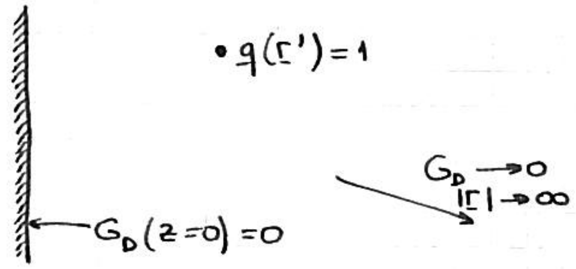
## Función de Green $G_D$ en un semiespacio




Buscamos $G_D(\underline{r},\underline{r}')$ tq

$$\begin{cases} \nabla^2 G_D = -4\pi\,\delta(\underline{r}-\underline{r}') \\ G_D(\underline{r},\underline{r}')\big|_S = 0 \end{cases}$$

Es el problema anterior pero con $q=1$ y la carga en un punto $\underline{r}'$ cualquiera

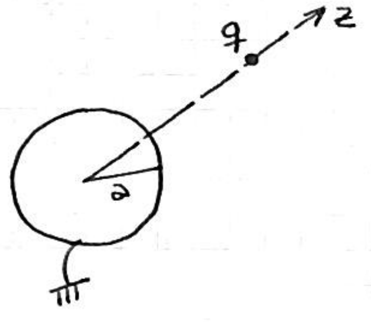
$$\Rightarrow G_D(\underline{r},\underline{r}') = \frac{1}{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}} - \frac{1}{\left[(x-x')^2+(y-y')+(z+z')^2\right]^{1/2}}$$

y dado cualquier problema electrostático con

$$\begin{cases} \rho(\underline{r}) \quad \text{en } z>0 \\ \varphi(z=0) = V(x,y) \end{cases} \qquad y \quad \varphi \xrightarrow[r\to\infty]{} 0$$

$$\Rightarrow \varphi(\underline{r}) = \int_V \rho(\underline{r}')\,G_D(\underline{r},\underline{r}')\,dV' - \frac{1}{4\pi}\int_S \varphi(\underline{r}')\,\frac{\partial G_D}{\partial n'}\,dS'$$

## Carga puntual frente a una esfera conductora




Veamos $\varphi(\underline{r})$ para $|\underline{r}|>a$.
Tomamos

$$\varphi(\underline{r}) = \frac{q}{|\underline{r}-d\hat{z}|} + \frac{q_{im}}{|\underline{r}-d_{im}\hat{z}|} \qquad (d_{im}<a)$$

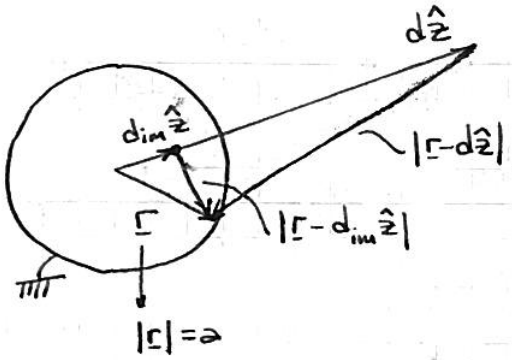
dentro de la esfera

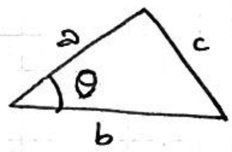
pues el sistema tiene simetría de revolución alrededor de z. De la cdc.

$$\varphi(|\underline{r}|=a) = 0$$

y usando el teo. del coseno

$$\varphi(r=a) = \frac{q}{(a^2+d^2-2ad\cos\theta)^{1/2}} + \frac{q_{im}}{(a^2+d_{im}^2-2ad_{im}\cos\theta)^{1/2}} = 0 \quad \forall\theta$$




$$c^2 = a^2+b^2-2ab\cos\theta$$

Luego $-\operatorname{sg}(q_{im}) = \operatorname{sg}(q)$

y elevando al cuadrado

$$q^2(a^2+d_{im}^2-2ad_{im}\cos\theta) = q_{im}^2(a^2+d^2-2ad\cos\theta)$$

y pedimos
$$\begin{cases} q^2(a^2+d_{im}^2) = q_{im}^2(a^2+d^2) & (1) \\ q^2\,\cancel{2a}d_{im}\cancel{\cos\theta} = \cancel{2a}d\,q_{im}^2\cancel{\cos\theta} \end{cases}$$

$$\Rightarrow \frac{q_{im}^2}{q^2} = \frac{d_{im}}{d}$$

y reemplazando en (1)

$$a^2+d_{im}^2 = \frac{d_{im}}{d}(a^2+d^2) \Rightarrow a^2d + dd_{im}^2 - a^2d_{im} - d^2d_{im} = 0$$

$$\Rightarrow d_{im} = \frac{(a^2+d^2) \pm \sqrt{(a^2+d^2)^2-4a^2d^2}}{2d} \longrightarrow \cancel{d_{im}=d} \quad (>a)$$

$$\longrightarrow \boxed{d_{im} = \frac{a^2}{d}}$$

y luego $\boxed{q_{im} = -\frac{qa}{d}}$

$$\Rightarrow \boxed{\varphi(\underline{r}) = \frac{q}{|\underline{r}-d\hat{z}|} - \frac{qa/d}{|\underline{r}-\frac{a^2}{d}\hat{z}|}} \qquad (r>a)$$

La carga inducida se obtiene de $\sigma = -\frac{1}{4\pi}\frac{\partial\varphi}{\partial r}\Big|_{r=a}$

Escribiendo

$$\varphi = \frac{q}{(r^2+d^2-2rd\,c\theta)^{1/2}} - \frac{qa/d}{\left[r^2+\left(\frac{a^2}{d}\right)^2+\frac{2ra^2}{d}c\theta\right]^{1/2}}$$

$$\Rightarrow \left.\frac{\partial\varphi}{\partial r}\right|_{r=a} = -\frac{1}{2}\frac{2q(a-dc\theta)}{(a^2+d^2-2adc\theta)^{3/2}} + \frac{1}{2}\frac{2aq/d\left(a-\frac{a^2}{d}c\theta\right)}{\left[a^2+\left(\frac{a^2}{d}\right)^2+2\frac{a^3}{d}c\theta\right]^{3/2}}$$

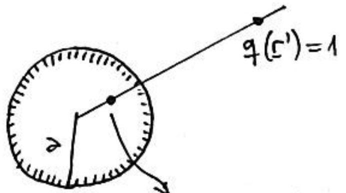
y como $\varphi(r=a)=0 \Rightarrow \frac{\cancel{q}}{(a^2+d^2-2adc\theta)^{1/2}} = \frac{\cancel{q}a/d}{\left[a^2+\left(\frac{a^2}{d}\right)^2+2\frac{a^3}{d}c\theta\right]^{1/2}}$

$$\Rightarrow \left.\frac{\partial\varphi}{\partial r}\right|_{r=a} = \frac{-q(a-dc\theta)+q\frac{d^2}{a^2}\left(a-\frac{a^2}{d}c\theta\right)}{(a^2+d^2-2ad\,c\theta)^{3/2}} = \frac{q}{a^2}\frac{(d^2/a^2-1)}{(1+d^2/a^2-2d/a\,c\theta)^{3/2}}$$

$$\Rightarrow \boxed{\sigma(\theta,\phi) = -\frac{q}{4\pi a^2}\left(\frac{a}{d}\right)\frac{(1-a^2/d^2)}{(1+a^2/d^2-2a/d\,c\theta)^{3/2}}}$$

Es max. en $\theta=0$ y minima en $\theta=\pi$

## Función de Green $G_D$ en la espéra




$q_{im} = -\frac{qa}{r'} = -\frac{a}{r'}$

Veamos la función de Green exterior con c.dc. de Dirichlet. Del problema anterior:

$$G_D(\underline{r},\underline{r}') = \frac{1}{|\underline{r}-\underline{r}'|} - \frac{a/r'}{\left|\underline{r}-\frac{a^2}{r'}\hat{r}'\right|}$$

y por teo. del coseno

$$\boxed{G_D(\underline{r},\underline{r}') = \frac{1}{(r^2+r'^2-2rr'c\gamma)^{1/2}} - \frac{a}{r'\left[r^2+\left(\frac{a^2}{r'}\right)^2-\frac{2ra^2}{r'}c\gamma\right]^{1/2}}}$$

con $\cos\gamma = s\theta s\theta' c(\phi-\phi') + c\theta c\theta'$

Sabiamos que $\frac{1}{|\underline{r}-\underline{r}'|} = \sum_{l,m}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\theta'\phi') Y_{lm}(\theta\phi)$

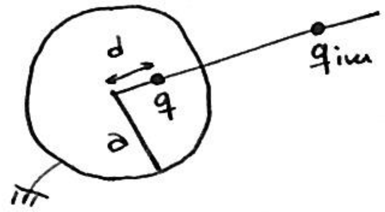
y $$\frac{a/r'}{\left|\mathbf{r}-\frac{a^2}{r'}\hat{\mathbf{r}}'\right|}=\sum_{\ell m}\frac{4\pi}{2\ell+1}\frac{a}{r'}\frac{r_<^{\ell}}{r_>^{\ell+1}}Y_{\ell m}^{*}(\theta'\phi')\,Y_{\ell m}(\theta\phi)$$

↳ ahora, afuera $\begin{cases} r_> = r \\ r_< = a^2/r' \end{cases}$

y luego reobtenemos el desarrollo

$$G_D(\mathbf{r},\mathbf{r}')=\sum_{\ell m}\frac{4\pi}{2\ell+1}\left(\frac{r_<^{\ell}}{r_>^{\ell+1}}-\frac{a^{2\ell+1}}{r'^{\ell+1}r^{\ell+1}}\right)Y_{\ell m}^{*}(\theta'\phi')\,Y_{\ell m}(\theta\phi)$$

## Carga puntual en el interior de una esfera conductora

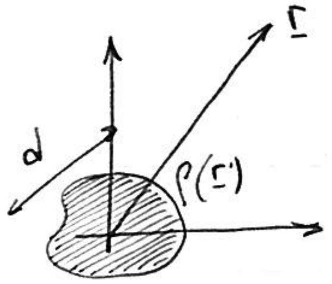



El problema es equivalente

$$\varphi(\mathbf{r})=\frac{q}{|\mathbf{r}-d\hat{\mathbf{z}}|}+\frac{q_{im}}{|\mathbf{r}-d_{im}\hat{\mathbf{z}}|}$$

con $\begin{cases} q_{im} = -q\dfrac{a}{d} \\ d_{im}=\dfrac{a^2}{d}\end{cases}$

## Desarrollo multipolar en esféricas




Tenemos

$$\varphi(\mathbf{r})=\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dV'$$

y queremos desarrollarlo para $|\mathbf{r}|\gg d$.

En cartesianas teníamos

$$\varphi(\mathbf{r})=\frac{Q}{r}+\frac{\mathbf{P}\cdot\mathbf{r}}{r^3}+\frac{1}{2}Q_{ij}\frac{r_ir_j}{r^5}+\dots$$

Calculemos el desarrollo en esféricas. Tenemos

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\sum_{\ell m}\frac{4\pi}{2\ell+1}\frac{r_<^{\ell}}{r_>^{\ell+1}}Y_{\ell m}^{*}(\theta'\phi')\,Y_{\ell m}(\theta\phi)$$

pero nos interesa $|\mathbf{r}|\gg d \Rightarrow r_<=r'\,,\ r_>=r$

Reemplazando

$$\varphi(\underline{r}) = \sum_{lm} \frac{4\pi}{2l+1} \frac{Y_{lm}(\theta\phi)}{r^{l+1}} \underbrace{\int Y^*_{lm}(\theta'\phi')\, r'^{l} \rho(\underline{r}')\, dV'}_{q_{lm} \quad \text{coef. del desarrollo}}$$

$$\Rightarrow \boxed{\varphi(\underline{r}) = \sum_{lm} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta\phi)}{r^{l+1}}}$$

con $q_{lm} = \int Y^*_{lm}(\theta'\phi')\, r'^{l} \rho(\underline{r}')\, dV'$ momentos multipolares

Veamos algunos ejemplos:

$$q_{00} = \frac{1}{\sqrt{4\pi}} \int \rho(\underline{r}')\, dV' = \frac{Q}{\sqrt{4\pi}} \Rightarrow \varphi^{(0,0)} = \frac{4\pi}{\sqrt{4\pi}} \frac{Q}{r\sqrt{4\pi}} = \frac{Q}{r}$$

$$q_{10} = \sqrt{\frac{3}{4\pi}} \int \rho(\underline{r}')\, \underbrace{r' c\theta}_{z}\, dV' = \sqrt{\frac{3}{4\pi}}\, P_z$$

$$q_{1,\pm1} = \mp\sqrt{\frac{3}{8\pi}} \int \rho(\underline{r}')\, r s\theta\, \underbrace{e^{\pm i\phi}}_{c\phi \pm i s\phi}\, dV' = \mp\sqrt{\frac{3}{8\pi}} (P_x \pm i P_y)$$

El campo eléctrico correspondiente a un momento multipolar puede escribirse usando

$$\underline{E}^{(l,m)} = -\underline{\nabla}\varphi^{(l,m)} = -\left( \frac{\partial\varphi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{\theta} + \frac{1}{r s\theta}\frac{\partial\varphi}{\partial\phi}\hat{\phi} \right) =$$

$$= \frac{4\pi}{2l+1} q_{lm} \left[ (l+1) \frac{Y_{lm}(\theta\phi)}{r^{l+2}}\hat{r} - \frac{1}{r^{l+2}} \frac{\partial}{\partial\theta} Y_{lm}(\theta\phi)\,\hat{\theta} - \frac{im}{r^{l+2} s\theta} Y_{lm}(\theta\phi)\,\hat{\phi} \right]$$

Por ejemplo, para un dipolo puntual en $\hat{z}$

$$E^{(1,0)} = \frac{4\pi}{3} q_{10} \left[ 2\sqrt{\frac{3}{4\pi}} \frac{c\theta\,\hat{r}}{r^3} + \frac{1}{r^3}\sqrt{\frac{3}{4\pi}}\, s\theta\,\hat{\theta} \right] =$$

$$= \frac{2pc\theta}{r^3}\hat{r} + \frac{ps\theta}{r^3}\hat{\theta}$$