

ESTRUCTURA DE LA MATERIA 4

2DO CUATRIMESTRE 2025

CLASE 8

RODOLFO SASSOT

CLASE 8: Covariancia y transformación de espinores

Covariancia: *todos tienen razón*



en S $x^\mu = (ct, \vec{x})$

$$x'^\mu = \Lambda^\mu_\nu x^\nu$$

en S' $x'^\mu = (ct', \vec{x}')$

$$x^\nu \rightarrow \begin{pmatrix} ct \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \Lambda \rightarrow \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{aligned} \gamma &\equiv 1/\sqrt{1-\beta^2} \\ \beta &\equiv v/c \end{aligned}$$

boost en x

vector $V_i(x) \longrightarrow V'_i(x') = R_{ij} V_j(x)$ R_{ij} matriz de 3x3

cuadrivector $A^\mu(x) \longrightarrow A'^\mu(x') = \Lambda^\mu_\nu A^\nu(x)$

$\psi(x)?$ $\psi(x) \longrightarrow \psi'(x') = S(\Lambda) \psi(x)$

CLASE 8: Covariancia y transformación de espinores

Covariancia: *todos tienen razón*



$$\text{en } S \quad \left(i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - mc \right) \psi(x) = 0 \quad (1)$$

$$\text{en } S' \quad \left(i\hbar \gamma^\mu \frac{\partial}{\partial x'^\mu} - mc \right) \psi'(x') = 0 \quad (2)$$

$$x'^\nu = \Lambda^\nu_\mu x^\mu$$

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu} \frac{\partial}{\partial x'^\nu} = \Lambda^\nu_\mu \frac{\partial}{\partial x'^\nu}$$

$$\psi(x) = S^{-1}(\Lambda) \psi'(x')$$

$$\text{en (1)} \quad \left(i\hbar \gamma^\mu \Lambda^\nu_\mu \frac{\partial}{\partial x'^\nu} - mc \right) S^{-1}(\Lambda) \psi'(x') = 0$$

$$S \rightarrow \quad \left(i\hbar S(\Lambda) \gamma^\mu \Lambda^\nu_\mu S^{-1}(\Lambda) \frac{\partial}{\partial x'^\nu} - mc \right) \psi'(x') = 0$$

$$S(\Lambda) \gamma^\mu \Lambda^\nu_\mu S^{-1}(\Lambda) = \gamma^\nu$$

$$S(\Lambda) \Lambda^\nu_\mu \gamma^\mu S^{-1}(\Lambda) = \gamma^\nu$$

$$\Lambda^\nu_\mu \gamma^\mu = S^{-1}(\Lambda) \gamma^\nu S(\Lambda)$$

CLASE 8: Covariancia y transformación de espinores

Ej. rotación en z:

para un espinor $S=1/2$ $\sim e^{-i\frac{\theta}{2}\vec{\sigma}\cdot\hat{n}}$



$$S_{rot_z}(\Lambda) = e^{-i\frac{\theta}{2}\Sigma_3} = \cos\frac{\theta}{2} - i\Sigma_3 \sin\frac{\theta}{2} = e^{i\gamma_1\gamma_2\frac{\theta}{2}}$$

$$\Sigma_3 \equiv \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = i\gamma_1\gamma_2$$

$$\sigma_{\mu\nu} \equiv \frac{i}{2}[\gamma_\mu, \gamma_\nu] = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) = i\gamma_\mu\gamma_\nu \quad (\mu \neq \nu)$$

$$S_{rot_z}(\Lambda) = e^{-i\sigma_{12}\frac{\theta}{2}}$$

Ej. boost en x:

$$\begin{cases} x'^1 = x^1 \cos\theta - x^2 \sin\theta \\ x'^2 = x^2 \cos\theta + x^1 \sin\theta \end{cases} \quad \begin{cases} x'^0 = \gamma x^0 - \gamma\beta x^1 = x^0 \cosh\omega - x^1 \sinh\omega \\ x'^1 = \gamma x^1 - \gamma\beta x^0 = x^1 \cosh\omega - x^0 \sinh\omega \end{cases}$$

$$\begin{aligned} \cosh\omega &\equiv \gamma \\ \sinh\omega &\equiv \gamma\beta \quad \text{tgh}\omega = \beta = v/c \end{aligned}$$

$$\begin{aligned} S_{boost_x}(\Lambda) &= e^{-i\sigma_{01}\frac{\omega}{2}} \\ &= e^{-\alpha_1\frac{\omega}{2}} \end{aligned}$$

$$-i\sigma_{01} = \gamma_0\gamma_1 = -\gamma^0\gamma^1 = -\beta\alpha_1 = -\alpha_1$$

$$\begin{aligned} S_{boost_x}(\Lambda) &= \cosh\frac{\omega}{2} - \alpha_1 \sinh\frac{\omega}{2} \\ &= \cosh\frac{\omega}{2} - \gamma^0\gamma^1 \sinh\frac{\omega}{2} \end{aligned}$$

CLASE 8: Covariancia y transformación de espinores

inversa: $S^{-1}(\Lambda) \quad \theta \rightarrow -\theta \quad \omega \rightarrow -\omega \quad S^{-1}S = \mathbb{1}$



para boosts: $S_{boost_x}(\Lambda) = \cosh \frac{\omega}{2} - \gamma^0 \gamma^1 \sinh \frac{\omega}{2}$

$$\begin{aligned} S^{-1}S &= \left(\cosh \frac{\omega}{2} + \gamma^0 \gamma^1 \sinh \frac{\omega}{2} \right) \left(\cosh \frac{\omega}{2} - \gamma^0 \gamma^1 \sinh \frac{\omega}{2} \right) \\ &= \left(\cosh^2 \frac{\omega}{2} - \gamma^0 \gamma^1 \gamma^0 \gamma^1 \sinh^2 \frac{\omega}{2} \right) = \left(\cosh^2 \frac{\omega}{2} + \underset{(+1)(-1)}{\gamma^0 \gamma^0 \gamma^1 \gamma^1} \sinh^2 \frac{\omega}{2} \right) = \mathbb{1} \end{aligned}$$

para rotaciones: $S_{rot_z}(\Lambda) = \cos \frac{\theta}{2} + \gamma^1 \gamma^2 \sin \frac{\theta}{2}$

$$\begin{aligned} S^{-1}S &= \left(\cos \frac{\theta}{2} - \gamma^1 \gamma^2 \sin \frac{\theta}{2} \right) \left(\cos \frac{\theta}{2} + \gamma^1 \gamma^2 \sin \frac{\theta}{2} \right) \\ &= \left(\cos^2 \frac{\theta}{2} - \gamma^1 \gamma^2 \gamma^1 \gamma^2 \sin^2 \frac{\theta}{2} \right) = \left(\cos^2 \frac{\theta}{2} + \underset{(-1)(-1)}{\gamma^1 \gamma^1 \gamma^2 \gamma^2} \sin^2 \frac{\theta}{2} \right) = \mathbb{1} \end{aligned}$$

CLASE 8: Covariancia y transformación de espinores



$$S^{-1}(\Lambda) \gamma^\mu S(\Lambda) = \Lambda^\mu_\nu \gamma^\nu$$

para rotaciones:

$$(\cos \frac{\theta}{2} - \gamma^1 \gamma^2 \sin \frac{\theta}{2}) \begin{pmatrix} \gamma^0 \\ \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix} (\cos \frac{\theta}{2} + \gamma^1 \gamma^2 \sin \frac{\theta}{2}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma^0 \\ \gamma^1 \\ \gamma^2 \\ \gamma^3 \end{pmatrix}$$

1ra fila $\gamma^0 S^{-1} S = \gamma^0$

4ta fila $\gamma^3 S^{-1} S = \gamma^3$

2da fila $\gamma^1 (\cos \frac{\theta}{2} + \gamma^1 \gamma^2 \sin \frac{\theta}{2}) (\cos \frac{\theta}{2} + \gamma^1 \gamma^2 \sin \frac{\theta}{2}) = (\gamma^1 \cos \theta - \gamma^2 \sin \theta)$

$$\gamma^1 (\cos^2 \frac{\theta}{2} + \gamma^1 \gamma^2 \gamma^1 \gamma^2 \sin^2 \frac{\theta}{2} + 2 \gamma^1 \gamma^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2})$$

$$\gamma^1 (\cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} + 2 \gamma^1 \gamma^2 \sin \frac{\theta}{2} \cos \frac{\theta}{2})$$

$$\gamma^1 (\cos \theta + \gamma^1 \gamma^2 \sin \theta)$$

$$\gamma^1 \cos \theta - \gamma^2 \sin \theta$$

CLASE 8: Covariancia y transformación de espinores



$$S^\dagger \gamma^0 = \gamma^0 S^{-1} \quad S^\dagger = \gamma^0 S^{-1} \gamma^0$$

para rotaciones:

$$S = \left(\cos \frac{\theta}{2} + \gamma^1 \gamma^2 \sin \frac{\theta}{2} \right)$$

$$\begin{aligned} S^\dagger &= \left(\cos \frac{\theta}{2} + \gamma^{2\dagger} \gamma^{1\dagger} \sin \frac{\theta}{2} \right) = \left(\cos \frac{\theta}{2} - \gamma^1 \gamma^2 \sin \frac{\theta}{2} \right) = (\gamma^0 \gamma^0 \cos \frac{\theta}{2} - \gamma^0 \gamma^0 \gamma^1 \gamma^2 \sin \frac{\theta}{2}) \\ &= (\gamma^0 \cos \frac{\theta}{2} \gamma^0 - \gamma^0 \gamma^1 \gamma^2 \gamma^0 \sin \frac{\theta}{2}) = \gamma^0 \left(\cos \frac{\theta}{2} - \gamma^1 \gamma^2 \sin \frac{\theta}{2} \right) \gamma^0 \end{aligned}$$

para boosts:

$$S = \cosh \frac{\omega}{2} - \gamma^0 \gamma^1 \sinh \frac{\omega}{2}$$

$$\begin{aligned} S^\dagger &= \cosh \frac{\omega}{2} - \gamma^{1\dagger} \gamma^{0\dagger} \sinh \frac{\omega}{2} = \cosh \frac{\omega}{2} - \gamma^0 \gamma^1 \sinh \frac{\omega}{2} = \gamma^0 \gamma^0 \cosh \frac{\omega}{2} - \gamma^0 \gamma^0 \gamma^0 \gamma^1 \sinh \frac{\omega}{2} \\ &= \gamma^0 \cosh \frac{\omega}{2} \gamma^0 + \gamma^0 \gamma^0 \gamma^1 \sinh \frac{\omega}{2} \gamma^0 = \gamma^0 \left(\cosh \frac{\omega}{2} + \gamma^0 \gamma^1 \sinh \frac{\omega}{2} \right) \gamma^0 \end{aligned}$$

CLASE 8: Covariancia y transformación de espinores



$$S^\dagger \gamma^0 = \gamma^0 S^{-1} \quad S^\dagger = \gamma^0 S^{-1} \gamma^0$$

$$\bar{\psi} \equiv \psi^\dagger \gamma^0$$

$$\bar{\psi}' = ? \quad \psi' = S\psi$$

$$\psi'^\dagger = (S\psi)^\dagger = \psi^\dagger S^\dagger$$

$$\bar{\psi}' = \psi'^\dagger \gamma^0 = \psi^\dagger S^\dagger \gamma^0 = \psi^\dagger \gamma^0 S^{-1} = \bar{\psi} S^{-1}$$

$$J'^\mu = \bar{\psi}' \gamma^\mu \psi' = \bar{\psi} S^{-1} \gamma^\mu S \psi = \Lambda_\nu^\mu \bar{\psi} \gamma^\nu \psi = \Lambda_\nu^\mu J^\nu \quad \partial_\mu J^\mu = 0$$

$$\rho^0 = \bar{\psi} \gamma^0 \psi = \psi^\dagger \psi \quad \text{no es invariante}$$

$$\bar{\psi}' \psi' = \bar{\psi} S^{-1} S \psi = \bar{\psi} \psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2$$

CLASE 8: Covariancia y transformación de espinores

Paridad Intrínseca:

$$\vec{x} \longrightarrow -\vec{x} \quad \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$S_p = ? \quad S^{-1}(\Lambda) \gamma^\mu S(\Lambda) = \Lambda^\mu_\nu \gamma^\nu$$

$$S_p^{-1} \gamma^0 S_p = \gamma^0 \quad \gamma^0 S_p = S_p \gamma^0$$

$$S_p^{-1} \gamma^i S_p = -\gamma^i \quad \gamma^i S_p = -S_p \gamma^i$$

$$S_p = \gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

partículas y antipartículas tienen paridades opuestas

