

CHAPTER II

APPLICATION OF QUANTUM MECHANICS TO LIQUID HELIUM

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1. Introduction

Liquid helium exhibits quantum mechanical properties on a large scale in a manner somewhat differently than do other substances. No other substance remains liquid to a temperature low enough to exhibit the effects. These effects have long been a puzzle. It is supposed that they can all be ultimately understood in terms of the properties of Schrödinger's equation. We cannot expect a rigorous exposition of how these properties arise. That could only come from complete solutions of the Schrödinger equation for the 10^{23} atoms in a sample of liquid. For helium, as for any other substance today we must be satisfied with some approximate understanding of how, in principle, that equation could lead to solutions which indicate behavior similar to that observed.

Since the discovery of liquid helium considerable progress has been made in understanding its behavior from first principles. Some of the properties are more easily understood than others. The most difficult of these concern the resistance to flow above critical velocity. If we permit some conjectures of Onsager¹, however, perhaps a start has been made in understanding even these. The aim of this article is to describe those physical ideas which have been suggested to explain the behavior of helium which can most easily be related to properties of the Schrödinger equation.

We shall omit references to the phenomena involved in the Rollin

film. It appears that the film can be understood as being maintained by van der Waals attraction to the wall. The flow properties of the film are interpreted as a special case of flow properties of helium in leaks in general.

The article falls naturally into two main sections. First, there are phenomena in which the superfluid velocity is irrotational. Here we can give a fairly complete picture. The second part concerns the case in which vorticity of the superfluid exists. Our position here is less satisfactory and more uncertain. It is described here in considerable detail because of the interesting problems it presents.

2. Summary of the Theoretical Viewpoint

The first striking way that helium differs from other substances is that it is liquid even down to absolute zero. Classically at absolute zero all motion stops, but quantum mechanically this is not so. In fact the most mobile substance known is one at absolute zero, where on the older concepts we should expect hard crystals. Helium stays liquid, as London² has shown, because the inter-atomic forces are very weak and the quantum zero point motion is large enough, since the atomic mass is small, to keep it fluid even at absolute zero. In the other inert gases the mass is so much higher that the zero-point motion is insufficient to oppose the crystalizing effect of the attractive forces. In hydrogen the intermolecular forces are very much stronger, so it, too, is solid. In liquid ^4He there is a further transition at 2.2°K , the λ -transition, between two liquid states of different properties. A transition is expected (at 3.2°K) for such atoms if the interatomic forces are neglected, as Einstein³ noticed. London⁴ has argued that the λ -transition corresponds to the transition which occurs even in the ideal Einstein-Bose gas. The inter-atomic forces alter the temperature and, in a way as yet only imperfectly understood^{5,6} the order of the transition, but qualitatively the reason for the transition is understood. We will concern ourselves here, only with the liquid He II, below the λ -point, and shall try to elucidate the qualitative reasons for some of its strange behavior. Also we explicitly limit our considerations to a liquid made purely of ^4He atoms so that the wave function must be symmetric for interchange of the atoms. We do not mean to imply anything about liquid helium ^3He , nor about superconductors, either by analogy nor by contrast. That is, we shall use the fact that the wave function is symmetric in many arguments without

stopping to inquire whether the symmetry is necessary part of the argument.

The central feature which dominates the properties of helium II is the scarcity of available low energy excited states in the Bose liquid ^{7, 8}. There do exist excited states of compression (i.e.: phonons) but states involving stirring or other internal motions which do not change the density cannot be excited without expenditure of an appreciable excitation energy. This is because, for quantum energies to be low, long wave lengths or long distances are necessary. But the wave function cannot depend on large scale modifications of the liquid's configuration. For a large scale motion, or stirring, which does not alter the density, only moves some atoms away to replace them by others *. It is essentially equivalent to a permutation of one atom for another, and the wave function must remain unchanged by a permutation of atoms, because ⁴He obeys the symmetrical statistics. The only wave functions available are those which change when atoms move in a way which is not reproducible by permutation, and therefore either, (1) movements accompanied by change in density (phonons), (2) movements over distances less than an atomic spacing, therefore of short wave length and high energy (rotons and more complex states), or (3) movements resulting in a change in the position of the containing walls (flow). We shall discuss these states in detail presently.

The scarcity of low energy excited states is the seat of many of the phenomena in the liquid. This has been known since the work of Landau who developed a theory of the liquid on the assumption of such scarcity. The specific heat is very low at low temperature and only rises rapidly above about 1°K when enough thermal energy is available to excite an appreciable number of the higher energy states (rotons). There are so few states excited that the excitations may be localized in the fluid like wave packets. These move about, collide with each other and the walls, and imitate the appearance that in the perfect background fluid there is another fluid or gas. This "gas" of excitations carries all the entropy of the liquid, may carry waves of number den-

* In the ideal gas the low excitations are those in which one or two atoms are excited to low states. These involve density changes and are more analagous to phonon states (but are even lower in energy than in the liquid because the ideal gas has infinite compressibility, and therefore vanishing sound velocity). The interatomic forces in the liquid make it more imperative that if atoms are moved away from one point others move in to take their place, if high repulsive energies between nearby atoms are to be avoided.

sity (second sound, analogous to sound in an ordinary gas), finds it difficult to diffuse through long thin channels, tries to even up uneven velocity distributions among its roton "molecules" (viscosity), and acts in many ways as a normal fluid. Meanwhile the background in which the rotons travel, that is, the total body of fluid itself, can flow. It flows, at low velocity, without resistance through small cracks. The reason is that to have resistance, flow energy and momentum must go into heat, that is internal excitations (eg. rotons). The energy required to form a roton is not available (at the necessary momentum change) unless the fluid velocity is very high.

Actually it appears likely that helium in flow doesn't form rotons directly at all. Resistance sets in at a relatively low velocity (critical velocity) because apparently a kind of turbulence begins in the perfect fluid*. This cannot occur at lower velocities because energy is needed to create vorticity. And, if we accept Onsager's suggestion, the vorticity is quantized, the line integral of the momentum per atom (mass of atom times fluid velocity) around a closed circuit must be a multiple of h . Below the critical velocity not enough kinetic energy is available in the fluid to produce the minimum vortex lines.

We shall discuss first the way that the scarcity of states accounts for many of the properties of the liquid. Here we are summarizing work of many others, particularly Landau. It is thought best to reemphasize this viewpoint, since it is the one which is directly supported by quantum mechanics. Furthermore, in this way we are starting over the more familiar ground. Next we discuss the quantum mechanical view of the reason for the scarcity of states. Finally in the second part of the paper we discuss the quantized vortex lines proposed by Onsager.

3. Landau's Interpretation of the Two Fluid Model

One of the most fruitful ideas in interpreting the behavior of the liquid is the two fluid model. It was developed by Tisza⁹ from analogy to the structure of an ideal Bose gas. It is often spoken of as a vague association of two penetrating fluids. Landau¹⁰ has interpreted it in a definite manner. We review his interpretation here, although an excellent review by Dingle¹¹ already exists. He has strongly emphasized the fact that one might picture the helium as a background fluid in which excitations move. At absolute zero one has a perfect

* The author now considers his statement (reference 7) that the reason for flow resistance "cannot very well be a kind of turbulence", to be in error.

ideal fluid which may flow frictionlessly with potential flow. If heated, the heat energy excites the liquid. This it does by creating here and there within it excitations of some sort. These excitations can make their way from one place to another, collide with the walls and with each other, and give to helium some properties associated with the so-called normal fluid component, such as viscosity. Landau as a result of his study of quantum hydrodynamics was led to suppose the excitations to be of two kinds. Of lowest energy are the phonons, or quantized sound waves, whose energy E equals pc where p is the momentum and c the speed of sound. Above these separated by an energy gap Δ are those of another kind, called rotons. At first he supposed the energy of these to be given by $\Delta + p^2/2\mu$ if they have momentum p , where μ is an effective mass. Later he found that this did not agree with the experiments of Peskhov on second sound, and he proposed instead the formula

$$E_{\text{rot}} = \Delta + (p - p_0)^2/2\mu \quad (1)$$

where p_0 is some constant. He went further and suggested that all these excitations really are of the same class and differ only in momen-

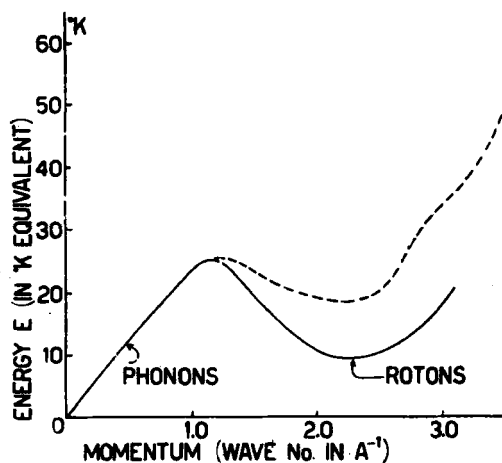


Fig. 1. The energy of excitations as a function of their momentum. Solid line as envisaged by Landau with parameters set to fit specific heat data; dotted line, an approximate curve derived from quantum mechanics. Excitations in linear section for low momentum correspond to phonons. Those near the minimum of the curve are called rotons.

tum. The energy $E(p)$ of those of momentum p , depends only on the magnitude of p , rising at first linearly as pc , but later falling to a minimum at p_0 and rising again, as in Fig. 1, solid curve. The curve in the vicinity of the minimum is given by (1). At the low temperatures encountered in He II only the states near $p = 0$, and those close to the minimum are excited. Therefore we do not have to know the rest of the curve accurately. Furthermore, the only important excitations are one of the two classes, phonons and rotons.

Supposing the excitations to obey Bose statistics the number, at temperature T , of momentum in the range d^3p is, according to statistical mechanics,

$$n_p = (\exp \beta E - 1)^{-1} d^3p (2\pi\hbar)^3 \quad (2)$$

with $\beta = (kT)^{-1}$ and $E = E(p)$. From this the average energy $E(p)$ and the specific heat can be calculated. In agreement with experiment it begins at low temperature as T^3 as expected, according to Debye, since only phonons are excited. At higher temperatures the higher energy roton excitations become excited, and the specific heat rises much more rapidly. The thermodynamic properties are in excellent agreement with the theory if¹²

$$\begin{aligned} c &= 240 \text{ meters/sec} \\ \Delta/k &= 9.6^\circ\text{K} \\ p_0/\hbar &= 2.0 \text{ \AA}^{-1} \\ \mu &= 0.77 m \end{aligned}$$

where m is the atomic mass of helium.

The hydrodynamic equations of the two fluid model arise as follows. Suppose the fluid at absolute zero has density ρ_0 and velocity \mathbf{v}_s . In the first part of this paper we shall take \mathbf{v}_s to be irrotational $\nabla \times \mathbf{v}_s = 0$. Later we discuss the problem of local circulation. The mass current density is $\rho_0 \mathbf{v}_s$ and the kinetic energy is $\frac{1}{2} \rho_0 \mathbf{v}_s^2$. Suppose that as a result of a rise in temperature a limited number of excitations are formed in the fluid. Landau has shown that the energy to form excitations in a moving fluid is not $E(p)$ but is

$$E = E(p) + \mathbf{p} \cdot \mathbf{v}_s \quad (3)$$

This results from simple considerations of the relations in moving and still frames of reference. The mass current density equals the momentum density of the fluid since all of the atoms have the same mass. It now is

$$\mathbf{j} = \rho_0 \mathbf{v}_s + \langle \mathbf{p} \rangle \quad (4)$$

where $\langle \mathbf{p} \rangle$ is the mean momentum of the excitations per unit volume. Now the mean $\langle \mathbf{p} \rangle$ depends on how the excitations drift. If they are in equilibrium with the fixed walls of the vessel the mean \mathbf{p} is *not* zero. The energy is $E(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v}_s$. It is lower than $E(\mathbf{p})$ for those excitations, whose momentum is directed oppositely to \mathbf{v}_s . Therefore in equilibrium more excitations align oppositely to \mathbf{v}_s than parallel to it. For this reason the mean \mathbf{p} is directed oppositely to \mathbf{v}_s and for small \mathbf{v}_s is proportional to it, let us say $\langle \mathbf{p} \rangle = -\rho_n \mathbf{v}_s$. This defines ρ_n . If ρ_s is defined as $\rho_0 - \rho_n$ we have a total current $\rho_s \mathbf{v}_s$ in a situation in which the excitations are in equilibrium with fixed walls. The equilibrium is established by collisions of the excitations with the walls and with each other.

The number of excitations of momentum \mathbf{p} is again determined by (2) but now with E given by (3) so that the average \mathbf{p} is

$$\langle \mathbf{p} \rangle = \int \mathbf{p} (\exp \beta(E(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v}_s) - 1)^{-1} d^3 \mathbf{p} (2\pi\hbar)^{-3}$$

or expanding to first order in \mathbf{v}_s , find $\langle \mathbf{p} \rangle = -\rho_n \mathbf{v}_s$ where

$$\rho_n = -\frac{\beta}{3} \int p^2 (\exp(\beta E(\mathbf{p})) - 1)^{-2} \exp \beta E(\mathbf{p}) d^3 \mathbf{p} (2\pi\hbar)^{-3} \quad (5)$$

The density ρ_n determined from experiments in second sound is in reasonable agreement with this expression (evaluated with the constants given above it fits above 1°K, but below 1°K the values $p_0/\hbar = 2.3 \text{ \AA}^{-1}$ and $\mu = 0.40 m$ fit better, and do not alter the good fit to the thermodynamic data). This explicitly shows that ρ_n is a derived concept, and does not represent the density of anything which has microscopic meaning.

The excitations can drift also. The distribution for equilibrium in a drifting gas is, according to statistical mechanics,

$$n(E) = (\exp \beta(E - \mathbf{p} \cdot \mathbf{u}) - 1)^{-1} \quad (6)$$

where \mathbf{u} is a parameter. In this case the mean momentum is

$$\langle \mathbf{p} \rangle = -\rho_n (\mathbf{v}_s - \mathbf{u}).$$

If we write $\mathbf{u} = \mathbf{v}_n$ we have for the current

$$\mathbf{j} = \rho_0 \mathbf{v}_s - \rho_n (\mathbf{v}_s - \mathbf{u}) = \rho_s \mathbf{v}_s + \rho_n \mathbf{v}_n \quad (7)$$

This can be interpreted macroscopically as saying that the current is like that in a mixture of two fluids, one of density ρ_s moving at velocity \mathbf{v}_s , the other of density ρ_n and velocity \mathbf{v}_n .

Actually (6) is not an equilibrium distribution unless the walls move at velocity \mathbf{u} , and furthermore \mathbf{u} is constant throughout the liquid. It is generally taken as a good approximation in the case that \mathbf{u} , that is, \mathbf{v}_n , is not constant. The lack of equilibrium in this case produces irreversible effects, such as viscosity, which can be associated with the "normal fluid component". The distribution is in equilibrium even if \mathbf{v}_s is not constant.

The entropy of the system is that of the excitations. It is easily verified that the mean group velocity of the excitations (the mean of $\delta(E(\mathbf{p}) + \mathbf{p} \cdot \mathbf{v}_s) / \delta \mathbf{p}$) is just \mathbf{v}_n . The entropy can therefore be considered as flowing with the "normal fluid".

It is also possible to work out the expected value of the energy of the system. If one calculates the internal energy and subtracts the internal energy the system would have at the same entropy but with $\mathbf{v}_s = \mathbf{v}_n = 0$ the excess expanded to the second order in the velocities can be written $\frac{1}{2} \rho_s \mathbf{v}_s^2 + \frac{1}{2} \rho_n \mathbf{v}_n^2$. This is just what the two fluid model would expect.

Therefore Landau shows that a liquid system with excitations as described will behave in many ways like a mixture of two fluids.

Furthermore, considerable progress has been made by Landau and Khalatnikov¹⁸ in the interpretation of many irreversible phenomena, such as viscosity, attenuation of second sound, etc. from the kinetic theory implied by such excitations. It is not possible as yet to find the crosssection for collision, say between two rotons, from first principles. But if a few such quantities are considered as unknown parameters, a great deal can be said. The number of rotons varies very rapidly with temperature, in the manner given by (2). For this reason the mean free path for collision and the resultant viscosity resulting from roton-roton collisions has a known temperature dependence. In a similar way the contribution of collisions between rotons and phonons or between phonons can be worked out. There are also collisions in which the number of excitations change. The results are often in excellent agreement with experiment.

There is, therefore, little doubt that in liquid helium there are such excitations, with the energy spectrum that Landau suggests, and that this picture supplies the complete interpretation of the two fluid model for helium II.

4. The Reason for the Scarcity of Low Energy States

The next question that concerns us is to try to see from first principles why the excitations of the helium fluid have these characteristics.

Landau has, in fact, tried to obtain some justification for the spectrum from a study of quantum hydrodynamics. This is not a completely detailed atomic approach. One attempts to describe the liquid by a few quantities such as density and current, or velocity. Then one makes these quantities operators with reasonable commutation relations, and tries to find the excitation energies of the fluid. The problem has not been analyzed in sufficient detail to establish the energy spectrum (1). Such an approach cannot give us an ultimate detailed understanding for two reasons. First, the numerical values of Δ , ρ , μ show these quantities to be characteristic of the atomic structure of the liquid. A theory which describes the fluid simply by average variables and which therefore cannot represent the fact that the liquid does in fact have atomic structure cannot lead to definite values for excitation energy. A more serious problem is this. It is necessary to show not only that the excitations $E(\rho)$ exist, but that there are not a host of other possible excitations lying lower. If we describe the liquid with average variables we have no assurance that there are no excitations at a level below the coarseness of our averages. Possibly excitations exist which represent no gross density variation and no mean current. If many other lower excitations exist they dominate the specific heat curve and the properties of the fluid. (Perhaps in ^3He we have an example of a system capable of excitations at an atomic level which are not describable by the variables used in quantum hydrodynamics).

However it is possible from first principles to see why there are no other excitations but those supposed by Landau and why the energy spectrum of these excitations has, qualitatively, the form which he supposed.^{7, 8}

In order to do so, we should, rigorously, have to solve the Schrödinger equation for the system.

$$H\psi = -\frac{\hbar^2}{2m} \sum_i \nabla_i^2 \psi + \sum_{ij} V(\mathbf{R}_{ij})\psi = E\psi$$

where m is the atomic mass, $V(\mathbf{R}_{ij})$ is the mutual potential of two atoms separated by the distance $\mathbf{R}_{ij} = \mathbf{R}_i - \mathbf{R}_j$, and ∇_i^2 represents the Laplacian with respect to the coordinates \mathbf{R}_i of the i^{th} atom.

The sums must be taken over all of the N atoms in the liquid. We cannot solve this equation directly but we can make surprising headway in guessing the characteristics of the wave functions ψ which satisfy it.

We shall have to picture the wave function ψ . It differs from one state to another. But we will consider its value for only one state at a time. Then it is a definite but complicated function $\psi(\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_N)$ of the $3N$ variables \mathbf{R}_i . To picture it we must have a scheme by which we clearly represent it in our minds. Now such a function is a number associated with every set \mathbf{R}^N of values \mathbf{R}_i , or, as we shall say, with every configuration of the atoms. We can represent a configuration \mathbf{R}^N by imagining each of the N atoms in the vessel containing the liquid to be located with its center at one of the \mathbf{R}_i . That is, each configuration is represented, as classically, as a particular definite location for each of the atoms. Then $\psi(\mathbf{R}^N)$ is a number associated with each such arrangement of the atoms. We can call it the amplitude of the configuration. For a given state, this amplitude for some atomic arrangements is large – these arrangements then have large probability – for others small and the configuration is unlikely. When we wish to speak of how the amplitude changes as the values of \mathbf{R}_i change, we shall use the more vivid language of asking how the amplitude changes as the atoms are “moved” about. Such motions are not directly related to any real classical motions, of course. In fact we cannot describe classical motions directly. All such classical ideas must be interpreted in terms of the mathematical behavior of ψ , if we are to be consistent with quantum mechanical principles. Most of our task, therefore, is trying to describe the ψ functions which correspond to the various kinds of states of energy, or motion, of which the liquid is capable.

Start by considering the ground state wave function which we shall call Φ . We use the intuition which we have acquired from knowing the solutions of the Schrödinger equations for simpler systems. For stationary states, ψ can be taken to be a real number. The lowest state always has no nodes (except for the exclusion principle, which does not operate here). Therefore Φ is everywhere positive. It is symmetrical, that is, Φ depends only on where atoms are, not on which is which. The energy $V(\mathbf{R})$ of interaction of two helium atoms, as worked out by Slater and Kirkwood,¹⁴ for example, consists of a very weak attraction at large distances, but a powerful repulsion inside of 2.7 Å (see Fig. 2). The atoms in liquid helium at the normal density have

a volume of 45 cubic Ångströms each so they are not tightly squeezed together. If one wishes a rough approximation, consider the atoms as impenetrable spheres of 2.7 Å diameter, and forget the attraction, whose effect is, after all, mainly just to hold the liquid together at the normal density even if the external pressure falls to zero. Then configurations of atoms in which some overlap each other, that is, are closer than about 2.7 Å, are of very small amplitude. In the most likely configurations the atoms are well spaced. As for a particle in a box whose wave function bows highest in the center and falls gradually to zero at the walls, we may imagine the amplitude highest for good separation and falling toward zero if a pair of atoms approach

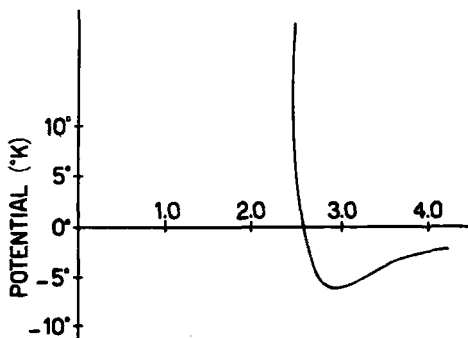


Fig. 2. The potential of interaction of two helium atoms as function of their separation as worked out from quantum mechanics by Slater and Kirkwood.

too closely. Our structure is a liquid, as a consequence of the zero-point energy, so that no particular lattice arrangement is strongly preferred. All configurations for which the spacing is ample have high probability. We can get from one arrangement to another without ever crossing a forbidden configuration of overlapping atoms because of the large spacing (cube root of atomic volume is 3.6 Å). Although not crystalline, there is a little local order induced by the tendency of atoms to stay apart, so that X-ray or neutron scattering experiments show a structure very similar to that of other simple liquids like liquid argon.

For the configurations of high amplitude the density is fairly uniform, at least until we look over such small volumes that we can see the fine grain atomic structure. If the density in a region is raised the atoms come closer together so that the "bow" on the wave function

which occurs as one atom is moved from contact with a neighbour on one side to one of the other side, is confined to a smaller space. The increased curvature represents increased kinetic energy and it is not as likely to find a configuration in which such an energy barrier is penetrated. As a matter of fact, this feature is easily analyzed quantitatively. Long range density fluctuations are sound waves. The rise of energy on compression is described by the compressibility coefficient, or equivalently by the speed of sound. Classically, standing density waves oscillating as a normal mode behave as an harmonic oscillator. Likewise, in quantum mechanics these are quantum oscillators and have zero point motions, although the most likely configuration is that of uniform constant density. The wave function for the zero-point motion of an oscillator is a gaussian so that the amplitude Φ for a given kind of density fluctuation falls off exponentially with the square of the fluctuation. To summarize, the ground state function is large for any configuration in which the atoms are well spaced from one another at nearly constant average density. It falls off if these conditions are violated.

Next we turn to the excited states. Right away one obvious excitation is that of the standing sound wave. If the classical frequency is ω the quantum excitation energy of such a mode is $\hbar\omega$. Usually one prefers by linear combinations to make states of running waves, or phonons. If the wave number is k , the energy is $\hbar kc$ if c is the sound velocity.

We may readily obtain the wave function for such a phonon excitation. If the density is $\rho(\mathbf{R})$ the classical normal coordinate going with such a mode is

$$q_{\mathbf{k}} = \int \rho(\mathbf{R}) \exp(i\mathbf{k} \cdot \mathbf{R}) d^3\mathbf{R} \quad (8)$$

Quantum mechanically for an oscillator the wave function for the ground state is a gaussian, and the first excited state is just the coordinate times this gaussian (the first hermite polynomial $H_1(x)$ is just x). Hence the wave function is

$$\psi_{\text{phonon}} = q_{\mathbf{k}} \Phi \quad (9)$$

if Φ is the ground state wave function of the system, which we have described in the preceding paragraphs. We have not bothered to normalize our function. The liquid consists of many atoms so if \mathbf{R}_i is the position of the i^{th} , the density in any configuration is

$$\varrho(\mathbf{R}) = \sum_i \delta(\mathbf{R} - \mathbf{R}_i) \quad (10)$$

the sum extending over all the atoms. Putting this in (8) and then (9) we find

$$\psi_{\text{phonon}} = (\sum_i \exp. i\mathbf{k} \cdot \mathbf{R}_i) \Phi \quad (11)$$

This is valid if the wave length ($2\pi/k$) is much larger than the atomic spacing, for then our description by compressional waves is adequate. The state energy is $\hbar kc$. Since $\hbar k$ is the momentum p of the state, this means $E = pc$. Since the wave length can be very long this energy can be exceedingly low.

The central problem is to see why no states other than these phonons can have such low energies. We try to construct the wave function ψ of an excitation which should be as low in energy as possible and yet not represent a phonon. We must associate a number which may now be positive or negative with each configuration. In fact, since ψ must be orthogonal to the ground state Φ which is everywhere plus, ψ must be plus for half the configurations and minus for the other half. Furthermore, ψ must be orthogonal to all the phonon states. This simply means that ψ must vary from plus to minus for changes in the configurations which do not appreciably alter the large scale density. Configurations can alter without variation of mean density by simply stirring the atoms about. Of course, since ψ must represent as low an energy as possible we must give low amplitude to configurations in which atoms seriously overlap, just as in the ground state Φ .

The function ψ takes on its maximum positive value for some configuration of the atoms. Let us call this configuration A , and the particular locations of the atoms α -positions. We said that the α -positions must be well spaced so that the atoms do not overlap, and further that they are, on a large scale, at roughly uniform density. Equally, call configuration B , with atomic positions β , that for which ψ has its largest negative value. Now we want B to be as different as possible from A . We want it to require as much readjustment over as long distances as possible to change A to B . Otherwise ψ changes too rapidly and easily from plus to minus, our wave function has a high gradient, and the energy of the state is not as low as possible.

Try to arrange things so that A requires a large displacement to be turned into B . At first you might suppose it is easy. For example (see Fig. 3) in A take some atom in the left side of the box containing the liquid and move it way over to the other side of the vessel, and

call the resulting configuration B . One objection to this is that an atom is moved from one side to another, so a hole remains at the left and an extra atom is at the right. This represents a density variation. To avoid this we may imagine that another atom has been moved at the same time from right to left, and the various holes and tight squeezes have

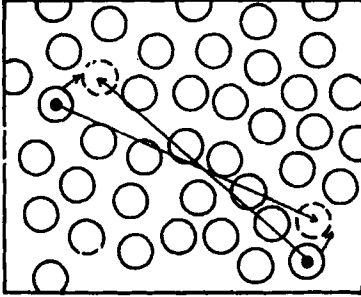


Fig. 3. Two configurations (solid and dotted) that result from large displacements (long arrows) of the atoms, can actually be accomplished by much smaller adjustments (short arrows) because of the identity of the atoms

been ironed out by some minor adjustments of several of the neighbouring atoms. This movement of two atoms each a distance of the size of the vessel, one from left to right and the other from right to left, is certainly a long displacement, so B and A are very different. But they are not.

The atoms must be considered as identical, the amplitude must not depend on which atom is which. One cannot allow ψ to change if one simply permutes atoms. The long displacements can be accomplished in two steps. In the first step permute the atoms you wish to move to those α -positions closest to the ultimate position they are to occupy in the final configuration B . This step does not change ψ because all the atoms are still in the same configuration of α -positions. Then the change to the B configuration is made by small readjustments, no atom moving more than half the atomic separation. In this minor motion ψ must change quickly from plus to minus and the energy cannot be low. For the reason that the wave function is unchanged by permutation of the atoms it is impossible to get a B configuration very far from the A configuration. No very low energy excitations can appear (other than phonons) at all.

In the phonon case we consider configurations in which, as ψ changes sign, the density distribution changes. A change in density cannot be accomplished by permuting atoms. That is why the Bose statistics does not affect phonon states. But it leaves them isolated as the lowest states of the system, so the specific heat approaches zero as T approaches zero according to Debyes T^3 law. This is the key argument for the understanding of the properties of liquid helium. It is given in

somewhat more detail in reference 7. Since it is a negative argument, attempting to prove that a low energy state does *not* exist, it is difficult to convey conviction in a few words. The reader should try to invent wave functions of low energy for himself. After a few attempts he will see much more clearly what we have tried to explain here.

5. Rotons

The qualitative argument is complete in itself. Nevertheless it is gratifying that it may be pushed even further to produce a quantitative estimate of the energy of these other excited states. We give only a summary of the considerations here (see reference 8 for details). We try to clarify our picture of the wave function ψ , until we can write a mathematical expression for it. This expression put into the energy integral $\int \psi^* H \psi d^N V / \int \psi^* \psi d^N V$ will give us an estimate for the energy.

As we said, in order to get the energy as low as possible we wish the gradients of ψ to be small. Therefore the configuration B (where ψ is maximum negative) must be as far as possible from configuration A . Yet we noted that no β -site is more than half the atomic spacing from an α -site. The two configurations are generally nearly the same. They are furthest from each other if as many atoms as possible must be moved. That is accomplished when, as illustrated in Fig. 4, all the β -sites are between α -sites, so every atom must move. To completely specify ψ , of course, we must give its value for all configurations, not only for A and B when all atoms are on α -sites, or all

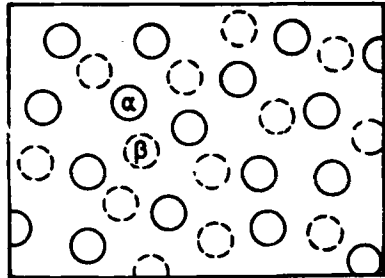


Fig. 4. The excited state wave function must be positive for one configuration, solid circles (α -positions) and negative for another. They are separated as far as possible if the negative configuration leaves no atom unmoved, dotted circles (β -positions)

on β -sites. The lowest energy results if the transition from plus to minus (hence A to B) is as gradual as possible. First for configurations in which each of the atoms is either on an α or on a β -position, this is most naturally accomplished if ψ is proportional to the number on α -sites minus the number on β -sites. This difference passes smoothly from plus to minus. It can be expressed mathematically this way: Consider a function, $f(\mathbf{R})$, of position, which is $+1$ if \mathbf{R} is at an α -

site and -1 if \mathbf{R} is at a β -site. Then $\sum_i f(\mathbf{R}_i)$ summed on all the atoms is just the desired number on α -sites minus number on β -sites. For intermediate positions ψ will vary as smoothly as possible if $f(\mathbf{R})$ is taken to vary in some smooth way between its extreme values of $+1$ and -1 , which it takes on at α and β -sites. This suggests that we take ψ to be of the form

$$\psi = \sum_i f(\mathbf{R}_i).$$

But this is incomplete for we tacitly assumed that in all the configurations the atoms did not overlap, the mean density did not vary very much and so on, just as in the ground state. This feature can be taken into account if we take instead

$$\psi_{\text{roton}} = \sum_i f(\mathbf{R}_i) \Phi \quad (12)$$

where Φ is the ground state function.¹⁵ Then ψ will fall rapidly if the atoms overlap, etc. We actually do not know what the function $f(\mathbf{R})$ is but we expect it to vary rapidly, so that if expanded in a Fourier integral the dominant wave lengths would be the atomic spacing.

According to the variational principle the best wave function is that which minimizes the energy integral. In this way, by variation of $f(\mathbf{R})$ it is readily found (see reference 8) that the minimum results if the function is

$$f(\mathbf{R}) = \exp(i\mathbf{k} \cdot \mathbf{R}) \quad (13)$$

and that the corresponding energy is

$$E(k) = \hbar^2 k^2 / 2mS(k) \quad (14)$$

where m is the atomic mass. The function $S(k)$ is the form factor for the scattering of neutrons from the liquid. That is, it is the Fourier transform of the function $p(R)$ which gives the probability per unit volume of finding an atom at a distance R from a given atom in the liquid in the ground state.

The local partial order of the liquid in the ground state shows up as in other liquids as a ring in the diffraction pattern (of neutrons, or X-rays). That is to say, there is a maximum in the function $S(k)$, which occurs when k represents a wavelength near the nearest neighbour spacing. The maximum in $S(k)$ represents a minimum in $E(k)$ here. This confirms the expectation that the low excitation would have wave numbers in this vicinity.

The state (12) and (13) has the momentum $\mathbf{p} = \hbar\mathbf{k}$. Ordinarily not

every value of a parameter in a wave function has significance in the variational method. But states of different momenta are orthogonal, and the energies (14) are significant not only for k near the minimum, but also in the neighbourhood of this value. The range of values for which (14) is useful is limited only by the range for which (12) can be expected to be a good wave function. For small k , (12) is identical to the wave function (11) representing phonon excitation, and (14) can be shown to give $\hbar kc$ in that region. Therefore the expression should be reasonable not only for k near the reciprocal atomic spacing, but for low k as well. It predicts a spectrum at first linear in p ($= \hbar k$) then falling to a minimum, just as anticipated by Landau, and in agreement with experiment.

The curve $S(k)$ taken from neutron data of Henshaw and Hurst¹⁶, or from the X-ray scattering data of Reekie¹⁷ agree. The $E(k)$ which results is shown in Fig. 1 by the dashed line. The general behavior and minimum are clearly shown.

The actual value of the energy at the minimum is twice too high to agree with the experimental value (solid line) for Δ . The theoretical value lies above the true value, as it should according to the variational principle.

The inaccuracy of the wave function (12) prevents us from giving a complete description of what the roton wave function must look like. The function (12) does not satisfy the conservation of current. It appears as though a more accurate function would represent a current distribution large and unidirectional in one region, with a field of return currents surrounding it, somewhat in the nature of a smoke ring. These and other arguments suggest a trial function of the type

$$\psi = \sum_i \exp i\mathbf{k} \cdot \mathbf{R}_i \cdot \exp i \sum_j g(\mathbf{R}_j - \mathbf{R}_i) \Phi \quad (15)$$

with the g , representing the back flow, to be determined. It is very hard to perform the integrals required in the variation problem with (15), so it has not been verified whether (15) represents a substantial improvement.

One way to understand the low energy for k near the reciprocal atomic spacing is this. One might consider these as sound waves of very short wave length. To obtain a density variation of long wave length is hard. To make the compression work must be done against opposing forces. For wave lengths closer to atomic spacing, however,

such density variations are easier to arrange. In fact, one can create variations of wave length equal to the atomic spacing simply by arranging the atoms, doing no appreciable work against repulsions, the energy being purely kinetic $\hbar^2 k^2/2m$. Actually the energy is even lower ($S(k)$ at maximum is 1.3) for there is a positive tendency in the liquid to have such variations; if some atoms are correctly arranged the others are more likely to be also satisfactory because of the local order. Therefore the energy does not continue to rise as $\hbar kc$ but falls lower for wave lengths near the atomic separation.

It is easy to verify that these excitations behave in just the way that has been assumed in developing the statistical mechanics and the two fluid model. To represent a state with two excitations, say with momenta \mathbf{k}_1 and \mathbf{k}_2 one has the approximate wave function

$$\psi = (\sum_i \exp i\mathbf{k}_1 \cdot \mathbf{R}_i) (\sum_j \exp (i\mathbf{k}_2 \cdot \mathbf{R}_j)) \Phi$$

and so on. Since the order of the factors is irrelevant this is the same state if \mathbf{k}_1 and \mathbf{k}_2 are reversed. The excitations obey the Bose statistics. In moving fluid the energy of the excitations can be shown to be (3).

6. Irrotational Superfluid Flow

So far we have only described the wave function for states representing internal excitation. We turn next to a description of the wave function which represents the state of the fluid when macroscopically we say it is flowing. We will assume that the flow velocity does not vary appreciably over distances of the order of an atomic spacing.

It is not difficult to represent by wave functions states which represent the motion of the superfluid. Suppose the system is at absolute zero so there are no excitations. If the entire system moves forward as a body, since the center of gravity coordinate can be separated out from Schrödinger's equation, the wave function is

$$\psi = (\exp i\mathbf{k} \cdot (\sum_i \mathbf{R}_i)) \Phi$$

where $N\hbar\mathbf{k}$ is the momentum of the system, if there are N atoms. In case the velocity is not uniform we can construct a wave function somewhat as follows: If the velocity varies only slowly from place to place, those atoms temporarily in a macroscopic region where the velocity is, say, \mathbf{v} must surely have a wave function very much the same as though the liquid in the region were isolated and moving at a uniform velocity. This suggests that the phase contains a term

$\hbar^{-1}m\mathbf{v} \cdot \sum_i \mathbf{R}_i$, the sum being taken only over those atoms in the region. Other regions where \mathbf{v} differs make similar contributions to the phase so the total phase is $m \sum_i \mathbf{v}(\mathbf{R}_i) \cdot \mathbf{R}_i$ where $\mathbf{v}(\mathbf{R})$ is the velocity at \mathbf{R} . This suggests a wave function of the form

$$\psi_{\text{flow}} = [\exp i(\sum_i s(\mathbf{R}_i))] \Phi \quad (16)$$

where $s(\mathbf{R})$ is a function which varies only very little over distances as small as the atomic spacing. We have suggested that it is $\hbar^{-1}m\mathbf{v}(\mathbf{R}) \cdot \mathbf{R}$, but as is usual for waves whose wave length varies with position, the momentum is the gradient of the phase, not the coefficient of \mathbf{R} . Thus (16) does represent the helium flowing, but the velocity is given by

$$\mathbf{v} = \hbar m^{-1} \nabla s. \quad (17)$$

It is readily verified that the current density is $\rho_0 \mathbf{v}$, and the energy (from the variational integral) is $\frac{1}{2} \rho_0 \mathbf{v}^2$, as expected classically. There is no change in density, as in (16) we have not allowed these small effects to be represented.

If excitations exist in the moving fluid the wave function is (16) multiplied by the factor $\sum_i f(\mathbf{R}_i)$ in (12). The excitation energy turns out to be (3) as expected, interpreting \mathbf{v} as the superfluid velocity \mathbf{v}_s .

Equation (17) implies that the motion is irrotational, that is, $\nabla \times \mathbf{v}_s = 0$. In a simply connected region this has only one solution for given motion of the boundaries. For fixed boundaries it is $\mathbf{v}_s = 0$. In a multiply connected region the situation is different. Since $\nabla \times \mathbf{v}_s = 0$, the circulation about any closed curve which can be shrunk to a point is zero. On the other hand, in the case of a toroidal region, if the curve encloses the hole the circulation need not vanish. Although the wave function must be single valued, s may be of the nature of the azimuthal angle, increasing by 2π , or a multiple thereof if one goes around the hole. That is, for a circuit enclosing a hole (into which liquid may not freely flow) the circulation must be an integral multiple n of a quantized unit $2\pi\hbar/m$,

$$\oint \mathbf{v}_s \cdot d\mathbf{s} = 2\pi\hbar m^{-1} \cdot n = 2\pi n \cdot 1.5 \times 10^{-4} \text{ cm}^2/\text{sec} \quad (18)$$

These states do not influence the previous statistical mechanical argument. There are too few of them. The velocity may be considered as a macroscopic variable, such as density. For a macroscopic torus even the lowest of the states given by (18) is very much higher than

a roton energy Δ . Thus if the torus area is A , radius R , the mass moving is $m A \cdot (2\pi R)/d^3$ where d^3 is the atomic volume. It moves at velocity given by $v_s \cdot 2\pi R = 2\pi\hbar m^{-1}$, from (18), so the kinetic energy is $(\hbar^2/2md^2) \cdot (2\pi A/Rd)$. The factor $\hbar^2/2md^2$ is an energy of the order of a roton, but the second factor is very large, being the torus dimension over the atomic spacing. Incidentally the total angular momentum is \hbar per atom.

If the fluid must flow irrotationally, at first sight, it cannot lose energy, unless it is moving very rapidly. This has been pointed out by Landau. If a body of fluid is moving at velocity v , and loses a small energy δE , it must do so (to keep the flow irrotational) by the entire fluid changing its velocity. Let the change in v be δv . If M is the effective fluid mass the momentum change δp is $M\delta v$ and $\delta E = Mv\delta v = v\delta p$. Now this energy loss must go into heat; that is, into internal excitations of rotons. But if the momentum transferred to excitations is δp the energy cannot be small. It must be at least about $(\delta p/p_0)\Delta$ where Δ and p_0 are the energy and momentum of an individual roton. That is, δE must be at least $(\Delta/p_0)\delta p$ and energy cannot be lost unless v exceeds Δ/p_0 , about 70 meters per second. (More accurately v must be high enough that a line drawn from the origin at slope v can cut the $E(p)$ vs p curve). This suggests the reason for the frictionless flow of superfluid. But we have proved too much, for in actuality the resistance sets in at velocities a few hundred times smaller.

The only way that gross slowing down can occur for lower velocities is for small parts of the fluid to stop or slow down without the entire fluid having to slow down at once. That is, energy loss must be accompanied by flow which is not irrotational; that is, flow which involves local circulation. To understand such effects we must add a new element to our picture of phonons, rotons and potential flow. These are the quantized vortex lines suggested by Onsager.¹ We proceed to describe them.

7. Rotation of the Superfluid

The problem which now faces us is to extend (16) so that we can also represent states for which $\nabla \times \mathbf{v}$ does not vanish, or at least where there is circulation in the superfluid. We analyze the situation at absolute zero for simplicity. We must present ourselves a problem in which such circulation is necessary and try to find the lowest energy state. The situation first considered by the author was the slip-stream

between two regions of fluid moving at different velocity, but it is easier to arrive at the result by considering the problem of helium with high angular momentum in a cylindrical vessel. Suppose, for example, the helium at absolute zero is initially under such pressure that it is solid and is set into rotation, then the pressure is released so that it liquifies. What is the final state of the helium? We ask then for the lowest state of a quantity of helium which has a definite, macroscopically high, total angular momentum.

For a system of given angular momentum the kinetic energy is least if the angular velocity ω is a constant throughout the liquid. This motion is not rotation free for $\nabla \times \mathbf{v} = 2\omega$. But it is very difficult for helium to manage a state of local circulation. In fact, without high excitation energy, local circulation is impossible. At first one might find it hard to see why the liquid cannot simply rotate as a rigid body. The energy is then low. But a liquid is not a rigid body. A part of it can turn independently of the whole. In a rough way of speaking the liquid may be thought of as made up of many quasi-independent units of nearly atomic dimensions. Any motion of the body can be compounded of motions of the tiny parts. But to set any small part into a rotational state requires a high energy because the moment of inertia is so small. If only a limited energy is available nearly all the "parts" must be frozen out in their ground states. That is, nearly everywhere the local angular momentum is zero, i.e., $\nabla \times \mathbf{v}_i = 0$. It takes energy to create circulation and, furthermore, we can expect this circulation not to be distributed uniformly throughout the fluid. The rigid body type of rotation where $\nabla \times \mathbf{v}_i \neq 0$ everywhere is not possible, or if at all, only with an enormous expenditure of energy, an expenditure far higher than that gained by the uniform distribution of angular velocity.

Another possibility that suggests itself is that the liquid, if the angular momentum is high, is not free of excitations like rotons and phonons even though the temperature is at absolute zero. These excitations could carry the angular momentum. That is, in the language of the two fluid model, perhaps there is at $T = 0$ a mixture of superfluid and normal fluid, with the superfluid component not rotating, and with the normal fluid carrying all of the angular momentum. The energy to maintain the normal fluid being sustained by the fact that if less normal fluid were present, for given angular momentum the kinetic energy would have to be larger. This turns out, for vessels of

centimeter dimensions turning at about one radian per second, to be a state of nearly 10^4 times the energy of a rigid body rotating at the same angular velocity. Surely nature can find some lower state for the helium.

We know (see 18) that if there is a hole in the liquid, circulation can exist. Therefore another solution suggests itself. The liquid circulates around a hole with constant circulation as in a free vortex (familiar from rotation of water around an emptying drain). The velocity varies inversely as the radius, rising to such heights near the center as to be able to maintain the hole free of liquid by centrifugal force. Such a solution would be easy to verify in a striking manner by looking at the surface of the liquid. Instead of the usual parabola it would be the curve of the surface of a free vortex. The energy is still quite a bit higher than the rigid body case, because the velocity instead of being distributed proportionally to the radius, actually falls as the radius increases. Nevertheless it is orders of magnitude below the mixture of normal fluid suggested above.

However, this is still not the lowest possible energy state, and the striking experiment will not succeed. To show this we construct a lower state. Suppose that the liquid has not only one vortex at the center, but several vortices. For example, suppose beside the central one there were a number distributed about the circle of radius $R/2$, half that of the vessel R , and all turning the same way. Viewed grossly this is like a vortex sheet so the tangential velocity can jump as we pass from inside $R/2$ to outside. Then the velocity can be arranged a little more like the linear curve by two sections, each of which is a $1/r$ curve. The gain in energy resulting from this improved distribution may more than compensate the energy needed to make the additional holes (and, further, the central vortex need not now be so large and energetic).

Continuing in this way with ever more vortices it soon becomes apparent that the energy can always be reduced if more vortices form. However there is a limit. Due to the quantization (18) of the vortex strength the smallest vortex has circulation $2\pi\hbar m^{-1}$. The lowest energy results if a large number of minimum strength vortex lines (which we shall call unit lines) form throughout the fluid at nearly uniform density. The lines are all parallel to the axis of rotation. Since the curl of the velocity is the circulation per unit area, and the curl is 2ω , there will be

$$2m\omega/2\pi\hbar = 2.1 \times 10^8 \omega \text{ lines per cm}^2 \quad (19)$$

with ω in radians per second. For $\omega = 1$ rad per second the lines are about 0.2 mm apart so that the velocity distribution is practically uniform.

Such weak lines will not form actual macroscopic holes. In fact, if one neglects atomic structure and assumes a classical continuous liquid with surface tension, a unit line makes a hole opposed by surface tension which figures out to be only 0.4 Å in radius. That means that there is no real hole in the liquid. Around such a unit line, for example a straight one along the z -axis, the wave function off the axis is roughly

$$\psi = (\exp i \sum_i \varphi_i) \Phi \quad (20)$$

where φ_i is the angle about the z -axis. This does not hold close to the axis. On the axis $\exp i\varphi$ is meaningless, and close to it has enormous gradients. A particle on the axis cannot have angular momentum, yet (20) implies that each atom has angular momentum \hbar , nor can there be exceptions because the Bose statistics implies that they are equivalent. Therefore a more accurate expression than (20) would be this expression multiplied by a factor which is unity except if any one of the atoms comes very close to the axis, in which case it falls rapidly to zero. The density of fluid falls to zero on the axis. This is the remnant of the classical hole. Actually quantum mechanically the line will not remain perfectly straight in one spot but will have some zero point motion of wandering and waving to and fro.

It is not hard to get a reasonable estimate of the energy contained in these lines. First consider an isolated unit line along the axis of a cylinder of length L , radius b . The velocity at radius r is \hbar/mr and if ρ_0 is the fluid density in atoms per cc ($\rho_0 = 1/45 \text{ Å}^3$) the kinetic energy is the integral

$$K. E. = \frac{1}{2} \int \rho_0 m (\hbar/mr)^2 \cdot 2\pi r dr \cdot L.$$

The upper limit of the integral is b . It diverges at the lower limit, but within about the atom spacing the velocity formula is meaningless. Furthermore, inside this radius some of the energy is potential, required to keep to density down near the axis (that is, to make the partial "hole"). Therefore the energy needed to form such a line, per unit length, is

$$\begin{aligned} \text{Line energy per unit length} &= \rho_0 \pi \hbar^2 m^{-1} \ln(b/a) \\ &= 10^{-8} \ln(b/a) \text{ ergs/cm.} \end{aligned} \quad (21)$$

Here a is a length of order of the atomic spacing. Its exact determination would require solving the difficult quantum mechanical problem. In almost all applications the ratio b/a will be very large, and the logarithm large enough to be insensitive to the exact value of a . For this reason we will not attempt a detailed evaluation, but simply choose a to be close to the atomic spacing. We arbitrarily take $a = 4.0 \text{ \AA}$. In more complicated geometrical situations the lower limit will be the same, but the upper limit b will be some other characteristic dimension of the apparatus, or more usually the spacing between vortex lines, etc. It can be found by integrating the velocity distribution as determined for the given distribution of singular vortex lines.

For a cylinder of liquid rotating at angular velocity $\omega = 1 \text{ rad/sec}$ the vortices are about 0.02 cm apart. This is 0.5×10^8 times a if $a = 4 \text{ \AA}$, so we can take the $\ln(b/a)$ in this case to be about $\ln(0.5 \times 10^8)$ or 14. Neglecting the variation of this logarithm with ω we find for the energy of all of the lines:

$$\text{Total line energy per unit volume} = \rho_0 \omega^2 \hbar \cdot \ln(b/a)$$

where we have estimated $\ln(b/a)$ as 14. The ratio of this to the kinetic energy for a rigid body is $4\hbar m^{-1} R^{-2} \omega^{-1} \ln(b/a)$ if the cylinder radius is R . For $R = 1 \text{ cm}$, $\omega = 1 \text{ rad/sec}$ this ratio is 10^{-2} . For macroscopic laboratory dimensions the excess energy to form the lines is small. They would form if rotating solid helium is melted by releasing the pressure, the angular velocity distribution would differ imperceptibly from uniformity, and the surface should appear parabolic.

It is not self-evident that there is no state of appreciably lower energy, and that the energy of the rotating liquid is correctly estimated. This subject has not yet been analyzed any more deeply than is reported here. Therefore this part of the paper is not on as firm a foundation as the rest. We must therefore still consider it conjectural whether the considerations on rotational flow reported here are actually correct. It is interesting that all the conclusions were arrived at independently by the author without knowledge of Onsager's previous work (with which they are in exact concordance).

8. Properties of Vortex Lines

In a situation more general than uniform rotation, in which the curl of the velocity is not constant, we can imagine a similar situation. We have a situation instantaneously with many vortex lines. Some are

closed on themselves in rings, and others terminate with their ends on the fluid boundaries. Viewed from a continuum approximation in which atomic structure is neglected, a velocity \mathbf{v}_s can be defined at every point. The curl of this is zero everywhere, except at one of the vortex lines where it is infinite. These lines are real quantized vortex lines. The circulation around a small circuit surrounding only one line is $2\pi\hbar m^{-1}$. The lines have a sense depending on the direction of rotation. The circulation about any curve whatsoever is given by

$$\oint \mathbf{v}_s \cdot d\mathbf{s} = 2\pi\hbar m^{-1}n$$

where n is always an integer, being the net number of lines linked by the circuit, account being taken to the sign of each.

If $\nabla \times \mathbf{v}_s$ is averaged over a large enough region that many lines are included, the number of lines per cm^2 must be at least $\langle \nabla \times \mathbf{v}_s \rangle > m/2\pi\hbar$ and the energy of these lines per unit volume is at least

$$\frac{1}{2} |\langle \nabla \times \mathbf{v}_s \rangle| \rho_s \hbar \ln(b/a) \quad (22)$$

where b is the spacing between lines, $1/b^2 = \langle \nabla \times \mathbf{v}_s \rangle > m/2\pi\hbar$. This shows that in our liquid it takes energy to create circulation. Actually in real, complex situations the energy might exceed greatly the value in (22). There may be great complex activity with many lines twisting and turning so that several lines of opposite senses are close together. In this case, the case of developed turbulence, the number of lines present may be bigger than the average $\nabla \times \mathbf{v}_s$ would indicate. (Probably in such a case it would be hard to define the average $\nabla \times \mathbf{v}_s$, because the result may depend on the size of the region over which the average is taken).

The discussion of the rotating cylinder of liquid with which we introduced the lines is rather special. We shall try to give a more complete and general description of the state of the superfluid with circulation. We continue to study the case at absolute zero. Let us try to characterize the state of a fluid in which we desire two things (which, it will turn out, are mutually incompatible). We want (a) the liquid to be flowing with a velocity \mathbf{v}_s , which is a smooth function of position without singularities (on a scale of distances large compared to atomic dimensions) and (b) we want $\nabla \times \mathbf{v}_s$ not to vanish.

Suppose the liquid in an element of volume ΔV (large compared to the atomic volume) is moving at velocity \mathbf{v} . Then as we have seen the wave function should depend on the position of the atoms, if they are

within ΔV as $\exp(i\mathbf{mv} \cdot \sum_i \mathbf{R}_i) \Phi$. That is, if a number of atoms in the region are displaced, each by $\Delta \mathbf{R}_i$ from one allowed (by Φ) configuration to another allowed one, the main effect is that the wave function must change phase by

$$\sum_i (\mathbf{mv} \cdot \Delta \mathbf{R}_i) \hbar^{-1} \quad (23)$$

This can also be seen in another way. If a region of fluid can be considered to have a velocity \mathbf{v} it has a momentum density $\rho_0 \mathbf{mv}$. It is characteristic of momentum in quantum mechanics, that if the center of mass is changed the wave function changes phase by an amount proportional to the momentum and to the displacement of the center of mass. Now if the atoms are displaced by $\Delta \mathbf{R}_i$ the center of mass moves so the phase change (23) results. This is true at least if the displacement makes no other important change in the wave function. We will suppose that both before and after the displacement the atoms

are well spaced and there are no gross density fluctuations, etc. such that in case the liquid were not in motion both configurations would have essentially the same amplitude.

The same argument goes for atoms in other regions, etc. so the phase shifts accumulate to a sum in (23) over displacements of atoms all over the liquid, if \mathbf{v} is now considered as a function of \mathbf{R}_i . The displacements $\Delta \mathbf{R}$ must be small compared to the distances over which \mathbf{v} varies. We shall apply the formula in a case in which $\Delta \mathbf{R}$ is the separation between atoms.

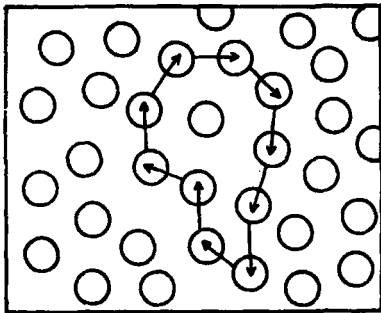


Fig. 5. The wave function must not change as a result of a permutation. If all the atoms are displaced around a ring, as shown, the phase change must be a multiple of 2π

Select, in a given configuration, a very long closed chain of atoms each of which is a nearest neighbor of the next in line (see Fig. 5). The last should have the first as nearest neighbor. The chain may consist of very large numbers of atoms and may even be so long that it passes through regions of varying velocity. Consider a displacement of each atom to its nearest neighbor next in line. The wave function cannot change, for it is simply a special permutation of the atoms. Further we will suppose that if all the displacements are made together

a little at a time, each intermediate configuration is allowed. This sliding of the chain along itself is not prevented by potential barriers, especially if we allow small temporary displacement of other atoms adjacent to the ring to permit passage in tight places. In the final configuration all atoms have returned to their original positions, except those of the ring which have moved one over. We suppose, because of the ease in which the displacement can be made that we can assume the wave function does not vanish for any intermediate position during the displacement. Then its phase shift is given by (23), but this must represent no change in the wave function. It is therefore necessarily an integral multiple of 2π . We conclude that

$$\oint \mathbf{v}_s \cdot d\mathbf{s} = 2\pi\hbar m^{-1}n \quad (24)$$

where n is an integer and the integral is taken over any path which goes from one atom to the next neighbor, etc. If \mathbf{v}_s is now assumed continuous at an atomic scale, the path can be smoothed out to any continuous curve. Of course, it is impossible that (24) holds for all continuous paths if n is an integer (depending on the path) if \mathbf{v}_s is free of singularities and continuous unless $n = 0$ (in a simply connected region). Because any path can be deformed continuously into an infinitesimal path, the left side changing continuously to zero. The right side cannot change left continuously so it must be zero for all paths. Likewise for a toroidal region n must be the same for all paths which surround the hole.

We see therefore that \mathbf{v}_s cannot be continuous if we are to have circulation. There must be places where \mathbf{v}_s is discontinuous, and places in the fluid where a displacement of an atom to its neighbor may not be possible without passing through a node in wave function. In the neighborhood of such a node the probability of finding an atom is reduced. This decrease in density requires energy to maintain it. We shall therefore try to arrange conditions so that such places are as infrequent as possible. Under those conditions, for nearly every conceivable ring of atoms the atoms can be moved over to the next adjacent atom without the wave function vanishing. Its phase change must be a multiple of 2π . If two adjacent rings have a phase change which is different, differing by 2π say, then between them somewhere must lie a very small ring of three or four atoms for which the circulation is $2\pi\hbar/m$. For example, suppose for a certain ring A the phase is zero, but for a nearby ring B it is 2π . Then shift ring B by a few

atoms at a time until it gets as close to ring *A* as possible, but still has phase shift 2π . Likewise shift *A* until it is as close to *B* as possible but so that it has still shift 0. Then *A* and *B* will contain many atoms in common and only differ by a few, as illustrated in Fig. 6. Then

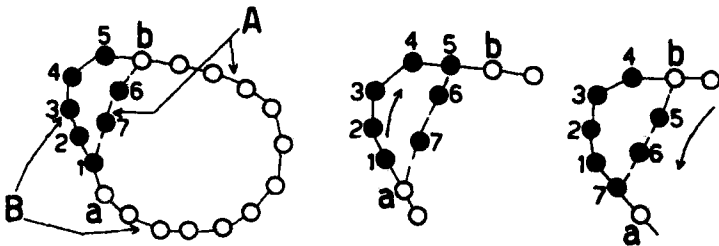


Fig. 6. A displacement along ring *B* followed by a reverse displacement along an adjacent ring *A* with many atoms in common is equivalent to a displacement around the ring *C*, indicated by black circles (except for an inconsequential permutation 5-*b*).

consider a permutation consisting of shifting *B* forward, then shifting *A* backward. It is readily verified that this change is the same as a shift of atoms around the very small ring *C* consisting of those parts of *A* and *B* which are not common, plus one of the common atoms. But the change in phase is 2π when *B* shifts and 0 when *A* shifts back, so that it must be 2π for the very small ring *C*. * This represents a highly concentrated angular momentum. Somewhere in the middle of ring *C* is a nodal point. It is readily appreciated geometrically that these nodal points must essentially form lines through the fluid. They are quantized vortex lines. It must be admitted that this argument is far from complete. We should consider states in which the location of the vortex line is uncertain, that is, a superposition of states with various locations for the line. Such a state would have a lower energy. Possibly we make a serious error in imagining that the velocity can be defined right up to atomic distance from the axes, if this axis itself does not have a definite location. Onsager has remarked, in private communication, on the possibility that these quantum effects might lower the energy to such an extent that the logarithm in (24) should

* The change in phase cannot be determined only from the initial and final configuration, but requires a description of the amplitude for intermediate configurations as well. Therefore this argument is not complete unless it is also assumed that partial rotations of *C* consisting of displacements of less than one atom spacing can also be roughly imitated by partial displacements of *B* and *A*.

be absent. At any rate, although our energy estimates may be incorrect, quantized vortex lines probably exist. We continue our discussion of the consequences of this assumption.

On a large scale according to the theorem of Helmholtz, vorticity moves with the fluid in such a way that the strength of a vortex filament remains constant. This means that if the fluid drifts the lines drift with it, maintaining their quantized strength. This is true, at least, if no forces act directly on the vortex line. In general the force per unit length on a vortex line equals the density, $\rho_0 m$, times the vector cross product of the circulation, $2\pi\hbar m^{-1}$, and the velocity of fluid where the vortex is.

9. Critical Velocity and Flow Resistance

We next turn to the role such vortex lines may play in the resistance to flow found at sufficiently high velocities.

We have suggested that this resistance cannot be understood in terms of a direct creation of rotons, the superfluid otherwise being in perfect flow. Let us consider what would happen if liquid is flowing out of an orifice, or tube, into a reservoir of fluid at rest. In Fig. 7 is illustrated the distribution of flow for irrotational motion. A very high velocity develops near the corners and large accelerations develop there. An ordinary fluid, such as water,

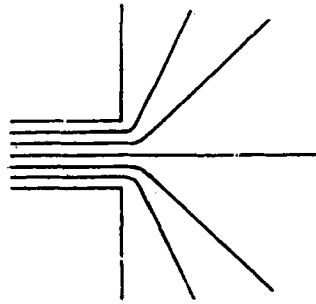


Fig. 7. Ideal potential flow from an orifice

flows in a complicated manner such as illustrated in Fig. 8 (a few moments after flow starts). The water shoots out straight into the nearly still fluid in the reservoir, forming a vortex sheet, which is unstable and curls around, eventually in an extremely complex manner. Let us see how helium might try to imitate some of the features of the type of flow illustrated in Fig. 8. Just for rough orientation and estimate suppose the fluid tries to go out in a jet, let us say at first of the same width and velocity as in the tube. Take the case that the tube is a long slot perpendicular to the paper, and the flow is roughly two dimensional.

Then circulation is implied for the velocity is v in the jet and 0 outside. This requires the formation of vortex lines, perhaps as illustrated in Fig. 9. The spacing is x and if this is small compared to d ,

the slot width, the velocity distribution is roughly uniform inside the jet. Taking a line integral along the jet for unit distance, and returning outside the jet, the circulation is v so the number of lines per centimeter is

$$\frac{1}{x} = v/2\pi\hbar m^{-1}.$$

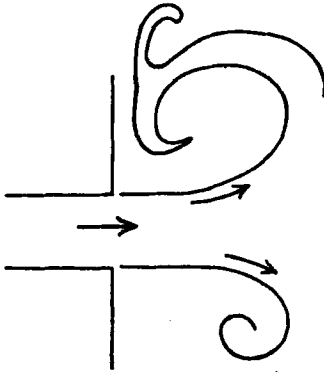


Fig. 8. Real flow from an orifice for ordinary liquids, producing an unstable vortex sheet

It takes energy to form these lines. If there is not enough kinetic energy in the fluid to supply the energy to make the lines, no resistance will appear. Once the lines can be formed they are, in a manner we shall soon discuss, ultimately dissipated as heat and a resistance appears. Let us see what order of critical velocity we would estimate in this way. The lines move out at the velocity of the fluid at their own location, which is $v/2$. Another way to see the necessity for this is to realize that as the fluid passes from inside to outside the pipe vorticity is created, so new lines must continually come rolling out of the ends of the orifice. In our case v/x lines are created per second. The energy needed to create these is (per unit length of slot)

$$\frac{v^2}{2\pi\hbar m^{-1}} \rho_0 \pi \hbar^2 m^{-1} \ln(d/a)$$

where the argument in the logarithm is only approximate. The total kinetic energy available per cc of fluid is $\frac{m\rho_0 v^2}{2}$, so that per second $v d \frac{m\rho_0 v^2}{2}$ is available. If we define v_0 as that velocity for which the energy available is just large enough to create the vortices we find

$$v_0 d = \hbar m^{-1} \ln(d/a).$$

For example, for a slit of width $d = 10^{-5}$ cm, (which is about three

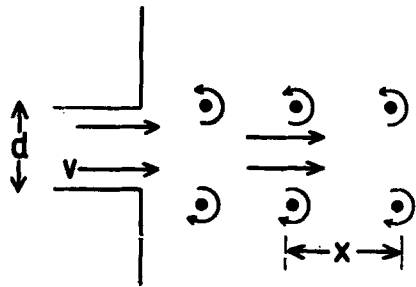


Fig. 9. Idealization of supposed vortex rings formed when superfluid helium issues at high speed from an orifice.

times the width of a Rollin film at a height of 1 cm) this gives $v_c = 100$ cm/sec, if $\ln(d/a)$ is taken as 6. This is somewhat higher than the critical velocities observed. The calculation is only meant as an estimate because the actual situation must be complicated. For one thing, near the critical condition α comes out about $3d$ so our picture of a uniform jet is poor. Further, the velocity in the jet must of course be reduced as a result of the energy needed to form the vortex line. Actually probably the situation near the critical point must be very complicated and irregular. The flow for short momentary periods may be much like Fig. 7 but irregularly vortex lines peel off of the edges of the slit, probably starting at one point along the slit and progressing to other places, or perhaps if the hole is circular, one or two vortex lines is fed out continuously in a form roughly like a helix. It is predicted that very close to the critical velocity when loss just begins, the resistance will be irregular and show fluctuations. These fluctuations are very small however and would be hard to detect. Possibly some sound may be generated by the flow irregularities. It is difficult to estimate its intensity. When helium is driven, just above critical velocity, through an emery powder superleak, some noise should be generated as the various vortex lines suddenly form and pass into the stream. The irregularities are a result of the unpredictable quantum transitions between states of no vortex line and one with a section of line.

Another possible source of vortex lines is the contact between flowing liquid and the walls. It is not necessary that all the loss occurs at the exit end of the tube. The walls of the pipe are irregular. Vortex lines may be created inside the pipe also.

It is difficult to go beyond this order of magnitude calculation in describing the conditions controlling the production of vortex lines. For example, if one studies the example given there are serious difficulties. As a particular vortex line leaves the end of the tube there are very great forces trying to pull it back resulting from its image in the tube wall. Let us imagine a line a distance b above the wall in a tube in which the velocity of flow is v_c . It is readily shown that the forces acting on the line are these. First a force pulling away from the surface of strength $2\pi\hbar\rho_0v_c$. Second, from the image, an attraction to the wall of strength $\pi\hbar^2\rho_0/mb$. A vortex line responds to forces by moving through the liquid to reduce the net force to zero. In this case it would drift upstream if the attraction is highest. But a vortex line

will interact with the wall, especially at its ends which go into the wall surface. Suppose this results in a frictional force which keeps the line from moving upstream. Then the response is to move closer to the wall. The vortex only moves away from the wall if $2\pi\hbar\varrho_0 v$ exceeds $\pi\hbar^2\varrho_0/m\dot{b}$. Even if v is 100 cm/sec this requires \dot{b} to exceed 10^{-6} cm or 20 atomic spacings. We might expect a vortex line to fluctuate away from the surface by a few atomic diameters. But how can we expect to penetrate the enormous potential barrier, to create a line so far away from the surface that the flow velocity can pull it further out and create eventual vorticity and energy loss?

More likely a line gets started somehow and has its ends tied on the wall. Then the forces of the fluid on the rest of the line cause it to wander about in such a way that more and more vortex line is fed out. It is not necessary to create bodily at one instant a complete section of line. For example, for the case of liquid issuing from a tube perhaps the vortex lines are helices with contact points at the edge of the hole which turn round and round while the helix moves outward. Similar things could happen inside tubes. If the tubes are very narrow the line will hit the other surface easily and be attracted by the walls. It can never get very far from a wall. Even if started somehow it will fall back into the tube walls unless the velocity v_0 suffices to keep it in the stream. Therefore the smallest tubes have the highest critical velocities.

10. Turbulence

The patterns of vortex lines which we have studied are well known to be unstable. In the case of the rotating cylinder this is not true if the cylindrical vessel containing the helium rotates also. But if the container is stopped the situation is altered. There are forces between the wall and vortex lines. (This is because the fluid density is altered near the line axis, so the interaction with the wall is not the same as the average for the rest of the helium). The lines at the outside drag past the stationary wall and as a result get distorted from their original vertical line position. This twists others, etc. Lines fall into the wall and others twist about each other in a complex way. It would be interesting to study this experimentally, to see how fast, and in what manner, the liquid eventually slows down.

In ordinary fluids flowing rapidly and with very low viscosity the phenomena of turbulence sets in. A motion involving vorticity is

unstable. The vortex lines twist about in an ever more complex fashion, increasing their length at the expense of the kinetic energy of the main stream. That is, if a liquid is flowing at a uniform velocity and a vortex line is started somewhere upstream, this line is twisted into a long complex tangle further down stream. To the uniform velocity is added a complex irregular velocity field. The energy for this is supplied by pressure head.

We may imagine that similar things happen in the helium. Except for distances of a few ångströms from the core of the vortex, the laws obeyed are those of classical hydrodynamics. A single line playing out from points in the wall upstream (both ends of the line terminate on the wall, of course) can soon fill the tube with a tangle of line. The energy needed to form the extra length of line is supplied by a pressure head. (The force that the pressure head exerts on the lines acts eventually on the walls through the interaction of the lines with the walls). The resistance to flow somewhat above critical velocity must be the analogue in superfluid helium of turbulence, and a close analogue at that.

There are some ways, however, in which the two cases differ. In a classical fluid there is a thin boundary layer near the wall of the pipe in which viscosity controls the situation. In this boundary layer there is a large vorticity, but it escapes into the stream to be amplified, only from the edge of the layer. Inside it is damped by viscosity. As the stream velocity falls the boundary layer thickens, for the amplification is less and the damping overpowers it ever further from the wall. Below a critical velocity the turbulence ceases altogether and the flow is laminar, but with vorticity, the viscosity keeping the vorticity from amplifying itself. That is, viscosity is the mechanism which determines whether vorticity will be amplified or not, and therefore whether turbulence is produced. If the viscosity goes to zero as a limit (and no other physical phenomena are added) a classical ideal liquid would exhibit turbulence at any velocity, no matter how small.

Superfluid helium is an ideal fluid of zero viscosity. It does not exhibit turbulence at low velocity because of another, quantum mechanical, effect. The vorticity is quantized and cannot begin at as low amount as desired. One must supply energy enough to get the first one or two vortex lines started before the amplification process of turbulence can take over. There will not be a boundary layer with a structure analogous to that in classical flow (although near the walls the flow will

be somewhat different because of the dragging forces between the moving vortex lines and the wall).

In a classical fluid, if the turbulent stream empties into a reservoir, the turbulent motion continues for a while, but as a result of the viscosity, it gradually slows up and dies out, the energy appearing eventually as heat.

What happens to a turbulent mass of superfluid left to itself? If there is normal fluid present the rotons and phonons will collide with the vortex lines and take energy from them, gradually turning this energy gain into more rotons and phonons (as a result of collisions among rotons the number of these may change). But an interesting question arises if the experiment is imagined at absolute zero. What can eventually become of the kinetic energy of the vortex lines?

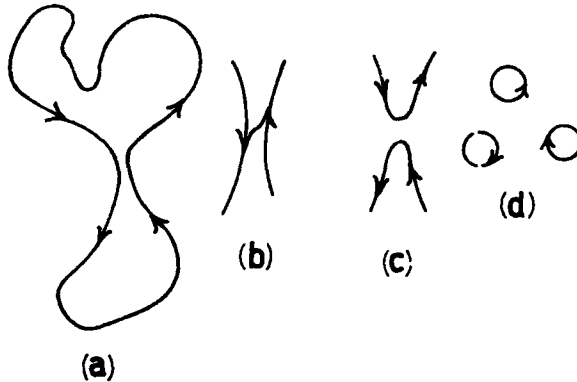


Fig. 10. A vortex ring (a) can break up into smaller rings if the transition between states (b) and (c) is allowed when the separation of vortex lines becomes of atomic dimensions. The eventual small rings (d) may be identical to rotons.

One possibility that suggests itself is this. Consider a large distorted ring vortex (Fig. 10a). If, in a place, two oppositely directed sections of line approach closely, the situation is unstable, and the lines twist about each other in a complicated fashion, eventually coming very close; in places, nearly within an atomic spacing. Consider two such lines (Fig. 10b). With a small rearrangement, the lines (which are under tension) may snap together and join connections a new way to form two loops (Fig. 10c). Energy released this way goes into further twisting and winding of the new loops. This continues until the single loop has become chopped into a very large number of small loops (Fig. 10d).

The smallest ring vortex that can exist must have a radius about half the atomic spacing. Let us guess that this is in fact a roton. Then all the energy of the vortex will eventually end by forming large numbers of rotons, that is, heat. Perhaps eventually it will be easier to understand the details of the complete transformation of organized flow energy into disorganized heat energy for liquid helium than for other substances.

11. Rotons as Ring Vortices

It is not unreasonable to guess that these smallest vortices are rotons. The velocity distribution around a roton, which is found by analytic means (ref. 8) is similar to that around a vortex ring. It is quite reasonable that a vortex ring can be only so small. To increase the curvature of a vortex line beyond that of radius roughly a may take energy. Let us imagine a roton to be the circular quantized vortex of lowest energy. A large circular vortex has (from (21)) energy

$$E = 2\pi R \cdot \frac{\pi\hbar^2}{m} \rho_0 \ln R/a.$$
 It carries momentum $p = \pi R^2 \cdot 2\pi\hbar\rho_0$. This momentum is that of a roton, p_0 , if $R = 2.2 \text{ \AA}$. The energy is the right order (it corresponds to replacing \ln by 1.6).

One might object that such a vortex drifts through the fluid, at velocity $v = (\hbar/2mR)\ln R/a$, so one would expect rotons not to have a zero group velocity. Actually this drift, of a large vortex, has its seat in the force tending to shrink the vortex to decrease the energy of the line. The response to the radially directed force is a perpendicular motion. It is analogous to the ornery response of a gyroscope. In fact, if a vortex line were a thin flexible mechanical tube with inertia, and were started with zero forward motion, it would first fall in a bit and then move forward in a halting fashion, like the nutation of a gyroscope, or the motion of an electron in crossed magnetic and electric fields. In a roton we imagine that the forces tending to contract the ring are already opposed by a kind of stiffness of the ring. It is already as small as possible. No drift motion results. In fact forward drift would expand it and raise the energy, while reverse drift would try to compress it to smaller size, again raising the energy. The lowest energy is at zero drift velocity. We may notice in passing that they can only drift in a direction perpendicular to their plane, that is, along, or opposite, the direction of the momentum. This agrees with a property derived for rotons from their energy-momentum relation

(1), that the group velocity $\delta E(\phi)/\delta\phi$ is in the direction of the momentum, (or opposite).

Having travelled so far making one unverified conjecture upon another we may have strayed very far from the truth. However imprudent it may be, there is one further observation we would like to make. A detailed picture is not available which describes physically just what goes on as the transition is approached from below. The free energy expression arising from (6) does not of itself describe the transition. The transition occurs when the number of rotons is very large. Some sort of interaction may occur between them, or there may be some limitation to the degrees of freedom.¹⁸ There is no doubt that it is the analogue of the transition in the ideal gas, but it would be nice if we could get a less mathematical and formal description of the events. Of the following I am not sure, but it does seem to be an interesting possibility.

If rotons are the smallest ring vortices, and those of lowest energy, Δ , then there are states of higher energy corresponding to larger rings. For example, a ring of twice the diameter may have twice the energy more or less. The relative number of these will be expected to be very low, however. Since Δ is 9.6°K, at the transition $\exp(-\Delta/kT)$ is 10^{-2} , so very few larger vortices will be expected in equilibrium. Certainly none whose length is 10^2 or 10^3 atoms! This neglects an important feature, however. For a long line there are an enormous number of shapes and orientations available. Such a line is not infinitely flexible, of course, for the curvature cannot well exceed a^{-1} . It may be likened to a chain of a finite number of links. Adding one link requires an energy ε , say of order Δ , but increases the number of orientations by some factor, asymptotically, say s . In equilibrium then, the number of chains of $n + 1$ links is a factor $s \exp(-\varepsilon/kT)$ times the number with n links. For low temperatures this is less than unity. No long chains are important. The excitations consist of rotons and a few other rings of slightly larger size. As the temperature rises, however, there comes a time when the factor $s \exp(-\varepsilon/kT)$ exceeds unity. Then suddenly the rings of very largest length are of importance. The state with one vortex line (or a very few) which winds and winds throughout the liquid like a near approximation to a Jordan curve, is no longer of negligible weight. The superfluid is pierced through and through with vortex line. We are describing the disorder of Helium I. At first the curve doesn't make full use of all of its orientations and higher

entropy. But as the temperature rises a little more it squeezes into the last corners and pockets of superfluid until it has no more degrees of flexibility available. The specific heat curve drops off from the transition to a smooth curve and the memory of the possibility the helium can exhibit quantum properties in a unique way is lost in the perfusion of states and in disorder, as it is for more usual liquids.

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- ¹⁸ R. P. Feynman, *Phys. Rev.* **94**, 262, (1954), especially the discussion preceding expression (28). There is a typographical error there. The relation (28) should read as an inequality, the left side not greater than the right side.