ESTRUCTURA DE LA MATERIA 4

CURSO DE VERANO 2021

CLASE 5

RODOLFO SASSOT

CLASE 5: Ecuación de Dirac

Temas: Ec. Klein Gordon, Ec. de Dirac, soluciones.

motivación: antipartículas, energías de los aceleradores, dinámica de los quarks….

O.Klein W. Gordon (1926)

$$\begin{cases} E \to i\hbar\frac{\partial}{\partial t} \\ \vec{p} \to -i\hbar\vec{\nabla} \end{cases} \qquad E^2 = c^2p^2 + m^2c^4 \qquad -\hbar^2\frac{\partial^2\phi}{\partial t^2} = -c^2\hbar^2\nabla^2\phi + m^2c^4\phi$$

$$\hbar = c = 1 \qquad \Box \equiv \partial_\mu\partial^\mu \qquad (\Box + m^2)\,\phi = 0$$

Schrödinger:

$$\begin{cases} E \to i\hbar\frac{\partial}{\partial t} \\ \vec{p} \to -i\hbar\vec{\nabla} \end{cases} \qquad E = \frac{p^2}{2m} \qquad i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi \qquad |\psi|^2 \equiv \rho \underset{\text{(p. libre)}}{=} |N|^2$$

$$i\hbar\,\psi^*\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\psi^*\nabla^2\psi = 0$$

$$-\quad -i\hbar\,\psi\frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\psi\nabla^2\psi^* = 0$$

$$i\hbar\,\frac{\partial(\psi^*\psi)}{\partial t} + \frac{\hbar^2}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) = 0$$

$$i\hbar\,\frac{\partial\rho}{\partial t} + \frac{\hbar^2}{2m}\vec{\nabla}\cdot(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) = 0$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \qquad \vec{J} \equiv \frac{\hbar}{2mi}(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*) \underset{\text{(p. libre)}}{=} \frac{\vec{p}}{m}|N|^2$$

$$-\frac{\partial}{\partial t}\int dV\,\rho = \int dV\,\vec{\nabla}\cdot\vec{J} = \int dS\,\vec{J}\cdot\hat{n}$$

CLASE 5: Ecuación de Dirac

$$\begin{cases} E \to i\hbar\frac{\partial}{\partial t} \\ \vec{p} \to -i\hbar\vec{\nabla} \end{cases} \qquad p^\mu = (\frac{E}{c}, \vec{p}) \qquad p_\mu = (\frac{E}{c}, -\vec{p})$$

$$x^\mu = (ct, \vec{x}) \qquad x_\mu = (ct, -\vec{x})$$

$$p_\mu \longrightarrow i\hbar\partial_\mu \equiv i\hbar\frac{\partial}{\partial x^\mu}$$

$$\mu = 0 \qquad \frac{E}{c} \longrightarrow i\hbar\frac{\partial}{\partial(ct)}$$

$$\mu = i \qquad -p_i \longrightarrow i\hbar\frac{\partial}{\partial x_i}$$

$$J^\mu \equiv i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)$$ "covariantosa"

$$\partial_\mu J^\mu = 0$$

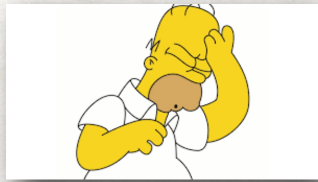
$$\phi^* KG - \phi KG^* \to \qquad \frac{\partial}{\partial t}\underbrace{\left[i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)\right]}_{\rho} + \vec{\nabla}\cdot\underbrace{\left[-i\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right)\right]}_{\vec{J}} = 0$$

partícula libre: $\phi = Ne^{i\vec{p}\cdot\vec{x} - iEt}$

$$\begin{cases} \rho = 2E|N|^2 \\ \vec{J} = 2\vec{p}\,|N|^2 \end{cases} \qquad J^\mu = 2p^\mu|N|^2$$

si $\rho = cte$ $\int d^3x\,\rho$ no es invariante! $(d^3x \longrightarrow \sqrt{1-v^2}\ d^3x)$

si $N = cte$ $\rho \sim E$ $E = \pm\sqrt{p^2+m^2}$



CLASE 5: Ecuación de Dirac

W. Pauli V. Weisskopf (1934)

$$J^{\mu} \equiv ie\left(\phi^{*}\partial^{\mu}\phi - \phi\partial^{\mu}\phi^{*}\right)$$

ρ densidad de carga

$\overrightarrow{J}$ corriente eléctrica

R. Feynman E. Stückelberg (1941)

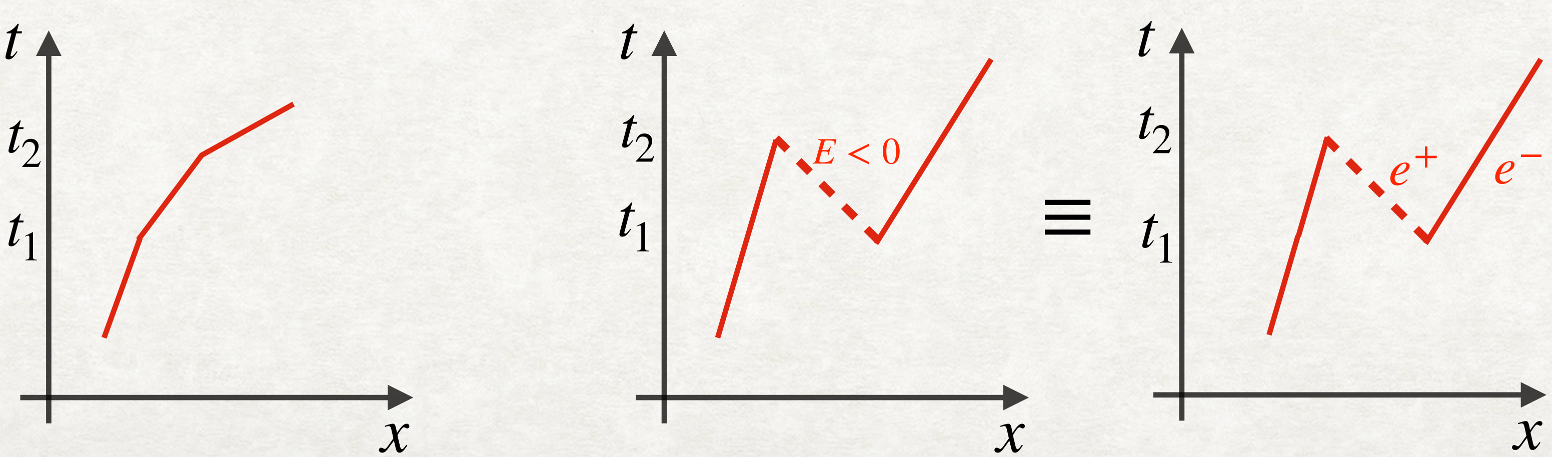
$e^{-}: \; -e, E, \overrightarrow{p} \qquad J^{\mu}_{e^-} = -2e\,(E, \overrightarrow{p})$

$e^{+}: \; e, E, \overrightarrow{p} \qquad J^{\mu}_{e^+} = 2e\,(E, \overrightarrow{p}) \; = -2e\,(-E, -\overrightarrow{p})$

e^{-} e^{+} dos "estados" de la misma entidad

$e^{-i(-E)(-t)} = e^{-iEt}$ solución $E<0$ que fuera hacia atrás en el tiempo

$\sim$ solución $E>0$ que va hacia adelante



creación/aniquilación ~
transiciones sgn(E_i) $\neq$ sgn(E_f)

CLASE 5: Ecuación de Dirac

P.A.M. Dirac (1927)

ec. K-G $$-\hbar^2\frac{\partial^2\phi}{\partial t^2} = -c^2\hbar^2\nabla^2\phi + m^2c^4\phi$$

ec. S $$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$$

$$H = \dots\ c\vec{p} + \dots mc^2$$
$$H = c\,\vec{\alpha}\cdot\vec{p} + \beta\,mc^2$$

$$\begin{cases} E \to i\hbar\frac{\partial}{\partial t} \\ \vec{p} \to -i\hbar\vec{\nabla} \end{cases}$$

$$i\hbar\frac{\partial\psi}{\partial t} = -i\hbar c\alpha_i\frac{\partial\psi}{\partial x^i} + \beta mc^2\psi$$

$$E^2 = c^2p^2 + m^2c^4$$

$$H^2\psi = \left(c\,\alpha_i p_i + \beta\,mc^2\right)\left(c\,\alpha_j p_j + \beta\,mc^2\right)\psi$$
$$= \left(c^2\alpha_i^2p_i^2 + (\alpha_i\alpha_j + \alpha_j\alpha_i)p_ip_j + (\alpha_i\beta + \beta\alpha_i)p_imc^3 + \beta^2m^2c^4\right)\psi$$

$$\alpha_i^2 = 1 \qquad \{\alpha_i, \alpha_j\} = 2\delta_{ij} \qquad \{\alpha_i, \beta\} = 0 \qquad \beta^2 = 1$$

α_i y β no son números: $\alpha_i\beta + \beta\alpha_i = 0 \quad \alpha_i = -\beta\alpha_i\beta \quad tr[\alpha_i] = -tr[\beta\alpha_i\beta]$

matrices hermíticas, autovalores± 1, traza nula, dimensión par, 4x4 $\quad tr[\alpha_i] = -tr[\alpha_i]$

CLASE 5: Ecuación de Dirac

representación de Dirac-Pauli

$$\alpha_i = \left(\begin{array}{c|c} 0 & \boldsymbol{\sigma}_i \\ \hline \boldsymbol{\sigma}_i & 0 \end{array}\right) \qquad \beta = \left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \end{array}\right)$$

representación de Weyl

$$\alpha_i = \left(\begin{array}{c|c} -\boldsymbol{\sigma}_i & 0 \\ \hline 0 & \boldsymbol{\sigma}_i \end{array}\right) \qquad \beta = \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \hline \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \end{array}\right)$$

$$H = c\,\overrightarrow{\alpha}\cdot\overrightarrow{p} + \beta\, mc^2$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

4 sol. l.i. (?) $\quad \psi^\dagger \equiv \left(\psi_1^*, \quad \psi_2^*, \quad \psi_3^*, \quad \psi_4^*\right)$

$\psi^\dagger\psi, \ \ \psi^\dagger\beta\psi, \ \ \psi^\dagger\alpha_i\psi \qquad \rho \geq 0$?

CLASE 5: Ecuación de Dirac

$$i\hbar\frac{\partial\psi}{\partial t} = -i\hbar c\alpha_i\frac{\partial\psi}{\partial x^i} + \beta mc^2\psi$$

$\longrightarrow \psi^\dagger D$

$$i\hbar\,\psi^\dagger\frac{\partial\psi}{\partial t} = -i\hbar c\,\psi^\dagger\alpha_i\frac{\partial\psi}{\partial x^i} + mc^2\,\psi^\dagger\beta\psi$$

$-$

$D^\dagger\psi \longleftarrow$

$$-i\hbar\,\frac{\partial\psi^\dagger}{\partial t}\psi = i\hbar c\,\alpha_i\frac{\partial\psi^\dagger}{\partial x^i}\psi + mc^2\,\psi^\dagger\beta\psi$$

restando m.a.m.

$$i\hbar\,\frac{\partial(\psi^\dagger\psi)}{\partial t} = -i\hbar c\,\frac{\partial(\psi^\dagger\alpha_i\psi)}{\partial x^i} = -i\hbar c\,\overrightarrow{\nabla}\cdot\psi^\dagger\overrightarrow{\alpha}\psi$$

$$\frac{\partial\rho}{\partial t} = -\overrightarrow{\nabla}\cdot\overrightarrow{J} \qquad \overrightarrow{J} \equiv c\,\psi^\dagger\overrightarrow{\alpha}\psi \qquad \rho \equiv \psi^\dagger\psi \geq 0$$

$\longrightarrow \frac{\beta}{\hbar}D$

$$i\beta\frac{\partial\psi}{\partial t} = -ic\beta\alpha_i\frac{\partial\psi}{\partial x^i} + \frac{mc^2}{\hbar}\psi$$

$\gamma^\mu \equiv (\beta, \beta\overrightarrow{\alpha})$ $\qquad (i\gamma^\mu\partial_\mu - \frac{mc^2}{\hbar})\psi = 0 \quad \xrightarrow[\hbar = c = 1]{} \quad (i\gamma^\mu\partial_\mu - m)\psi = 0$ "covariantosa"

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$\gamma^0 \equiv \beta \quad \gamma^{0\dagger} = \gamma^0 \quad (\gamma^0)^2 = \mathbb{1} \quad (\gamma^k)^2 = -\mathbb{1}$$

$$\gamma^{k\dagger} = (\beta\alpha_k)^\dagger = \alpha_k^\dagger\beta^\dagger = \alpha_k\beta = -\beta\alpha_k = -\gamma^k$$

$$\overline{\psi} \equiv \psi^\dagger\gamma^0$$

$$J^\mu \equiv \overline{\psi}\gamma^\mu\psi$$

$$\partial_\mu J^\mu = 0$$

$$\begin{cases} J^0 = \overline{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi = \rho \\ J^i = \overline{\psi}\gamma^i\psi = \psi^\dagger\gamma^0\gamma^i\psi = \psi^\dagger\alpha_i\psi = \overrightarrow{J} \end{cases}$$

CLASE 5: Ecuación de Dirac

soluciones de la ec. de Dirac $$i\hbar\frac{\partial\psi}{\partial t} = -\,i\hbar c\alpha_i\frac{\partial\psi}{\partial x^i} + \beta mc^2\psi$$

$$\overrightarrow{p} = 0 \qquad i\hbar\frac{\partial\psi}{\partial t} = \beta mc^2\psi \qquad \beta = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = -\,i\frac{mc^2}{\hbar}\begin{pmatrix}\psi_1\\ \psi_2\\ -\psi_3\\ -\psi_4\end{pmatrix}$$

4 sol. l.i. $$\psi^{(1)} = e^{-imc^2\frac{t}{\hbar}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \psi^{(2)} = e^{-imc^2\frac{t}{\hbar}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad E = mc^2$$ 2 sol. l.i. $E > 0$?

$$\psi^{(3)} = e^{+imc^2\frac{t}{\hbar}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad \psi^{(4)} = e^{+imc^2\frac{t}{\hbar}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \qquad E = -\,mc^2$$

CLASE 5: Ecuación de Dirac

soluciones de la ec. de Dirac $$i\hbar\frac{\partial\psi}{\partial t} = -\,i\hbar c\alpha_i\frac{\partial\psi}{\partial x^i} + \beta mc^2\psi$$

$$\vec{p} \neq 0 \qquad [H, \vec{P}] = 0 \qquad \begin{cases} \vec{P}\psi = \vec{p}\,\psi & -i\hbar\,\vec{\nabla}\,\psi = \vec{p}\,\psi \\ H\psi = E\psi & i\hbar\frac{\partial}{\partial t}\psi = E\psi \end{cases}$$

$$\psi(\vec{x}, t) = u(p, E)\, e^{\frac{i}{\hbar}\vec{p}\,\vec{x}}\, e^{-\frac{i}{\hbar}Et} \qquad u(p, E)$$ espinor de 4 componentes

$$= u(p, E)\, e^{-\frac{i}{\hbar}p_\mu x^\mu}$$

$$\left(c\,\vec{\alpha}\cdot\vec{p} + \beta\, mc^2\right) u(\vec{p}, E) = E\, u(\vec{p}, E) \qquad u(p, E) = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\begin{pmatrix} mc^2 & c\,\vec{\sigma}\cdot\vec{p} \\ c\,\vec{\sigma}\cdot\vec{p} & -mc^2 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = E\begin{pmatrix} u_A \\ u_B \end{pmatrix} \qquad \begin{matrix} mc^2 u_A + c\,\vec{\sigma}\cdot\vec{p}\,u_B = Eu_A & c\,\vec{\sigma}\cdot\vec{p}\,u_B = (E - mc^2)\,u_A \\ c\,\vec{\sigma}\cdot\vec{p}\,u_A - mc^2 u_B = Eu_B & c\,\vec{\sigma}\cdot\vec{p}\,u_A = (E + mc^2)\,u_B \end{matrix}$$

$$u_A = \frac{c\,\vec{\sigma}\cdot\vec{p}}{(E - mc^2)}\,u_B \qquad u_B = \frac{c\,\vec{\sigma}\cdot\vec{p}}{(E + mc^2)}\,u_A$$

$$u_A = \frac{c^2(\vec{\sigma}\cdot\vec{p})(\vec{\sigma}\cdot\vec{p})}{(E - mc^2)(E + mc^2)}\,u_A \qquad u_A = \frac{c^2p^2}{(E^2 - m^2c^4)}\,u_A \qquad E^2 = c^2p^2 + m^2c^4$$

$$(\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B} + i(\vec{A}\times\vec{B})\cdot\vec{\sigma}$$

CLASE 5: Ecuación de Dirac

soluciones de la ec. de Dirac

$$E > 0 \qquad u_A = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \qquad u_B = \frac{c\,\vec{\sigma}\cdot\vec{p}}{(E+mc^2)}\,u_A$$

$$\psi^{(1)}(\vec{x},t) = N\begin{pmatrix}1\\0\\ \left(\frac{c\,\vec{\sigma}\,\vec{p}}{E+mc^2}\right)\begin{matrix}1\\0\end{matrix}\end{pmatrix} e^{-\frac{i}{\hbar}p_\mu x^\mu} \qquad \psi^{(2)}(\vec{x},t) = N\begin{pmatrix}0\\1\\ \left(\frac{c\,\vec{\sigma}\,\vec{p}}{E+mc^2}\right)\begin{matrix}0\\1\end{matrix}\end{pmatrix} e^{-\frac{i}{\hbar}p_\mu x^\mu}$$

$$E < 0 \qquad u_B = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \qquad u_A = \frac{c\,\vec{\sigma}\cdot\vec{p}}{(E-mc^2)}\,u_B$$

$$\psi^{(3)}(\vec{x},t) = N\begin{pmatrix}\left(\frac{c\,\vec{\sigma}\,\vec{p}}{E-mc^2}\right)\begin{matrix}1\\0\end{matrix}\\ 1\\0\end{pmatrix} e^{-\frac{i}{\hbar}p_\mu x^\mu} \qquad \psi^{(4)}(\vec{x},t) = N\begin{pmatrix}\left(\frac{c\,\vec{\sigma}\,\vec{p}}{E-mc^2}\right)\begin{matrix}0\\1\end{matrix}\\ 0\\1\end{pmatrix} e^{-\frac{i}{\hbar}p_\mu x^\mu}$$

$$\left(\frac{c\,\vec{\sigma}\cdot\vec{p}}{E+mc^2}\right) \sim \mathcal{O}\left(\frac{v}{c}\right)$$

p.ej. $\vec{p} = p\hat{k}$

$$\vec{\sigma}\cdot\vec{p} = \sigma_3\,p$$
$$p \simeq mv$$
$$E \simeq mc^2$$

$$\left(\frac{c\,\vec{\sigma}\cdot\vec{p}}{E+mc^2}\right)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{c\,mv}{2mc^2}\\0\end{pmatrix} = \begin{pmatrix}\frac{v}{2c}\\0\end{pmatrix}$$

CLASE 5: Ecuación de Dirac

soluciones de la ec. de Dirac

2 sol. l.i. para el mismo valor de E y de $\overrightarrow{p}$: H y $\overrightarrow{P}$ no son un conjunto completo …

$$\overrightarrow{p} = 0 \qquad \Sigma_3 \equiv \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \qquad \begin{array}{l} \Sigma_3 \psi^{(1)} = +\psi^{(1)} \\ \Sigma_3 \psi^{(2)} = -\psi^{(2)} \end{array}$$

$$\overrightarrow{p} \neq 0 \qquad [H, \Sigma_i] \neq 0 \qquad \frac{d\overrightarrow{\Sigma}}{dt} = \frac{i}{\hbar}[H, \overrightarrow{\Sigma}] = -\frac{2c}{\hbar}\overrightarrow{\alpha} \times \overrightarrow{p}$$

$$\frac{d\overrightarrow{\Sigma} \cdot \hat{p}}{dt} = -\frac{2c}{\hbar}\overrightarrow{\alpha} \times \overrightarrow{p} \cdot \hat{p}$$

helicidad

$$[H, L_i] \neq 0 \qquad \overrightarrow{L} \equiv \vec{r} \times \overrightarrow{p}$$

$$\frac{d\overrightarrow{L}}{dt} = \frac{i}{\hbar}[H, \overrightarrow{L}] = c\overrightarrow{\alpha} \times \overrightarrow{p}$$

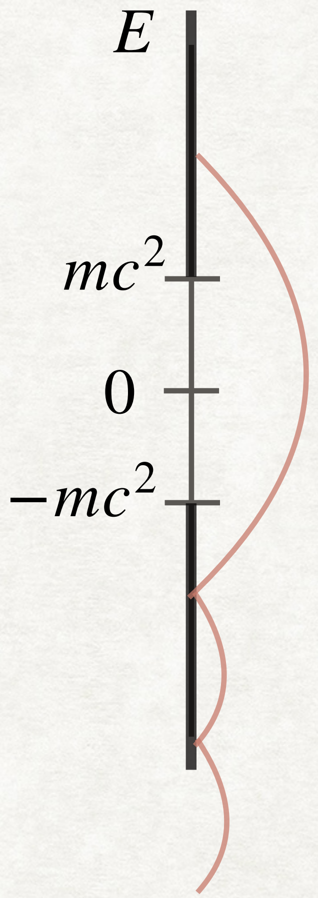
$$\overrightarrow{J} \equiv \overrightarrow{L} + \frac{\hbar}{2}\overrightarrow{\Sigma} \qquad [H, J_i] = 0$$

$\overrightarrow{\Sigma}$ ~ momento angular intrinseco

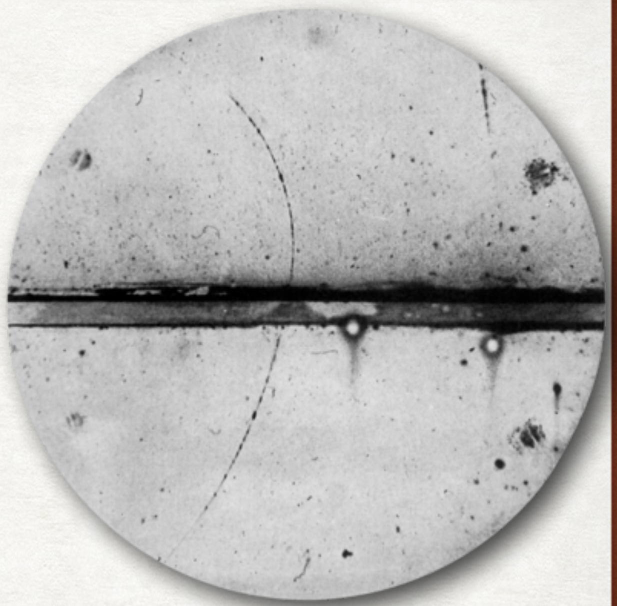
W. Pauli (1924)

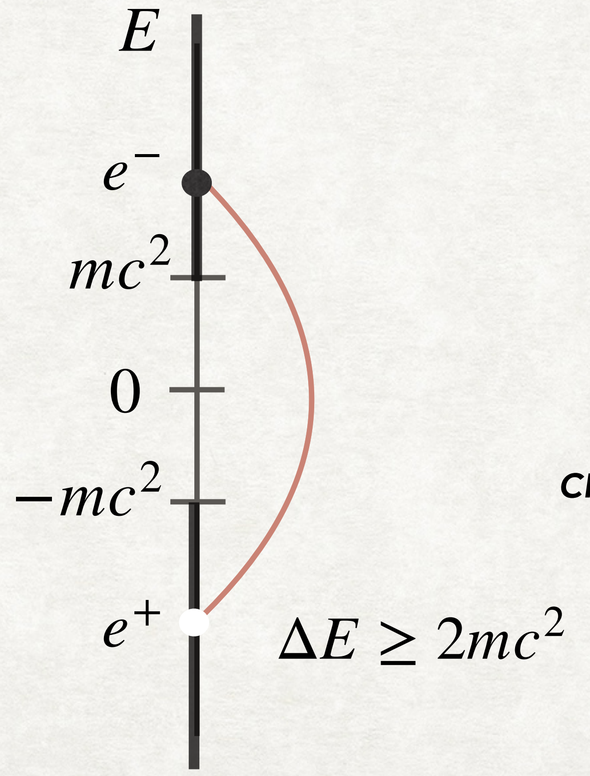
CLASE 5: Ecuación de Dirac

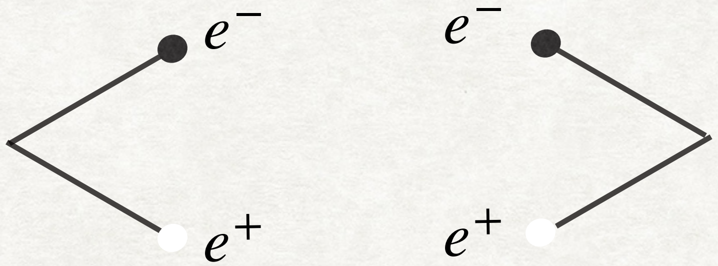
soluciones de energía negativa



todos los estados $E < 0$, y como son fermiones …







creación y aniquilación de pares partícula-antipartícula