=1}$$

Pero para obtener los A'_l, notar que $\int P_l P_{l'}\, dc\theta\, d\phi = 2\pi \frac{2}{2l+1}\delta_{ll'}$

y si $f = \sum A'_l P_l(c\theta) \Rightarrow A'_l = \frac{2l+1}{4\pi}\int f\, P_l(c\theta)\, dc\theta\, d\phi$

y $A_{l0} = \sqrt{\frac{2l+1}{4\pi}}\int f\, P_l(c\theta)\, dc\theta\, d\phi$

$$\Longrightarrow \quad A'_\ell = A_{\ell 0}\sqrt{\frac{2\ell+1}{4\pi}}$$

y luego $\boxed{A'_\ell = \frac{q}{d^{\ell+1}}}$ y $\boxed{B'_\ell = q d^\ell}$

$$\Longrightarrow \begin{cases} \varphi_{I} = q \sum_\ell \frac{r^\ell}{d^{\ell+1}} P_\ell(c\theta) & r < d \\ \varphi_{II} = q \sum_\ell \frac{d^\ell}{r^{\ell+1}} P_\ell(c\theta) & r > d \end{cases}$$

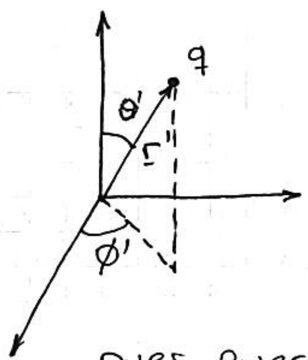
Se introduce notación $r_< = \min\{r, d\}$

$r_> = \max\{r, d\}$

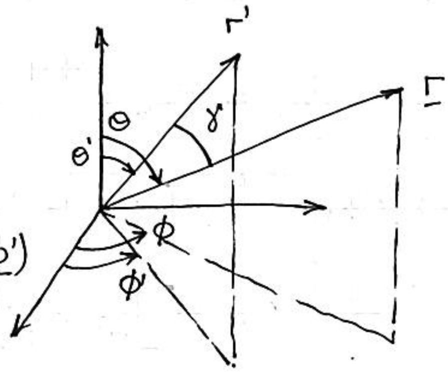
y $\boxed{\varphi(\underline{r}) = q \sum_\ell \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(c\theta) = \frac{q}{|\underline{r} - d\hat{\underline{z}}|}}$ Desarrollo del pot. debido a una carga puntual en $z=d$

Teorema de adición para armónicos esféricos

Supongamos que ahora queremos el potencial debido a una carga puntual fuera del eje z.



En general, necesitamos poder expresar los $Y_{\ell m}(\theta\phi)$ en términos de θ', ϕ', o equivalentemente, queremos $P_\ell(c\gamma)$



pues queremos resolver el problema con

$$\sigma(\theta,\phi) = q\,\frac{\delta(\theta-\theta')\,\delta(\phi-\phi')}{r'^2 \operatorname{sen}\theta}$$

Consideremos θ' y ϕ' fijos y desarrollemos

$$P_\ell(c\gamma) = \sum_{\ell'}\sum_m A_{\ell'm}(\theta'\phi')\, Y_{\ell'm}(\theta\phi)$$

pero cuando $\theta'=0$ tenemos $P_\ell(c\theta) = \sum_{\ell'm} A_{\ell'm}\, Y_{\ell'm}(\theta,\phi)$

y se sigue $\ell=\ell'$.

pues

$c\gamma = \hat r\cdot\hat r' = s\theta s\theta' c(\phi-\phi') + c\theta c\theta'$

usando $\hat r = (s\theta c\phi,\ s\theta s\phi,\ c\theta)$

$\hat r' = (s\theta' c\phi',\ s\theta' s\phi',\ c\theta')$

y $c(\phi-\phi') = c\phi c\phi' + s\phi s\phi'$

$$\Longrightarrow \qquad P_\ell(c\gamma) = \sum_m A_{\ell m}(\theta'\phi')\, Y_{\ell m}(\theta\phi)$$

con $A_{\ell m}(\theta'\phi') = \int Y^*_{\ell m}(\theta\phi)\, P_\ell(c\gamma)\, d\Omega$

que puede interpretarse como los coef. de $\sqrt{\frac{4\pi}{2\ell+1}}\, Y^*_{\ell m}(\theta\phi)$ en una serie de $Y_{\ell m'}(\gamma\beta)\,\delta_{m'0}$ con β el ángulo toroidal alrededor de $\underline{r}'$:

$$A_{\ell m}(\theta'\phi') = \sqrt{\frac{4\pi}{2\ell+1}} \int Y^*_{\ell m}(\theta\phi)\, \underbrace{\sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(c\gamma)}_{Y_{\ell m'}(\gamma\beta)\,\delta_{m',0}}\, d\Omega$$

luego

$$\sqrt{\frac{4\pi}{2\ell+1}}\left[Y^*_{\ell m}(\theta(\gamma,\beta),\phi(\gamma,\beta))\right]_{\gamma=0} = \sum_{\ell' m'} A_{\ell m}\sqrt{\frac{2\ell'+1}{4\pi}}\, P_{\ell'}(c\gamma)$$

$\uparrow$ pero nuevamente $\ell'=\ell$ y $m'=0$

y para $\gamma\to 0$ $\begin{cases} \theta(\gamma,\beta)\to\theta' \\ \phi(\gamma,\beta)\to\phi' \end{cases}$

$$\text{y} \qquad A_{\ell m}(\theta',\phi') = \frac{4\pi}{2\ell+1}\, Y_{\ell m}(\theta',\phi')$$

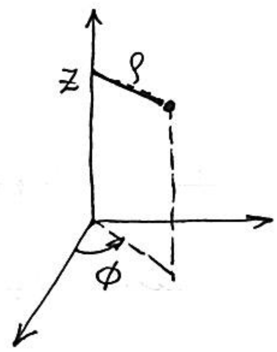
$$\Longrightarrow \boxed{P_\ell(c\gamma) = \sum_m \frac{4\pi}{2\ell+1}\, Y^*_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta\phi)}$$

Usando este resultado podemos escribir para la carga q fuera del eje z

$$\varphi(\underline{r}) = \frac{q}{|\underline{r}-\underline{r}'|} = 4\pi q \sum_{\ell,m} \frac{1}{2\ell+1} \frac{r_<^{\ell}}{r_>^{\ell+1}} Y_{\ell m}^{*}(\theta'\phi') Y_{\ell m}(\theta\phi)$$

Coordenadas cilíndricas

En cilíndricas $\quad \nabla^2\varphi = 0 \quad$ con (ρ,ϕ,z)



$$\frac{\partial^2\varphi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\varphi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\varphi}{\partial\phi^2} + \frac{\partial^2\varphi}{\partial z^2} = 0$$

Tomamos $\quad \varphi = R(\rho)\, Q(\phi)\, Z(z)$

$$\Rightarrow \underbrace{\frac{1}{R}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}\right)R + \frac{1}{\rho^2 Q}\frac{d^2Q}{d\phi^2}}_{= -\lambda} + \underbrace{\frac{1}{Z}\frac{d^2Z}{dz^2}}_{= \lambda} = 0$$

$$\Rightarrow \boxed{\frac{d^2Z}{dz^2} = \lambda Z} \qquad (1)$$

Luego $\quad \underbrace{\frac{\rho^2}{R}\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho}\right)R + \lambda\rho^2}_{= \beta} + \underbrace{\frac{1}{Q}\frac{d^2Q}{d\phi^2}}_{= -\beta} = 0$

$$\Rightarrow \boxed{\frac{d^2Q}{d\phi^2} = -\beta Q} \qquad (2)$$

y finalmente

$$\rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} = (\beta - \lambda\rho^2) R \qquad (3)$$

Veamos la solución en ϕ: De (2)

$$\text{si} \begin{cases} \beta = -\nu^2 & \Longrightarrow \quad Q_\nu = e^{\nu\phi},\ e^{-\nu\phi} \\ \beta = 0 & \Longrightarrow \quad 1,\ \phi \\ \beta = \nu^2 & \Longrightarrow \quad \cos\nu\phi,\ \operatorname{sen}\nu\phi \end{cases}$$

consideremos estos casos (ν entero)

La solución en Z, de (1)

$$Z(z) = \begin{cases} e^{kz}, e^{-kz} & \text{si } \lambda = k^2 \\ 1, z & \lambda = 0 \\ \operatorname{sen} kz, \cos kz & \lambda = -k^2 \end{cases}$$