Funciones de Green avanzadas y retardadas

Tenemos $\quad \nabla^2\chi - \frac{1}{c^2}\frac{\partial^2\chi}{\partial t^2} = -4\pi f(\underline{r},t)$

Transformando Fourier

$$\begin{cases} \chi(\underline{r},\omega) = \int \chi(\underline{r},t)\, e^{i\omega t}\, dt & \chi(\underline{r},t) = \frac{1}{2\pi}\int \chi(\underline{r},\omega)\, e^{-i\omega t}\, d\omega \\ f(\underline{r},\omega) = \int f(\underline{r},t)\, e^{i\omega t}\, dt & f(\underline{r},t) = \frac{1}{2\pi}\int f(\underline{r},\omega)\, e^{-i\omega t}\, d\omega \end{cases}$$

$$\Rightarrow \nabla^2\chi(\underline{r},\omega) + \frac{\omega^2}{c^2}\chi(\underline{r},\omega) = -4\pi f(\underline{r},\omega)$$

y las frec. quedan desacopladas. Introduzcamos ahora func. de Green

$$\begin{cases} (\nabla^2 + k^2)\, G_k(\underline{r},\underline{r}') = -4\pi\,\delta(\underline{r}-\underline{r}') \\ G_k \xrightarrow[|\underline{r}|\to\infty]{} 0 \end{cases} \qquad k^2 = \omega^2/c^2$$

Busquemos una sol. particular. Tomando $G_k(\underline{r},\underline{r}') = G_k(\underline{R})$ con $\underline{R} = \underline{r}-\underline{r}'$

$$\nabla^2 G_k(\underline{R}) + k^2 G_k(\underline{R}) = -4\pi\,\delta(\underline{R})$$

sin contornos es esféricamente simétrica alrededor de $\underline{r}'$

$G_k(\underline{R}) = G_k(R)$.

y $\nabla^2 = \frac{1}{R}\frac{d^2}{dR^2}(R\ \ )$

y tiene soluciones

$$\boxed{G_k(R) = \underbrace{A\,\frac{e^{ikR}}{R}}_{G_k^+(R)} + \underbrace{B\,\frac{e^{-ikR}}{R}}_{G_k^-(R)}}$$

representan ondas esféricas entrantes y salientes.

Ahora consideremos

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\underline{r},t;\underline{r}',t') = -4\pi\,\overbrace{\delta(\underline{r}-\underline{r}')\,\delta(t-t')}^{f(\underline{r},t)}$$

Transformando Fourier

$$(\nabla^2 + k^2)\, G_k(\underline{r},\underline{r}') = -4\pi\,\delta(\underline{R})\, e^{i\omega t'}$$

$$\Rightarrow \quad G_k^{\pm}(\underline{r},\underline{r}') = \frac{e^{\pm ikR}}{R}\, e^{i\omega t'}$$

Antitransformando

$$G^{\pm}(\underline{r},t;\underline{r}',t') = \frac{1}{2\pi}\int d\omega\, \frac{e^{\pm ikR}}{R}\, e^{i\omega t'} e^{-i\omega t} =$$

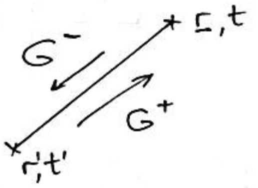
$$= \frac{1}{2\pi R}\int d\omega\, e^{i\omega\left(\pm\frac{R}{c}+t'-t\right)}$$

$$\Rightarrow \quad G^{\pm}(\underline{r},t;\underline{r}',t') = \frac{1}{R}\,\delta\left(\pm\frac{R}{c}+t'-t\right)$$

$$\boxed{G^{\pm}(\underline{r},t;\underline{r}',t') = \frac{\delta\left[t'-\left(t\mp\frac{|\underline{r}-\underline{r}'|}{c}\right)\right]}{|\underline{r}-\underline{r}'|}}$$

Func. de Green avanzadas (−) y retardadas (+).

G^+ es la sol. de la ec. de ondas inhomogénea en $(\underline{r},t)$ si $\exists$ una inhomogeneidad (fuente) en $(\underline{r}',t')$. Es no nula solo en $t = t' + \frac{|\underline{r}-\underline{r}'|}{c}$ $\Rightarrow$ describe la propagación de lo ocurrido en $(\underline{r}',t')$ hacia adelante en t con vel. c.



G^- es no nula en $t = t' - \frac{|\underline{r}-\underline{r}'|}{c}$.

Dada una fuente $f(\underline{r},t)$ tenemos sol part. de la ec.

$$\psi_{part}(\underline{r},t) = \int d^3r'dt'\; G^{\pm}(\underline{r},t;\underline{r}',t')\, f(\underline{r}',t')$$

La sol. gral es: $\quad \psi(\underline{r},t) = \psi_{part}(\underline{r},t) + \psi_{homog}(\underline{r},t)$

con sol. homogenea tq. satisfaga las cdc.

Notar que la elección conveniente de $G^{\pm}$ facilita satisfacer las cdc. Por ejemplo

$$\frac{e^{ikR}}{R} - \frac{e^{-ikR}}{R} = \frac{2i\,\mathrm{sen}\,kR}{R}$$

Ejemplo: Encendido de un dipolo en el origen

$\uparrow p\hat{z}$

$$\rho(\underline{r}',t') = -p\,\delta(x')\,\delta(y')\,\delta'(z')\,\Theta(t')$$

$$\text{y} \quad j_z(\underline{r}',t') = p\,\delta(x')\,\delta(y')\,\delta(z')\,\delta(t')$$

$$\Rightarrow \quad \varphi = \int G^+(\underline{r},t;\underline{r}',t')\,\rho(\underline{r},t)\,d^3r'dt'$$

Pues si una fuente se enciende en $t=t_0$, $\chi(\underline{r},t)=0$ para $t<t_0$. La sol. avanzada no satisface las cdc.

$$\varphi(\underline{r},t) = \int d^3r'dt' \frac{\rho(\underline{r}'t')\,\delta\left(t-t'-\frac{|\underline{r}-\underline{r}'|}{c}\right)}{|\underline{r}-\underline{r}'|} =$$

$$= \int d^3r' \frac{\rho\left(\underline{r}', t-\frac{|\underline{r}-\underline{r}'|}{c}\right)}{|\underline{r}-\underline{r}'|} =$$

$$= p\int d^3r' \frac{\delta(x')\,\delta(y')\,\delta'(z')\,\Theta\left(t-\frac{|\underline{r}-\underline{r}'|}{c}\right)}{|\underline{r}-\underline{r}'|} =$$

$$= p\frac{\partial}{\partial z'}\left[\frac{\Theta\left(t-\frac{|\underline{r}-\underline{r}'|}{c}\right)}{|\underline{r}-\underline{r}'|}\right]_{\underline{r}'=0} = p\,\Theta\left(t-\frac{r}{c}\right)\frac{z-z'}{|\underline{r}-\underline{r}'|^3}\Bigg|_{\underline{r}'=0} +$$

$$+ \frac{p}{r}\,\delta\left(t-\frac{r}{c}\right)\left(-\frac{1}{c}\right)\frac{\partial}{\partial z'}\left(|\underline{r}-\underline{r}'|\right)\Big|_{\underline{r}'=0}$$

$$\Rightarrow \boxed{\varphi(\underline{r},t) = \frac{pz}{r^3}\,\Theta\left(t-\frac{r}{c}\right) + \frac{pz}{cr^2}\,\delta\left(t-\frac{r}{c}\right)}$$

Idem $$\underline{A}(\underline{r},t) = \hat{z}\,\frac{1}{c}\int d^3r \frac{j_z\left(\underline{r},t-\frac{|\underline{r}-\underline{r}'|}{c}\right)}{|\underline{r}-\underline{r}'|}$$

y $$\boxed{A(\underline{r},t) = \frac{\hat{z}}{c}\,\frac{p}{r}\,\delta\left(t-\frac{r}{c}\right)}$$

Calculemos $\underline{E}$. Tenemos

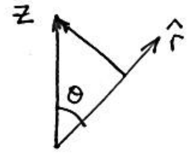
$$\underline{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial A}{\partial t}$$

Veamos término por término $\quad \underline{p} = p\hat{z}$

$$-\underline{\nabla}\left[\frac{pz}{r^3}\,\Theta\left(t-\frac{r}{c}\right)\right] = \left[\frac{3(\underline{p}\cdot\underline{r})\underline{r} - r^2\underline{p}}{r^5}\right]\Theta\left(t-\frac{r}{c}\right) + \frac{pz}{cr^3}\,\delta\left(t-\frac{r}{c}\right)\hat{r}$$

$$-\underline{\nabla}\left[\frac{pz}{cr^2}\,\delta\left(t-\frac{r}{c}\right)\right] = \frac{1}{c}\left[\frac{2(\underline{p}\cdot\underline{r})\underline{r} - r^2\underline{p}}{r^4}\right]\delta\left(t-\frac{r}{c}\right) + \underbrace{\frac{pz}{c^2r^2}\,\delta'\left(t-\frac{r}{c}\right)\hat{r}}_{1}$$

$$-\frac{1}{c}\frac{\partial \underline{A}}{\partial t} = -\frac{p}{c^2 r}\,\delta'\left(t-\frac{r}{c}\right)\hat{z} = ②$$



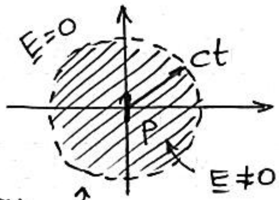
Nota que $\quad ① + ② = \dfrac{p}{rc^2}\,\delta'\left(t-\dfrac{r}{c}\right)\underbrace{\left(\cos\theta\,\hat{r}-\hat{z}\right)}_{\operatorname{sen}\theta\,\hat{\theta}}$

Sumando

$$\underline{E}(\underline{r},t) = \left[\frac{3(\underline{p}\cdot\underline{r})\underline{r}-r^2\underline{p}}{r^5}\right]\Theta\left(t-\frac{r}{c}\right) + \frac{pz}{cr^3}\,\delta\left(t-\frac{r}{c}\right)\hat{r} +$$

$$+\ \delta\left(t-\frac{r}{c}\right)\frac{1}{c}\left[\frac{2(\underline{p}\cdot\underline{r})\underline{r}-r^2\underline{p}}{r^4}\right] + \frac{p}{c^2 r}\,\delta'\left(t-\frac{r}{c}\right)\operatorname{sen}\theta\,\hat{\theta}$$

representa el retardo
en la llegada de la información
$\underline{E}=0$ si $t-\frac{r}{c}<0$



decae como $1/r$
y apunta en $\hat{\theta}$: campo
de radiación

Los demás términos representan
la perturbación en la frontera

Idem, para $\underline{B} = \underline{\nabla}\times\underline{A} = \left(\dfrac{\partial A_z}{\partial y}\hat{x},\ -\dfrac{\partial A_z}{\partial x}\hat{y},\ 0\right)$

$$B(\underline{r},t) = \frac{p}{cr^2}\,\delta\left(t-\frac{r}{c}\right)\hat{\varphi} + \frac{p}{c^2 r}\,\delta'\left(t-\frac{r}{c}\right)\hat{\varphi}$$

campo de
radiación