FISICA DE LAS INTERACCIONES FUNDAMENTALES

1ER CUATRIMESTRE 2026

CLASE 5

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CLASE 5: Ecuación de Dirac

Temas: Ec. Klein Gordon, Ec. de Dirac, soluciones.

motivación: antipartículas, energías de los aceleradores, dinámica de los quarks….

O.Klein W. Gordon (1926)

$$\begin{cases} E \to i\hbar \frac{\partial}{\partial t} \\ \vec{p} \to -i\hbar \vec{\nabla} \end{cases} \qquad E^2 = c^2 p^2 + m^2 c^4 \qquad -\hbar^2 \frac{\partial^2 \phi}{\partial t^2} = -c^2 \hbar^2 \nabla^2 \phi + m^2 c^4 \phi$$

$$\hbar = c = 1 \qquad \square \equiv \partial_\mu \partial^\mu \qquad (\square + m^2)\,\phi = 0$$

Schrödinger (1925):

$$\begin{cases} E \to i\hbar \frac{\partial}{\partial t} \\ \vec{p} \to -i\hbar \vec{\nabla} \end{cases} \qquad E = \frac{p^2}{2m} \qquad i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi \qquad |\psi|^2 \equiv \rho \underset{\text{(p. libre)}}{=} |N|^2$$

$$i\hbar\, \psi^* \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \psi^* \nabla^2 \psi = 0$$

$$-$$

$$-i\hbar\, \psi \frac{\partial \psi^*}{\partial t} + \frac{\hbar^2}{2m} \psi \nabla^2 \psi^* = 0$$

$$i\hbar\, \frac{\partial (\psi^* \psi)}{\partial t} + \frac{\hbar^2}{2m} (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) = 0$$

$$i\hbar\, \frac{\partial \rho}{\partial t} + \frac{\hbar^2}{2m} \vec{\nabla} \cdot (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) = 0$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0 \qquad \vec{J} \equiv \frac{\hbar}{2mi} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*) \underset{\text{(p. libre)}}{=} \frac{\vec{p}}{m} |N|^2$$

$$-\frac{\partial}{\partial t} \int dV \rho = \int dV\, \vec{\nabla} \cdot \vec{J} = \int dS\, \vec{J} \cdot \hat{n}$$

CLASE 5: Ecuación de Dirac

$$\begin{cases} E \to i\hbar \frac{\partial}{\partial t} \\ \vec{p} \to -i\hbar\vec{\nabla} \end{cases} \qquad p^\mu = (\frac{E}{c}, \vec{p}\,) \qquad p_\mu = (\frac{E}{c}, -\vec{p}\,)$$

$$x^\mu = (ct, \vec{x}\,) \qquad x_\mu = (ct, -\vec{x}\,)$$

$$p_\mu \longrightarrow i\hbar\partial_\mu \equiv i\hbar\frac{\partial}{\partial x^\mu}$$

$$\mu = 0 \qquad \frac{E}{c} \longrightarrow i\hbar\frac{\partial}{\partial(ct)}$$

$$\mu = i \qquad -p_i \longrightarrow i\hbar\frac{\partial}{\partial x_i}$$

$$J^\mu \equiv i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)$$ "covariantosa"

$$\partial_\mu J^\mu = 0$$

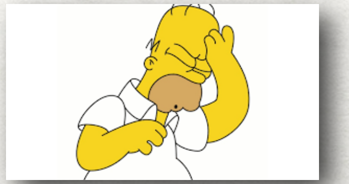
$$\phi^* KG - \phi KG^* \to \qquad \frac{\partial}{\partial t}\underbrace{\left[i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right)\right]}_{\rho} + \vec{\nabla}\cdot\underbrace{\left[-i\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right)\right]}_{\vec{J}} = 0$$

partícula libre: $\phi = Ne^{i\vec{p}\cdot\vec{x} - iEt}$

$$\begin{cases} \rho = 2E|N|^2 \\ \vec{J} = 2\vec{p}\,|N|^2 \end{cases} \qquad J^\mu = 2p^\mu|N|^2$$

si $\rho = cte$ $\quad\int d^3x\,\rho$ no es invariante! $\quad (d^3x \longrightarrow \sqrt{1-v^2}\;d^3x)$

si $N = cte$ $\qquad \rho \sim E \qquad E = \pm\sqrt{p^2 + m^2}$



CLASE 5: Ecuación de Dirac

W. Pauli V. Weisskopf (1934)

$$J^\mu \equiv ie\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)$$

ρ densidad de carga

$\overrightarrow{J}$ corriente eléctrica

R. Feynman E. Stückelberg (1941)

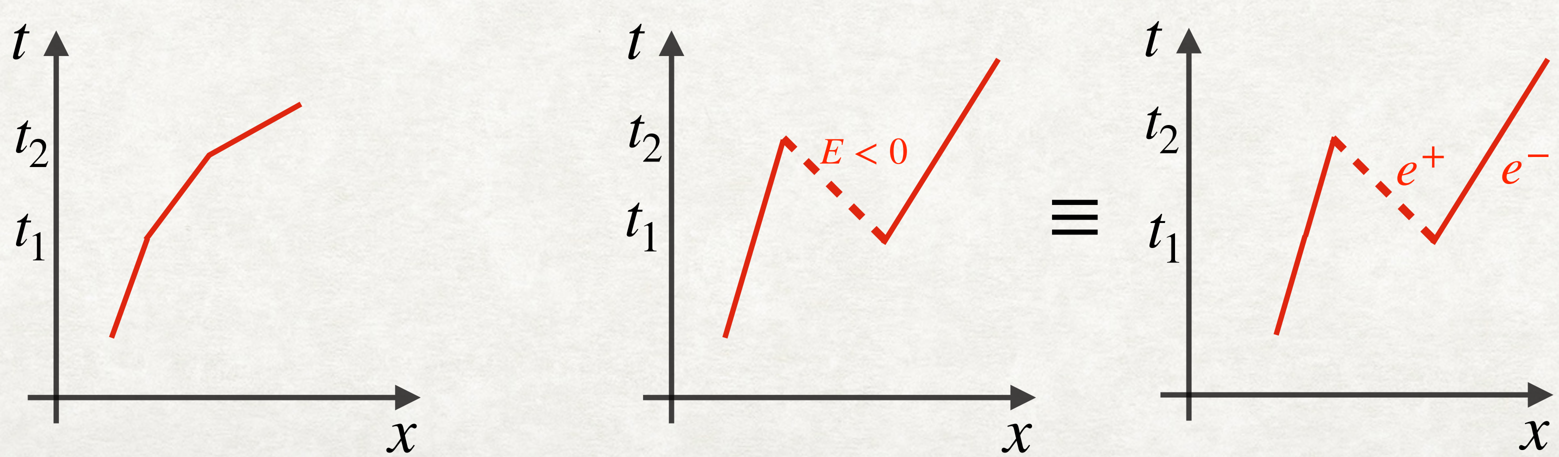
$e^-: \; -e, E, \overrightarrow{p}$ $\quad J^\mu_{e^-} = -2e\,(E, \overrightarrow{p})$

$e^+: \; e, E, \overrightarrow{p}$ $\quad J^\mu_{e^+} = 2e\,(E, \overrightarrow{p}) \; = -2e\,(-E, -\overrightarrow{p})$

e^- e^+ dos "estados" de la misma entidad

$e^{-i(-E)(-t)} = e^{-iEt}$ solución $E < 0$ que fuera hacia atrás en el tiempo

$\sim$ solución $E > 0$ que va hacia adelante



creación/aniquilación ~

transiciones sgn(E_i) $\neq$ sgn(E_f)

CLASE 5: Ecuación de Dirac

P.A.M. Dirac (1927)

ec. K-G $-\hbar^2\frac{\partial^2\phi}{\partial t^2} = -c^2\hbar^2\nabla^2\phi + m^2c^4\phi$

ec. S $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi$

$$H = \dots\ c\vec{p} + \dots mc^2$$
$$H = c\,\vec{\alpha}\cdot\vec{p} + \beta\,mc^2$$

$$\begin{cases} E \to i\hbar\frac{\partial}{\partial t} \\ \vec{p} \to -i\hbar\vec{\nabla} \end{cases}$$

$$i\hbar\frac{\partial\psi}{\partial t} = -i\hbar c\alpha_i\frac{\partial\psi}{\partial x^i} + \beta mc^2\psi$$

$$E^2 = c^2p^2 + m^2c^4$$

$$H^2\psi = \left(c\,\alpha_i p_i + \beta\,mc^2\right)\left(c\,\alpha_j p_j + \beta\,mc^2\right)\psi$$
$$= \left(c^2\alpha_i^2 p_i^2 + c^2(\alpha_i\alpha_j + \alpha_j\alpha_i)p_ip_j + (\alpha_i\beta + \beta\alpha_i)p_imc^3 + \beta^2m^2c^4\right)\psi$$

$$\alpha_i^2 = 1 \qquad \{\alpha_i, \alpha_j\} = 2\delta_{ij} \qquad \{\alpha_i, \beta\} = 0 \qquad \beta^2 = 1$$

α_i y β no son números: $\alpha_i\beta + \beta\alpha_i = 0 \quad \alpha_i = -\beta\alpha_i\beta \quad tr[\alpha_i] = -tr[\beta\alpha_i\beta]$

matrices hermíticas, autovalores± 1, traza nula, dimensión par, 4x4 $\quad tr[\alpha_i] = -tr[\alpha_i]$

CLASE 5: Ecuación de Dirac

representación de Dirac-Pauli

$$\alpha_i = \left(\begin{array}{c|c} 0 & \boldsymbol{\sigma}_i \\ \hline \boldsymbol{\sigma}_i & 0 \end{array}\right) \qquad \beta = \left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \end{array}\right)$$

representación de Weyl

$$\alpha_i = \left(\begin{array}{c|c} -\boldsymbol{\sigma}_i & 0 \\ \hline 0 & \boldsymbol{\sigma}_i \end{array}\right) \qquad \beta = \left(\begin{array}{c|c} 0 & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \hline \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} & 0 \end{array}\right)$$

$$H = c\,\overrightarrow{\alpha}\cdot\overrightarrow{p} + \beta\, mc^2$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

4 sol. l.i. (?)

$$\psi^\dagger \equiv \left(\psi_1^*, \quad \psi_2^*, \quad \psi_3^*, \quad \psi_4^*\right)$$

$$\psi^\dagger\psi, \;\; \psi^\dagger\beta\psi, \;\; \psi^\dagger\alpha_i\psi$$

$\rho \geq 0$?

CLASE 5: Ecuación de Dirac

$$i\hbar\frac{\partial\psi}{\partial t} = -i\hbar c\alpha_i\frac{\partial\psi}{\partial x^i} + \beta mc^2\psi$$

$\longrightarrow \psi^\dagger D$
$$i\hbar\,\psi^\dagger\frac{\partial\psi}{\partial t} = -i\hbar c\,\psi^\dagger\alpha_i\frac{\partial\psi}{\partial x^i} + mc^2\,\psi^\dagger\beta\psi$$

$-$

$D^\dagger\psi \longleftarrow$
$$-i\hbar\,\frac{\partial\psi^\dagger}{\partial t}\psi = i\hbar c\,\alpha_i\frac{\partial\psi^\dagger}{\partial x^i}\psi + mc^2\,\psi^\dagger\beta\psi$$

restando m.a.m.
$$i\hbar\,\frac{\partial(\psi^\dagger\psi)}{\partial t} = -i\hbar c\,\frac{\partial(\psi^\dagger\alpha_i\psi)}{\partial x^i} = -i\hbar c\,\vec{\nabla}\cdot\psi^\dagger\vec{\alpha}\psi$$

$$\frac{\partial\rho}{\partial t} = -\vec{\nabla}\cdot\vec{J} \qquad \vec{J}\equiv c\,\psi^\dagger\vec{\alpha}\psi \qquad \rho\equiv\psi^\dagger\psi\geq 0$$

$\longrightarrow \frac{\beta}{\hbar}D$
$$i\beta\frac{\partial\psi}{\partial t} = -ic\beta\alpha_i\frac{\partial\psi}{\partial x^i} + \frac{mc^2}{\hbar}\psi$$

$$\gamma^\mu\equiv(\beta,\beta\vec{\alpha}) \qquad (i\gamma^\mu\partial_\mu - \frac{mc^2}{\hbar})\psi = 0 \quad \xrightarrow[\hbar=c=1]{} \quad (i\gamma^\mu\partial_\mu - m)\psi = 0$$

"covariantosa"

$$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$$\gamma^0\equiv\beta \quad \gamma^{0\dagger}=\gamma^0 \quad (\gamma^0)^2 = \mathbb{1} \quad (\gamma^k)^2 = -\mathbb{1}$$

$$\gamma^{k\dagger} = (\beta\alpha_k)^\dagger = \alpha_k^\dagger\beta^\dagger = \alpha_k\beta = -\beta\alpha_k = -\gamma^k$$

$$\bar{\psi}\equiv\psi^\dagger\gamma^0$$

$$J^\mu\equiv\bar{\psi}\gamma^\mu\psi$$

$$\partial_\mu J^\mu = 0$$

$$\begin{cases} J^0 = \bar{\psi}\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi = \rho \\ J^i = \bar{\psi}\gamma^i\psi = \psi^\dagger\gamma^0\gamma^i\psi = \psi^\dagger\alpha_i\psi = \vec{J} \end{cases}$$

CLASE 5: Ecuación de Dirac

soluciones de la ec. de Dirac $$i\hbar\frac{\partial\psi}{\partial t} = -\,i\hbar c\alpha_i\frac{\partial\psi}{\partial x^i} + \beta mc^2\psi$$

$$\overrightarrow{p} = 0 \qquad i\hbar\frac{\partial\psi}{\partial t} = \beta mc^2\psi \qquad \beta = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

$$\frac{\partial}{\partial t}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = -\,i\frac{mc^2}{\hbar}\begin{pmatrix}\psi_1\\ \psi_2\\ -\psi_3\\ -\psi_4\end{pmatrix}$$

4 sol. l.i. $$\psi^{(1)} = e^{-i\,mc^2\frac{t}{\hbar}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \psi^{(2)} = e^{-i\,mc^2\frac{t}{\hbar}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad E = mc^2$$ 2 sol. l.i. $E > 0$?

$$\psi^{(3)} = e^{+i\,mc^2\frac{t}{\hbar}}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad \psi^{(4)} = e^{+i\,mc^2\frac{t}{\hbar}}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} \qquad E = -\,mc^2$$

CLASE 5: Ecuación de Dirac

soluciones de la ec. de Dirac

2 sol. l.i. para el mismo valor de E y de $\vec{p}$: H y $\vec{P}$ no son un conjunto completo …

$$\vec{p} = 0 \qquad \Sigma_3 \equiv \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \qquad \begin{matrix} \Sigma_3 \psi^{(1)} = +\psi^{(1)} \\ \Sigma_3 \psi^{(2)} = -\psi^{(2)} \end{matrix}$$

$$\vec{p} \neq 0 \qquad [H, \Sigma_i] \neq 0 \qquad \frac{d\vec{\Sigma}}{dt} = \frac{i}{\hbar}[H, \vec{\Sigma}] = -\frac{2c}{\hbar}\vec{\alpha} \times \vec{p}$$

$$\frac{d\vec{\Sigma} \cdot \hat{p}}{dt} = -\frac{2c}{\hbar}\vec{\alpha} \times \vec{p} \cdot \hat{p}$$

helicidad

$$[H, L_i] \neq 0 \qquad \vec{L} \equiv \vec{r} \times \vec{p}$$

$$\frac{d\vec{L}}{dt} = \frac{i}{\hbar}[H, \vec{L}] = c\,\vec{\alpha} \times \vec{p}$$

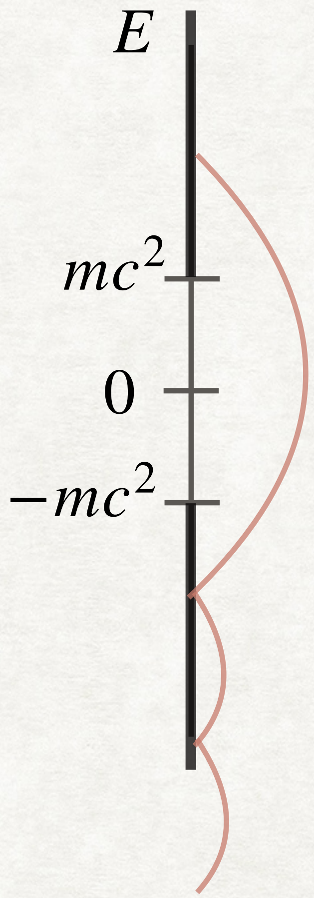
$$\vec{J} \equiv \vec{L} + \frac{\hbar}{2}\vec{\Sigma} \qquad [H, J_i] = 0$$

$\vec{\Sigma}$ ~ momento angular intrínseco

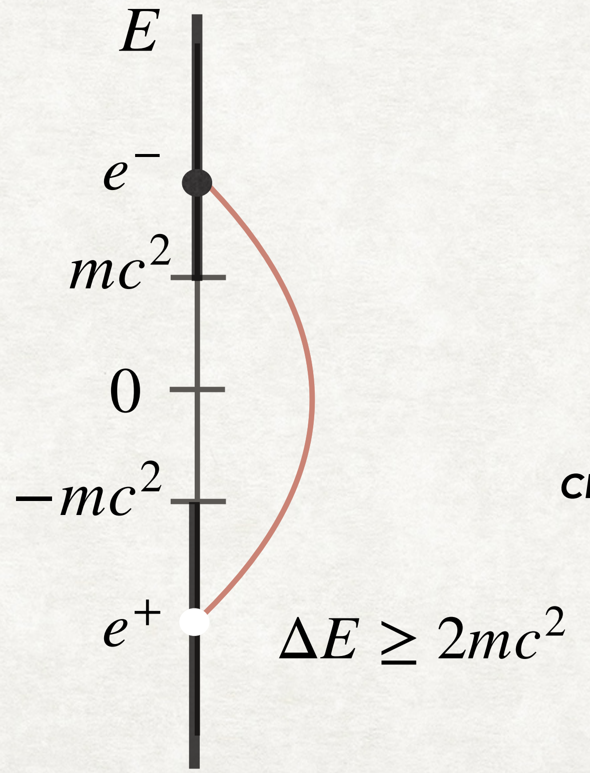
W. Pauli (1924)

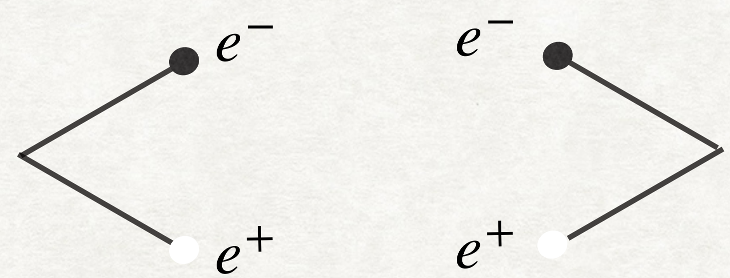
CLASE 5: Ecuación de Dirac

soluciones de energía negativa



todos los estados $E < 0$, y como son fermiones …





creación y aniquilación de pares partícula-antipartícula