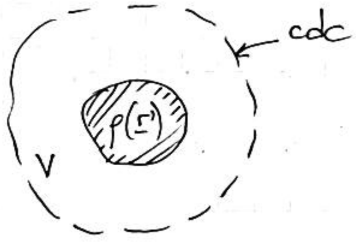
Función de Green:



Queremos hallar φ en V

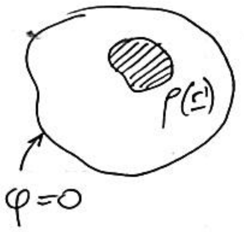
Consideremos el caso con cdc de Dirichlet, $\varphi)_s$ dato.

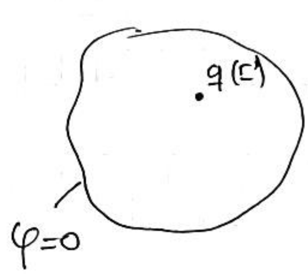
Tomamos $\varphi = \varphi_1 + \varphi_2$

con φ_1 tq $\quad \nabla^2\varphi_1 = -4\pi\rho \quad$ y $\quad \varphi_1\big|_S = 0$

y $\quad \varphi_2$ tq $\quad \nabla^2\varphi_2 = 0 \quad$ y $\quad \varphi_2\big|_S = \varphi\big|_S = V(\underline{r})$

Veremos que resolver φ_1 es facil si se resolver





La sol. del segundo problema es

$$\nabla^2 G_D(\underline{r},\underline{r}') = -4\pi\,\delta(\underline{r}-\underline{r}')$$

Func. de Green tipo Dirichlet

con cdc $\qquad G_D(\underline{r},\underline{r}')\big|_S = 0$

Por ejemplo, para el caso $S \to \infty$ tenemos

$G_D(\underline{r},\underline{r}') = \dfrac{1}{|\underline{r}-\underline{r}'|}$.

Dada la geometría de la región de interés, $G_D(\underline{r},\underline{r}')$ está univocamente definida por estas cond. para cada $\underline{r}'$.

Luego $\qquad \varphi_1(\underline{r}) = \int \rho(\underline{r}')\, G_D(\underline{r},\underline{r}')\, dV'$

$\Rightarrow$ es automático que

$$\begin{cases} \varphi_1\big|_S = \int \rho(\underline{r}')\, G_D(\underline{r},\underline{r}')\big|_S\, dV' = 0 \\ \nabla^2\varphi_1 = \int \rho(\underline{r}')\, \underbrace{\nabla^2 G_D(\underline{r},\underline{r}')}_{-4\pi\delta(\underline{r}-\underline{r}')}\, dV' = -4\pi\rho(\underline{r}) \end{cases}$$

Prop: G_D es simétrica en $\underline{r}, \underline{r}'$ $\qquad G_D(\underline{r}, \underline{r}') = G_D(\underline{r}', \underline{r})$

Para verlo usemos el teorema de Green

$$\int_V (\phi \nabla^2 \chi - \chi \nabla^2 \phi)\, dV = \int_{S(V)} \left(\phi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \phi}{\partial n}\right) dS$$

Tomando

$$\phi = G_D(\underline{x}, \underline{r}) \qquad \text{con } \underline{x} \text{ variable de integración}$$
$$\chi = G_D(\underline{x}, \underline{r}')$$

$$\Rightarrow \int_V \left[G_D(\underline{x}, \underline{r})\left(-4\pi\,\delta(\underline{x} - \underline{r}')\right) - G_D(\underline{x}, \underline{r}')\left(-4\pi\,\delta(\underline{r} - \underline{x})\right)\right] d^3x =$$

$$= \int_{S(V)} \Big[\underbrace{\cancel{G_D(\underline{x}, \underline{r})}}_{=0 \text{ sobre la sup } S} \frac{\partial}{\partial n} G_D(\underline{x}, \underline{r}') - \underbrace{\cancel{G_D(\underline{x}, \underline{r}')}}_{=0} \frac{\partial}{\partial n} G_D(\underline{x}, \underline{r})\Big] dS$$

$$\Rightarrow \quad -4\pi\, G_D(\underline{r}', \underline{r}) + 4\pi\, G_D(\underline{r}, \underline{r}') = 0$$

$$\Rightarrow \quad \boxed{G_D(\underline{r}, \underline{r}') = G_D(\underline{r}', \underline{r})}$$

Para tener sol. al problema, necesitamos también φ_2. Veamos que sale usando G_D: usamos Green, ahora con

$$\phi = \varphi_2(\underline{r})$$
$$\chi = G_D(\underline{r}, \underline{r}')$$

$$\int_V \varphi_2(\underline{r})\left(-4\pi\,\delta(\underline{r} - \underline{r}')\right) d^3r - \int_V G_D(\underline{r}, \underline{r}')\, \underbrace{\cancel{\nabla^2 \varphi_2}}_{=0}\, d^3r =$$

$$= \int_{S(V)} \Big[\varphi_2 \frac{\partial G_D}{\partial n}(\underline{r}, \underline{r}') - \underbrace{\cancel{G_D(\underline{r}, \underline{r}')}}_{=0} \frac{\partial \varphi_2}{\partial n}\Big] dS$$

$$\Rightarrow \quad -4\pi\, \varphi_2(\underline{r}') = \int_{S(V)} \varphi_2 \frac{\partial}{\partial n} G_D(\underline{r}, \underline{r}')\, dS$$

Intercambiando $\underline{r}$ con $\underline{r}'$

$$\varphi_2(\underline{r}) = -\frac{1}{4\pi}\int_{S(V)} \varphi_2(\underline{r}')\,\frac{\partial}{\partial n} G_D(\underline{r},\underline{r}')\,dS'$$

O sea que, dadas $\rho(\underline{r})$ y $\varphi\big|_S = V(\underline{r})$, la sol. formal del problema en términos de G_D se escribe

$$\boxed{\varphi(\underline{r}) = \int_V \rho(\underline{r}')\,G_D(\underline{r},\underline{r}')\,dV' - \frac{1}{4\pi}\int_{S(V)} \varphi(\underline{r}')\,\frac{\partial G_D}{\partial n}(\underline{r},\underline{r}')\,dS'}$$

φ_2 representa el potencial del sistema de cargas fuera de V

<u>Nota</u>: tenemos sol. integral

$$\varphi(\underline{r}) = \int_V \frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|}\,dV' + \frac{1}{4\pi}\int_{S(V)}\left[\frac{1}{|\underline{r}-\underline{r}'|}\frac{\partial\varphi}{\partial n'} - \varphi(\underline{r}')\frac{\partial}{\partial n'}\left(\frac{1}{|\underline{r}-\underline{r}'|}\right)\right]dS'$$

La elección $\quad \varphi = \varphi_1 + \varphi_2$

con $\quad \varphi_1 = G_D$

y $\quad \nabla^2\varphi_2 = 0 \quad$ equivale a definir

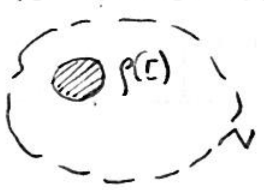
$$\nabla^2 G(\underline{r},\underline{r}') = -4\pi\,\delta(\underline{r}-\underline{r}')$$

con $\quad G = \varphi_1 + \varphi_2 \quad$: func. de Green con cond. de cont. arbitraria

y usar φ_2 para eliminar uno de los dos términos en S'.

<u>Función de Green con cdc. de Neumann</u>

Ahora tenemos $\quad \rho(\underline{r})$ en V



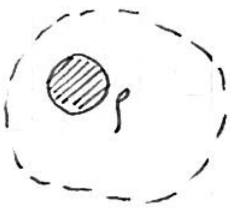
y cdc $\left.\frac{\partial\varphi}{\partial n}\right|_S$

Usemos el teo. de Green con $\begin{cases} \phi = \varphi(\underline{r}) \\ \chi = G_N(\underline{r}',\underline{r}) \;/\; \nabla^2 G_N = -4\pi\delta(\underline{r}-\underline{r}') \end{cases}$

$$\overbrace{\int_V \varphi(\underline{r}')\nabla^2 G_N(\underline{r}',\underline{r})\,dV'}^{-4\pi\varphi(\underline{r})} + 4\pi\int_V G_N(\underline{r}',\underline{r})\,\rho(\underline{r}')\,dV' =$$

$$= \underbrace{\int_{S(V)} \varphi(\underline{r}')\frac{\partial G_N}{\partial n'}(\underline{r}',\underline{r})\,dS'}_{\substack{\text{Queremos definir } G_N \\ \text{para eliminar este término}}} - \int_{S(V)} G_N(\underline{r}',\underline{r})\frac{\partial \varphi}{\partial n'}\,dS'$$

En este caso, no podemos elegir $\left.\frac{\partial G_N}{\partial n'}\right|_{S'} = 0$ pues no tenemos tanta libertad de elección en $\left.\frac{\partial \varphi}{\partial n'}\right|_{S'}$. Por Gauss



$$\oint_S \underline{E}\cdot\hat{n}\,dS = 4\pi\int \rho\,dV$$

$$\Rightarrow \int_S -\frac{\partial\varphi}{\partial n}\,dS = 4\pi\int\rho\,dV$$

y $\frac{\partial\varphi}{\partial n}$ y ρ deben ser compatibles.

En otras palabras, si tengo una carga puntual $q=1$ en V, debe valer que

$$\int_S -\frac{\partial G_N}{\partial n}\,dS = 4\pi$$

y lo mas simple que podemos pedir es $\boxed{\left.\frac{\partial G_N}{\partial n}\right|_S = -\frac{4\pi}{A}}$

Luego

$$\varphi(\underline{r}) = \int_V G_N(\underline{r}',\underline{r})\,\rho(\underline{r}')\,dV' - \underbrace{\frac{1}{4\pi}\int_{S(V)}\varphi(\underline{r}')\frac{4\pi}{A}\,dS'}_{\substack{\text{No depende de } \underline{r}' \\ \Rightarrow \text{es una cte} = \langle\varphi\rangle|_S}} + \frac{1}{4\pi}\int_{S(V)} G_N(\underline{r}',\underline{r})\frac{\partial\varphi}{\partial n'}\,dS'$$

pero como φ está definido salvo una cte, puedo tomarla nula y

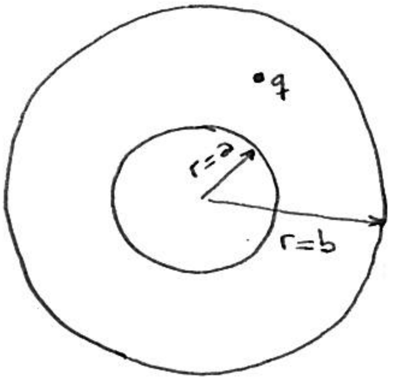
$$\boxed{\varphi(\underline{r}) = \int_V G_N(\underline{r}',\underline{r})\,\rho(\underline{r}')\,dV' + \frac{1}{4\pi}\int_{S(V)} G_N(\underline{r},\underline{r}')\,\frac{\partial\varphi}{\partial n'}\,dS'}$$

y conociendo $\rho(\underline{r}')$, $\left.\frac{\partial\varphi}{\partial n}\right|_S$, y G_N puedo resolver $\varphi(\underline{r})$.

<u>Expansión de la func. de Green en esféricas</u>

Cuando el problema es separable (esféricas, cilíndricas, o cartesianas) la func. de Green puede expandirse en términos de las func. del problema de Sturm-Liouville.

Consideremos el problema interior



$$\nabla^2 G_D(\underline{r},\underline{r}') = -4\pi\,\delta(\underline{r}-\underline{r}') = -\frac{4\pi}{r^2}\,\delta(r-r')\,\delta(\phi-\phi')\,\frac{\delta(\theta-\theta')}{\operatorname{sen}\theta}$$

Para $a \to 0$, $b \to \infty$ ya lo resolvimos:

$$G_D = \frac{1}{|\underline{r}-\underline{r}'|} = 4\pi \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{1}{2l+1}\frac{r_<^l}{r_>^{l+1}}\, Y^*_{lm}(\theta'\phi')\, Y_{lm}(\theta\phi)$$

Ahora queremos $\left.G_D\right|_{r=a} = \left.G_D\right|_{r=b} = 0$

Solo debería cambiar la parte radial. Planteando

$$\nabla^2 G_D = \nabla^2\left[4\pi \sum_{lm} g_{lm}(r,r')\, Y^*_{lm}(\theta'\phi')\, Y_{lm}(\theta\phi)\right] =$$

$$= -\frac{4\pi}{r^2}\,\delta(r-r')\,\delta(\phi-\phi')\,\frac{\delta(\theta-\theta')}{\operatorname{sen}\theta} = -\frac{4\pi}{r^2}\,\delta(r-r')\sum_{lm} Y^*_{lm}(\theta'\phi')\,Y_{lm}(\theta\phi)$$

completitud de los Y_{lm}

y usando que

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 s\theta}\frac{\partial}{\partial\theta}\left(s\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 s^2\theta}\frac{\partial^2}{\partial\phi^2} =$$

$$= \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\quad\right) + \frac{1}{r^2}\nabla^2_{\theta\phi} \qquad \text{con } \nabla^2_{\theta\phi} Y_{lm} = -l(l+1)\,Y_{lm}$$

$$\Rightarrow \nabla^2 G_D = \sum_{\ell m} 4\pi \left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(r C_{\ell m}) - \frac{\ell(\ell+1)}{r^2} C_{\ell m}\right] Y^*_{\ell m}(\theta'\phi') Y_{\ell m}(\theta\phi) =$$

$$= -\frac{4\pi}{r^2}\,\delta(r-r') \sum_{\ell m} Y^*_{\ell m}(\theta'\phi')\, Y_{\ell m}(\theta\phi)$$

luego
$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r C_{\ell m}) - \frac{\ell(\ell+1)}{r^2} C_{\ell m} = -\frac{4\pi}{r^2}\delta(r-r')$$

y notar que $C_{\ell m} = C_\ell$ (solo depende de ℓ). Además

$$C_\ell(r) = \begin{cases} A_\ell r^\ell + B_\ell r^{-(\ell+1)} & r<r' \\ A'_\ell r^\ell + B'_\ell r^{-(\ell+1)} & r>r' \end{cases}$$

partiendo el problema en regiones en las que la ec. es homogénea. Pidiendo

$$C_\ell(a) = 0 = A_\ell a^\ell + B_\ell a^{-(\ell+1)}$$
$$C_\ell(b) = 0 = A'_\ell b^\ell + B'_\ell b^{-(\ell+1)}$$

$$\Rightarrow C_\ell(r) = \begin{cases} A_\ell\left(r^\ell - \frac{a^{2\ell+1}}{r^{\ell+1}}\right) & r<r' \\ B'_\ell\left(\frac{1}{r^{\ell+1}} - \frac{r^\ell}{b^{2\ell+1}}\right) & r>r' \end{cases}$$