Podemos escribir, para tener continuidad en G_D

$$C_\ell(r) = D_\ell \left(r_<^\ell - \frac{a^{2\ell+1}}{r_<^{\ell+1}} \right) \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}} \right)$$

que hace al problema simétrico en r, r'. Falta determinar D: pedimos

$$- \left.\frac{\partial G_D}{\partial r}\right|_{r'+} + \left.\frac{\partial G_D}{\partial r}\right|_{r'-} = 4\pi\sigma$$

$$\cancel{4\pi} \sum_{\ell m} \left[- \left.\frac{dC_\ell}{dr}\right|_{r'+} + \left.\frac{dC_\ell}{dr}\right|_{r'-} \right] Y^*_{\ell m}(\theta'\phi') Y_{\ell m}(\theta\phi) = \frac{\cancel{4\pi}}{r'^2} \sum_{\ell m} Y^*_{\ell m}(\theta'\phi') Y_{\ell m}(\theta\phi)$$

$$\Longrightarrow \quad - \left.\frac{dC_\ell}{dr}\right|_{r'+} + \left.\frac{dC_\ell}{dr}\right|_{r'-} = \frac{1}{r'^2}$$

y reemplazando

$$\frac{1}{r'^2}=D_\ell\left[-\left(r'^{\ell}-\frac{a^{2\ell+1}}{r'^{\ell+1}}\right)\left(-\frac{\ell+1}{r'^{\ell+2}}-\frac{\ell r'^{\ell-1}}{b^{2\ell+1}}\right)+\left(\ell r'^{\ell-1}+\frac{(\ell+1)a^{2\ell+1}}{r'^{\ell+2}}\right)\left(\frac{1}{r'^{\ell+1}}-\frac{r'^{\ell}}{b^{2\ell+1}}\right)\right]$$

$$\Longrightarrow\quad D_\ell\left[\frac{\ell+1}{\cancel{r'^2}}-\left(\frac{a}{b}\right)^{2\ell+1}\frac{\ell}{\cancel{r'^2}}+\frac{\ell}{\cancel{r'^2}}-\left(\frac{a}{b}\right)^{2\ell+1}\frac{\ell+1}{\cancel{r'^2}}\right]=\frac{1}{\cancel{r'^2}}$$

$$\text{y}\quad D_\ell=\frac{1}{(2\ell+1)\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]}$$

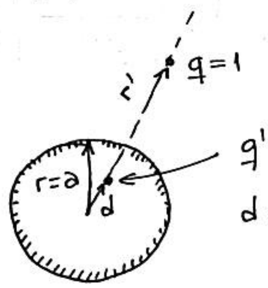
y

$$\boxed{G_D(\underline{r},\underline{r}')=4\pi\sum_{\ell,m}\frac{1}{(2\ell+1)\left[1-\left(\frac{a}{b}\right)^{2\ell+1}\right]}\left(r_<^{\ell}-\frac{a^{2\ell+1}}{r_<^{\ell+1}}\right)\left(\frac{1}{r_>^{\ell+1}}-\frac{r_>^{\ell}}{b^{2\ell+1}}\right)Y^*_{\ell m}(\theta'\phi')Y_{\ell m}(\theta\phi)}$$

Veamos el límite $b\to\infty$

$$G_D(\underline{r},\underline{r}')=4\pi\sum_{\ell,m}\frac{1}{2\ell+1}\left[\frac{r_<^{\ell}}{r_>^{\ell+1}}-\frac{1}{a}\left(\frac{a^2}{rr'}\right)^{\ell+1}\right]Y^*_{\ell m}(\theta'\phi')Y_{\ell m}(\theta\phi)$$

es la imagen de una carga puntual en la esfera de radio a
(q=1)



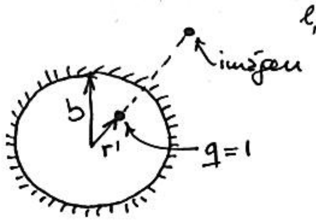
$q'=-\frac{a}{r'}q=-\frac{a}{r'}$

$d=\frac{a^2}{r'}$

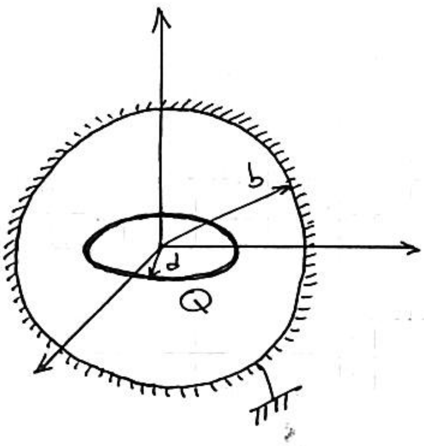
y en el límite $a\to 0$

$$G_D(\underline{r},\underline{r}')=4\pi\sum_{\ell m}\frac{1}{2\ell+1}\left[\frac{r_<^{\ell}}{r_>^{\ell+1}}-\frac{1}{b}\left(\frac{rr'}{b^2}\right)^{\ell}\right]Y^*_{\ell m}(\theta'\phi')Y_{\ell m}(\theta\phi)$$

es la imagen de la carga puntual (q=1) <u>interna</u> a la esfera de radio b



<u>Ejemplo</u>: usemos esto para resolver el problema de un anillo con carga q en el interior de una esfera



Tenemos $\varphi\big|_S = 0$

$$\Rightarrow \varphi(\underline{r}) = \int_V \rho(\underline{r}') \, G_D(\underline{r},\underline{r}') \, dV' +$$

$$+ \cancel{\frac{1}{4\pi}\oint_{S(V)} \frac{\partial G_D}{\partial n}(\underline{r},\underline{r}')} \underbrace{\varphi(\underline{r}')}_{=0} \, ds'$$

Escribamos $\rho(\underline{r}')$

$$\rho(\underline{r}') = C \, \delta(r'-d) \frac{\delta(\theta' - \pi/2)}{\operatorname{sen}\theta'} \quad \text{tq} \quad \int \rho(\underline{r}') \, dv' = Q$$

$$\Rightarrow \rho(\underline{r}') = \frac{Q}{2\pi d^2} \, \delta(r'-d) \frac{\delta(\theta'-\pi/2)}{\operatorname{sen}\theta'}$$

Luego
$$\varphi(\underline{r}) = \frac{Q}{2\pi d^2} \int \delta(r'-d) \frac{\delta(\theta'-\pi/2)}{\cancel{\operatorname{sen}\theta'}} 4\pi \sum_{\ell m} \frac{1}{2\ell+1} \left[\frac{r_<^\ell}{r_>^{\ell+1}} - \overbrace{\frac{1}{b}\left(\frac{r r'}{b^2}\right)^\ell}^{= \frac{r_<^\ell r_>^\ell}{b^{2\ell+1}}} \right] Y_{\ell m}^*(\theta'\phi') Y_{\ell m}(\theta\phi) \, r'^2 \cancel{\operatorname{sen}\theta'} \, dr' d\theta' d\phi'$$

Solo sobreviven los términos con $m = 0$

$$\Rightarrow \varphi(\underline{r}) = \frac{\cancel{2\pi} Q \cancel{d^2}}{\cancel{2\pi} \cancel{d^2}} \sum_\ell \frac{4\pi}{2\ell+1} Y_{\ell 0}^*\left(\frac{\pi}{2}, \phi'\right) Y_{\ell 0}(\theta\phi) \; r_<^\ell \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}} \right)$$

donde ahora
$$r_> = \max\{r, d\}$$
$$r_< = \min\{r, d\}$$

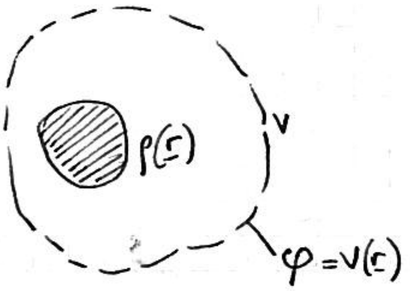
Usando
$$Y_{\ell 0}(\theta\phi) = \sqrt{\frac{2\ell+1}{4\pi}} \, P_\ell(\cos\theta)$$

$$\Rightarrow \varphi(\underline{r}) = Q \sum_\ell P_\ell(0) P_\ell(\cos\theta) \; r_<^\ell \left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^\ell}{b^{2\ell+1}} \right)$$

y usando que
$$P_{2n}(0) = \frac{(-1)^n (2n-1)!!}{2^n n!}$$
$$\text{y} \quad P_{2n+1}(0) = 0$$

$$\Rightarrow \varphi(\underline{r}) = Q \sum_{n=0}^{\infty} \frac{(-1)^n (2n-1)!!}{2^n n!} P_{2n}(\cos\theta) \; r_<^{2n} \left(\frac{1}{r_>^{2n+1}} - \frac{r_>^{2n}}{b^{4n+1}} \right)$$

Método de imágenes



Tenemos $\rho(\underline{r})$ en V y conocemos el valor de φ en la sup. que encierra a V. Escribimos

$$\varphi = \int \frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|} dV' + \varphi'$$

tq
$$\begin{cases} \nabla^2 \varphi' \Big|_{\underline{r}\in V} = 0 \\ \varphi\Big|_S = V(\underline{r}) = \int \frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|} dV'\Big|_S + \varphi'\Big|_S \end{cases}$$

$\Rightarrow$ φ' en V está determinado unívocamente pidiendo

$$\begin{cases} \nabla^2 \varphi' = 0 \\ \varphi'\Big|_S = V(\underline{r}) - \int \frac{\rho(\underline{r}')dV'}{|\underline{r}-\underline{r}'|}\Big|_S . \end{cases}$$

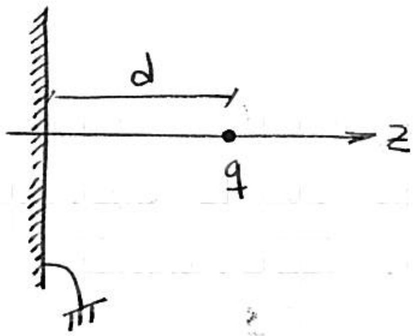
Un candidato es una función de la forma

$$\varphi'(\underline{r}) = \int \frac{\rho(\underline{r}')}{|\underline{r}-\underline{r}'|} dV'$$

con $\rho(\underline{r}') \neq 0$ afuera de V, y tq satisfaga la cdc. para $\varphi'\Big|_S$.

Notar que lo pedido no define unívocamente $\rho(\underline{r})$ afuera de V. Sin embargo, por los teo. de existencia y unicidad de sol., una vez encontrada φ tq. satisface las cdc., la sol. es única (en V).

Carga puntual frente a un plano infinito conductor



Queremos φ en $z>0$ sabiendo que

$$\begin{cases} \varphi(z=0)=0 \\ \varphi \xrightarrow[r\to\infty]{} 0 \end{cases}$$

Nos olvidamos del conductor y tomamos

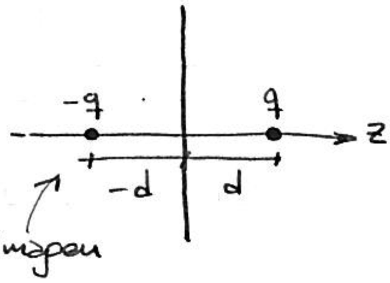
$$\varphi(\underline{r}) = \frac{q}{|\underline{r}-d\hat{z}|} + \underbrace{\frac{q_{im}}{|\underline{r}-\underline{r}'_{im}|}}_{\substack{\text{candidato} \\ \text{para } \varphi'}} \qquad \text{para } z>0$$

con $\underline{r}'_{im}$ en $z<0$. Debe satisfacer

$$\varphi\Big|_S = 0 = \frac{q}{\sqrt{x^2+y^2+(0-d)^2}} + \frac{q_{im}}{\sqrt{(x-x_{im})^2+(y-y_{im})^2+(0-z_{im})^2}}$$

$\Rightarrow$ tomamos

$$q_{im} = -q$$
$$x_{im} = y_{im} = 0$$
$$z_{im} = -d$$



La solución en todo el espacio es

$$\varphi(r) = \begin{cases} \dfrac{q}{|\underline{r}-d\hat{z}|} - \dfrac{q}{|\underline{r}+d\hat{z}|} & z>0 \\ 0 & z<0 \end{cases}$$

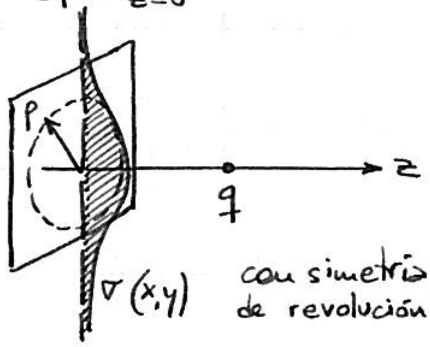
y conociendo φ en todo el espacio podemos hallar la verdadera distribución de cargas en la superficie:

$$4\pi\sigma = \hat{n}\cdot\left(E_2 - \cancel{E_1}\right)\Big|_{z=0} = -\frac{\partial\varphi}{\partial z}\Big|_{z=0}$$

$$\Rightarrow \boxed{\sigma = -\frac{1}{4\pi}\frac{\partial\varphi}{\partial z}\Big|_{z=0}}$$

luego $\sigma(x,y) = -\frac{1}{4\pi}\left(-\frac{q(z-d)}{|\underline{r}-d\hat{z}|^3} + \frac{q(z+d)}{|\underline{r}+d\hat{z}|^3}\right)\Big|_{z=0}$

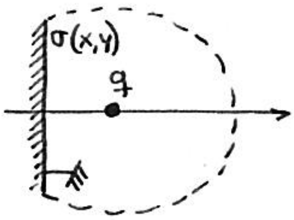
$$\boxed{\sigma(x,y) = -\frac{1}{4\pi}\frac{2qd}{(x^2+y^2+d^2)^{3/2}}}$$



Se puede ver que la carga inducida satisface

$$Q_{ind} = \int \sigma(x,y)\,dx\,dy = -q$$

O bien, tomando Gauss



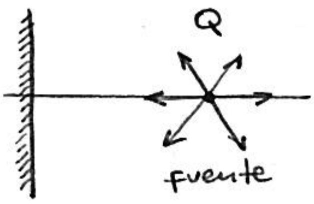
$$4\pi Q_{ENC} = \oint \underline{E}\cdot d\underline{S} = \int_{\text{conductor}} \underline{E}\cdot d\underline{S} + \int_{\substack{\text{sup}\\ r\to\infty}} \underline{E}\cdot d\underline{S} = 0$$

$E \to 1/r^3$

$dS \to r^2$

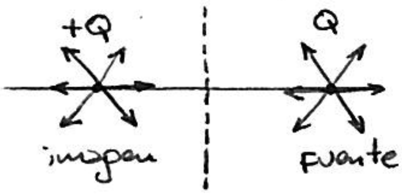
$\Rightarrow$ tiende a cero

Notar que la imagen depende de las cdc. Para un fluido en 2D



$$\begin{cases} \underline{\nabla}\cdot\underline{u} = 0 & \text{(excepto en la pos. de la fuente)} \\ \underline{\nabla}\times\underline{u} = 0 & \end{cases}$$

$\boxed{\nabla^2\varphi = 0}$



$\Rightarrow \underline{u} = \underline{\nabla}\varphi$ y la cdc es

$$\underline{u}\cdot\hat{n}\Big|_S = \frac{\partial\varphi}{\partial z}\Big|_S = 0$$