FISICA DE LAS INTERACCIONES FUNDAMENTALES

1ER CUATRIMESTRE 2026

CLASE 6

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CLASE 6: Ecuación de Dirac

recapitulación:

$$\begin{cases} E \rightarrow i\hbar\frac{\partial}{\partial t} \\ \overrightarrow{p} \rightarrow -i\hbar\overrightarrow{\nabla} \end{cases}$$

$$H = c\,\overrightarrow{\alpha}\cdot\overrightarrow{p} + \beta mc^2 \qquad E^2 = c^2p^2 + m^2c^4$$

$$\alpha_i^2 = 1 \quad \beta^2 = 1 \quad \{\alpha_i, \alpha_j\} = 2\delta_{ij} \quad \{\alpha_i, \beta\} = 0$$

$$H\psi = E\psi \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

2 sol. l.i. $E > 0$

2 sol. l.i. $E < 0$

$$\rho \equiv \psi^\dagger\psi \geq 0$$

$$\psi^{(1)} = e^{-imc^2\frac{t}{\hbar}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \psi^{(2)} = e^{-imc^2\frac{t}{\hbar}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \psi^{(3)} = e^{+imc^2\frac{t}{\hbar}}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \psi^{(4)} = e^{+imc^2\frac{t}{\hbar}}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$E = mc^2 \qquad\qquad E = -mc^2$$

CLASE 6: Ecuación de Dirac

soluciones de la ec. de Dirac $$i\hbar\frac{\partial\psi}{\partial t} = -\, i\hbar c\alpha_i\frac{\partial\psi}{\partial x^i} + \beta mc^2\psi$$

$$\overrightarrow{p} \neq 0 \qquad [H, \overrightarrow{P}] = 0 \qquad \begin{cases} \overrightarrow{P}\psi = \overrightarrow{p}\psi & -i\hbar\overrightarrow{\nabla}\psi = \overrightarrow{p}\psi \\ H\psi = E\psi & i\hbar\dfrac{\partial}{\partial t}\psi = E\psi \end{cases}$$

$$\psi(\overrightarrow{x}, t) = u(p, E)\, e^{\frac{i}{\hbar}\overrightarrow{p}\,\overrightarrow{x}} e^{-\frac{i}{\hbar}Et} \qquad u(p, E)$$ espinor de 4 componentes

$$= u(p, E)\, e^{-\frac{i}{\hbar}p_\mu x^\mu}$$

$$\left(c\,\overrightarrow{\alpha}\cdot\overrightarrow{p} + \beta\, mc^2\right) u(\overrightarrow{p}, E) = E\, u(\overrightarrow{p}, E) \qquad u(p, E) = \begin{pmatrix} u_A \\ u_B \end{pmatrix} \qquad \alpha_i = \left(\begin{array}{c|c} 0 & \sigma_i \\ \hline \sigma_i & 0 \end{array}\right) \qquad \beta = \left(\begin{array}{c|c} \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} & 0 \\ \hline 0 & \begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \end{array}\right)$$

$$\begin{pmatrix} mc^2 & c\,\overrightarrow{\sigma}\cdot\overrightarrow{p} \\ c\,\overrightarrow{\sigma}\cdot\overrightarrow{p} & -mc^2 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = E\begin{pmatrix} u_A \\ u_B \end{pmatrix} \qquad \begin{matrix} mc^2u_A + c\,\overrightarrow{\sigma}\cdot\overrightarrow{p}\,u_B = Eu_A \\ c\,\overrightarrow{\sigma}\cdot\overrightarrow{p}\,u_A - mc^2u_B = Eu_B \end{matrix} \qquad \begin{matrix} c\overrightarrow{\sigma}\cdot\overrightarrow{p}\,u_B = (E - mc^2)\,u_A \\ c\overrightarrow{\sigma}\cdot\overrightarrow{p}\,u_A = (E + mc^2)\,u_B \end{matrix}$$

$$u_A = \frac{c\overrightarrow{\sigma}\cdot\overrightarrow{p}}{(E - mc^2)}u_B$$

$$u_A = \frac{c^2(\overrightarrow{\sigma}\cdot\overrightarrow{p})(\overrightarrow{\sigma}\cdot\overrightarrow{p})}{(E - mc^2)(E + mc^2)}u_A \qquad u_A = \frac{c^2p^2}{(E^2 - m^2c^4)}u_A \qquad E^2 = c^2p^2 + m^2c^4$$

$$u_B = \frac{c\overrightarrow{\sigma}\cdot\overrightarrow{p}}{(E + mc^2)}u_A$$

$$(\overrightarrow{\sigma}\cdot\overrightarrow{A})(\overrightarrow{\sigma}\cdot\overrightarrow{B}) = \overrightarrow{A}\cdot\overrightarrow{B} + i(\overrightarrow{A}\times\overrightarrow{B})\cdot\overrightarrow{\sigma}$$

CLASE 6: Ecuación de Dirac

soluciones de la ec. de Dirac

$$E > 0 \qquad u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad u_B = \frac{c\,\vec{\sigma}\cdot\vec{p}}{(E+mc^2)}\,u_A$$

$$\psi^{(1)}(\vec{x},t) = N \begin{pmatrix} 1 \\ 0 \\ \left(\frac{c\,\vec{\sigma}\,\vec{p}}{E+mc^2}\right)\begin{matrix}1\\0\end{matrix} \end{pmatrix} e^{-\frac{i}{\hbar}p_\mu x^\mu} \qquad \psi^{(2)}(\vec{x},t) = N \begin{pmatrix} 0 \\ 1 \\ \left(\frac{c\,\vec{\sigma}\,\vec{p}}{E+mc^2}\right)\begin{matrix}0\\1\end{matrix} \end{pmatrix} e^{-\frac{i}{\hbar}p_\mu x^\mu}$$

$$E < 0 \qquad u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad u_A = \frac{c\,\vec{\sigma}\cdot\vec{p}}{(E-mc^2)}\,u_B$$

$$\psi^{(3)}(\vec{x},t) = N \begin{pmatrix} \left(\frac{c\,\vec{\sigma}\,\vec{p}}{E-mc^2}\right)\begin{matrix}1\\0\end{matrix} \\ 1 \\ 0 \end{pmatrix} e^{-\frac{i}{\hbar}p_\mu x^\mu} \qquad \psi^{(4)}(\vec{x},t) = N \begin{pmatrix} \left(\frac{c\,\vec{\sigma}\,\vec{p}}{E-mc^2}\right)\begin{matrix}0\\1\end{matrix} \\ 0 \\ 1 \end{pmatrix} e^{-\frac{i}{\hbar}p_\mu x^\mu}$$

$$\left(\frac{c\,\vec{\sigma}\cdot\vec{p}}{E+mc^2}\right) \sim \mathcal{O}\left(\frac{v}{c}\right)$$

p.ej. $\vec{p} = p\hat{k}$ $\qquad \vec{\sigma}\cdot\vec{p} = \sigma_3\, p$

$$p \simeq mv$$

$$E \simeq mc^2$$

$$\left(\frac{c\,\vec{\sigma}\cdot\vec{p}}{E+mc^2}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{c\,mv}{2mc^2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{v}{2c} \\ 0 \end{pmatrix}$$

CLASE 6: Ecuación de Dirac

límite no relativista

$$\left(c\,\vec{\alpha}\cdot\vec{p} + \beta\, mc^2\right)\psi = E\,\psi$$

$$\vec{p} \longrightarrow \vec{p} - \frac{q}{c}\vec{A} \qquad E \longrightarrow E - q\phi \qquad \psi = \begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$

$$\longrightarrow \left[c\,\vec{\alpha}\cdot\left(\vec{p} - \frac{q}{c}\vec{A}\right) + \beta\, mc^2\right]\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = (\epsilon_{NR} + mc^2 - q\phi)\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$

$$\equiv \vec{\pi}$$

$$\left[c\begin{pmatrix}0 & \vec{\sigma}\cdot\vec{\pi}\\ \vec{\sigma}\cdot\vec{\pi} & 0\end{pmatrix} + \begin{pmatrix}mc^2 & 0\\ 0 & -mc^2\end{pmatrix} + q\phi\right]\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix} = (\epsilon_{NR} + mc^2)\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$

$$c\,\vec{\sigma}\cdot\vec{\pi}\psi_B + \cancel{mc^2\psi_A} + q\phi\psi_A = (\epsilon_{NR} + \cancel{mc^2})\psi_A \qquad (\epsilon_{NR} - q\phi)\psi_A = c\,\vec{\sigma}\cdot\vec{\pi}\psi_B$$

$$c\,\vec{\sigma}\cdot\vec{\pi}\psi_A - mc^2\psi_B + q\phi\psi_B = (\epsilon_{NR} + mc^2)\psi_B \qquad (\epsilon_{NR} + 2mc^2 - q\phi)\psi_B = c\,\vec{\sigma}\cdot\vec{\pi}\psi_A$$

$$\psi_A = \frac{c\,\vec{\sigma}\cdot\vec{\pi}}{(\epsilon_{NR} - q\phi)}\psi_B \qquad \psi_A = \frac{c\,\vec{\sigma}\cdot\vec{\pi}}{(\epsilon_{NR} - q\phi)}\frac{c\,\vec{\sigma}\cdot\vec{\pi}}{(\epsilon_{NR} + 2mc^2 - q\phi)}\psi_A$$

$$\psi_B = \frac{c\,\vec{\sigma}\cdot\vec{\pi}}{(\epsilon_{NR} + 2mc^2 - q\phi)}\psi_A \qquad \psi_A = \frac{c^2\left(\pi^2 + i\,\vec{\sigma}\cdot(\vec{\pi}\times\vec{\pi})\right)}{(\epsilon_{NR} - q\phi)(\epsilon_{NR} + 2mc^2 - q\phi)}\psi_A \qquad \vec{\pi}\times\vec{\pi} \neq 0$$

CLASE 6: Ecuación de Dirac

límite no relativista

$$\vec{\pi}\times\vec{\pi} = (\vec{p}-\frac{q}{c}\vec{A})\times(\vec{p}-\frac{q}{c}\vec{A})$$

$$= \cancel{\vec{p}\times\vec{p}} - \frac{q}{c}\vec{p}\times\vec{A} - \frac{q}{c}\vec{A}\times\vec{p} + \frac{q^2}{c^2}\cancel{\vec{A}\times\vec{A}}$$

$$= \frac{q}{c}i\hbar\,(\vec{\nabla}\times\vec{A} + \vec{A}\times\vec{\nabla})$$

$$(\vec{\pi}\times\vec{\pi})\psi = \frac{q}{c}i\hbar\,(\vec{\nabla}\times\vec{A}\psi + \vec{A}\times\vec{\nabla}\psi) = \frac{q}{c}i\hbar\left[(\vec{\nabla}\times\vec{A})\psi + \cancel{\vec{\nabla}\psi\times\vec{A}} + + \cancel{\vec{A}\times\vec{\nabla}\psi}\right]$$

$$= \frac{q}{c}i\hbar\,(\vec{\nabla}\times\vec{A})\psi$$

$$\psi_A = \frac{c^2\left(\pi^2 + i\vec{\sigma}\cdot(\vec{\pi}\times\vec{\pi})\right)}{(\epsilon_{NR}-q\phi)(\epsilon_{NR}+2mc^2-q\phi)}\psi_A = \frac{c^2\left(\pi^2 + i\vec{\sigma}\cdot(i\hbar\frac{q}{c}(\vec{\nabla}\times\vec{A})\right)}{(\epsilon_{NR}-q\phi)(\epsilon_{NR}+2mc^2-q\phi)}\psi_A$$

$$\psi_A = \frac{\cancel{c^2}\left((\vec{p}-\frac{q}{c}\vec{A})^2 - \frac{q}{c}\hbar\vec{\sigma}\cdot\vec{B}\right)}{(\epsilon_{NR}-q\phi)(\cancel{\epsilon_{NR}}+2m\cancel{c^2}-\cancel{q\phi})}\psi_A \qquad \psi_A(\epsilon_{NR}-q\phi) = \frac{1}{2m}\left((\vec{p}-\frac{q}{c}\vec{A})^2 - \frac{q}{c}\hbar\vec{\sigma}\cdot\vec{B}\right)\psi_A$$

$$\left[\frac{1}{2m}\left((\vec{p}-\frac{q}{c}\vec{A})^2 - \frac{q}{c}\hbar\vec{\sigma}\cdot\vec{B}\right) + q\phi\right]\psi_A = \epsilon_{NR}\psi_A$$

CLASE 6: Ecuación de Dirac

límite no relativista

$$\left[\frac{1}{2m}\left((\overrightarrow{p}-\frac{q}{c}\overrightarrow{A})^2-\frac{q}{c}\hbar\overrightarrow{\sigma}\cdot\overrightarrow{B}\right)+q\phi\right]\psi_A=\epsilon_{NR}\psi_A$$

$$\psi_A=\psi_{schroedinger}\begin{pmatrix}1\\0\end{pmatrix}$$

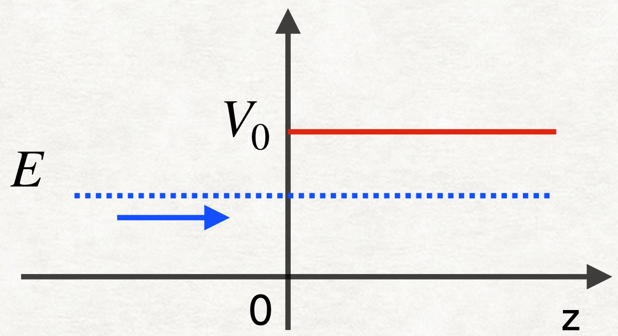
$$\frac{q}{2mc}\hbar\overrightarrow{\sigma}\cdot\overrightarrow{B}=\frac{gq}{2mc}\overrightarrow{S}\cdot\overrightarrow{B}\qquad g=2$$

$g=2.00321930436182(52)$ (NIST 2018)

↑ teoría

CLASE 6: Fenomenología de Dirac

paradoja de Klein: *cuando las componentes débiles no son tan débiles…*



$$\psi_{inc} = \begin{pmatrix} 1 \\ 0 \\ \frac{k}{E+m} \\ 0 \end{pmatrix} e^{-ikz} \qquad \vec{k} = k\hat{z} \quad S_z = 1/2 \quad \hbar = c = 1$$

$$\psi_{refl} = a \begin{pmatrix} 1 \\ 0 \\ \frac{-k}{E+m} \\ 0 \end{pmatrix} e^{ikz} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{k}{E+m} \end{pmatrix} e^{ikz}$$

$$E = \sqrt{c^2q^2 + m^2c^4} + V_0$$

$$E - V_0 = \sqrt{c^2q^2 + m^2c^4}$$

$$cq = \sqrt{(E-V_0)^2 - m^2c^4}$$

$$q = \sqrt{(E-V_0)^2 - m^2}$$

$$\psi_{trans} = c \begin{pmatrix} 1 \\ 0 \\ \frac{q}{E-V_0+m} \\ 0 \end{pmatrix} e^{-iqz} + d \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{-q}{E-V_0+m} \end{pmatrix} e^{-iqz}$$

continuidad de ψ en z=0 $\quad \psi_{inc}(z=0) + \psi_{refl}(z=0) = \psi_{trans}(z=0)$

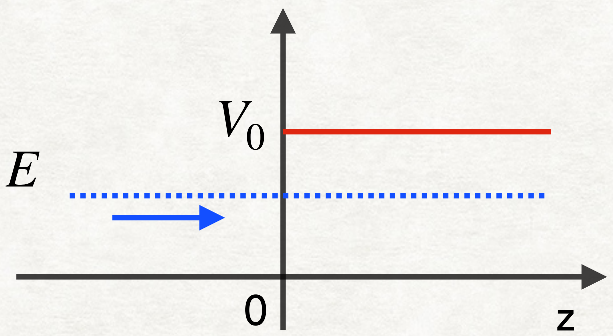
$$1 + a = c \qquad k(1-a)/(E+m) = cq/(E-V_0+m) \quad \rightarrow (1-a) = cr$$

$$b = d \qquad bk/(E+m) = -dq/(E-V_0+m) \quad \Rightarrow b = d = 0$$

$$r \equiv \frac{q}{k}\frac{E+m}{E-V_0+m}$$

CLASE 6: Fenomenología de Dirac

paradoja de Klein: *cuando las componentes débiles no son tan débiles…*



$$1 + a = c \qquad r \equiv \frac{q}{k}\frac{E+m}{E-V_0+m}$$

$$(1-a) = cr \qquad q = \sqrt{(E-V_0)^2 - m^2}$$

corrientes: $J^\mu \equiv \overline{\psi}\gamma^\mu\psi \qquad J^3 \equiv \overline{\psi}\gamma^3\psi = \psi^\dagger\gamma^0\gamma^3\psi = \psi^\dagger\beta\beta\alpha_3\psi \;\; = \psi^\dagger\alpha_3\psi$

$$\alpha_i = \left(\begin{array}{c|c} 0 & \sigma_i \\ \hline \sigma_i & 0 \end{array}\right)$$

$$J^3_{inc} = \left(1,0,\frac{k}{E+m},0\right)\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ \frac{k}{E+m} \\ 0 \end{pmatrix} = \frac{2k}{E+m}$$

$$J^3_{refl} = \frac{2a^2k}{E+m}$$

$$J^3_{inc} = J^3_{refl} + J^3_{trans}$$

$$J^3_{trans} = \frac{2c^2q}{E-V_0+m}$$

$$T \equiv \frac{J^3_{trans}}{J^3_{inc}} = \frac{4r}{(1+r)^2} \qquad R \equiv \frac{J^3_{refl}}{J^3_{inc}} = \frac{(1-r)^2}{(1+r)^2}$$

a) $V_0 < E + m \qquad r > 0 \qquad T > 0$ hay transmisión

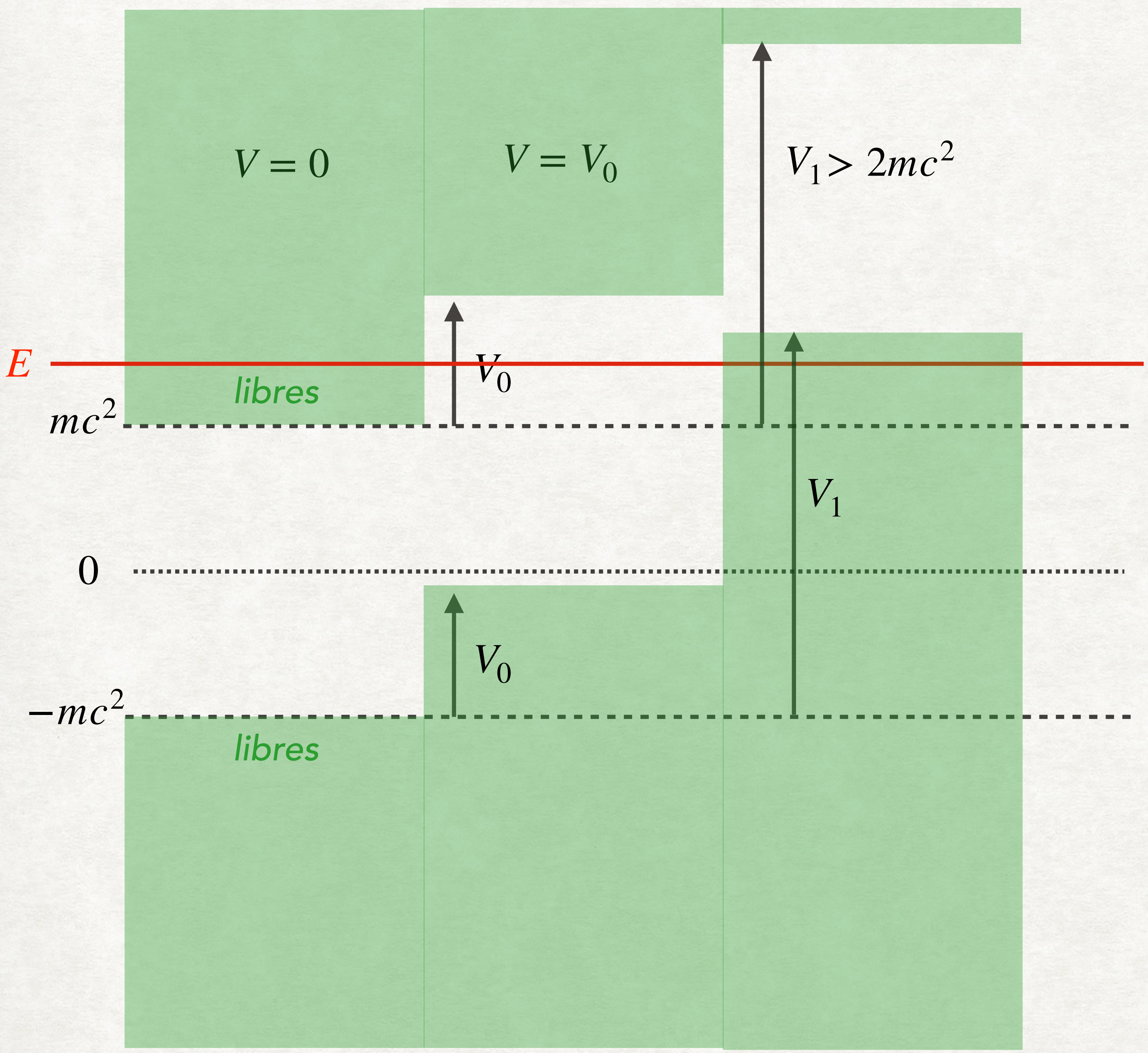
$R < 1$ se reflejan menos que las incidentes

b) $V_0 > E + m \qquad r < 0 \qquad T < 0$ transmitidas en dirección opuesta?

$R > 1$ más partículas reflejadas que las incidentes?

CLASE 6: Fenomenología de Dirac

paradoja de Klein: *cuando las componentes débiles no son tan débiles…*



Григорій Альфредович Брейт

CLASE 6: Fenomenología de Dirac

trayectoria de una partícula de Dirac: *zitterbewegung*

velocidad de una partícula de Dirac:

$$\dot{x}_k \equiv \frac{i}{\hbar}[H, x_k] = \frac{ic}{\hbar}[\alpha_j p_j, x_k] = \frac{ic\alpha_j}{\hbar}[p_j, x_k] = c\alpha_k$$

nunca va estar en autotestado de velocidad (salvo que $m = 0$)

$$H = c\,\vec{\alpha}\cdot\vec{p} + \beta\, mc^2 \qquad [x_i, p_j] = i\hbar\,\delta_{ij}$$

además, $[\alpha_i, \alpha_j] \neq 0$ ....

$$[H, p_k] = 0 \qquad [H, \alpha_k] \neq 0$$

para una partícula libre, la velocidad no es constante de movimiento (salvo que $m = 0$)

aceleración de una partícula de Dirac:

$$\dot{\alpha}_k \equiv \frac{i}{\hbar}[H, \alpha_k] = \frac{i}{\hbar}(-2\alpha_k H + 2cp_k) \neq 0$$

$$\{\alpha_i, \beta\} = 0 \quad \alpha_i\beta + \beta\alpha_i = 0 \quad \alpha_i\beta = -\beta\alpha_i \qquad \frac{i}{\hbar}[\beta\, mc^2, \alpha_k] = \frac{i\, mc^2}{\hbar}(\beta\alpha_k - \alpha_k\beta) = -2\frac{i}{\hbar}\,\alpha_k\beta mc^2$$

$$\frac{i}{\hbar}[c\,\vec{\alpha}\cdot\vec{p}, \alpha_k] = \underset{i\neq k}{\frac{i\,cp_i}{\hbar}[\alpha_i, \alpha_k]} = \underset{i\neq k}{\frac{i\,cp_i}{\hbar}(\alpha_i\alpha_k - \alpha_k\alpha_i)} = \underset{i\neq k}{-\frac{i\,cp_i}{\hbar}2\,\alpha_k\alpha_i} = \underset{i\neq k}{-2\,\frac{i}{\hbar}\alpha_k c\alpha_i p_i} = -2\frac{i}{\hbar}c\,\alpha_k\vec{\alpha}\cdot\vec{p} + 2\frac{i}{\hbar}c\,\alpha_k p_k \alpha_k$$

CLASE 6: Fenomenología de Dirac

trayectoria de una partícula de Dirac: *zitterbewegung*

velocidad de una partícula de Dirac:

$$\dot{x}_k \equiv \frac{i}{\hbar}[H, x_k] = \frac{ic}{\hbar}[\alpha_j p_j, x_k] = \frac{ic\alpha_j}{\hbar}[p_j, x_k] = c\alpha_k$$

nunca va estar en autotestado de velocidad (salvo que $m = 0$)

$$H = c\,\vec{\alpha}\cdot\vec{p} + \beta\, mc^2 \qquad [x_i, p_j] = i\hbar\,\delta_{ij}$$

además, $[\alpha_i, \alpha_j] \neq 0$ ….

$$[H, p_k] = 0 \qquad [H, \alpha_k] \neq 0$$

para una partícula libre, la velocidad no es constante de movimiento (salvo que $m = 0$)

aceleración de una partícula de Dirac:

$$\dot{\alpha}_k \equiv \frac{i}{\hbar}[H, \alpha_k] = \frac{i}{\hbar}(-2\alpha_k H + 2cp_k) \neq 0$$

partícula libre…

$$\alpha_k(t) = \frac{cp_k}{H} + (\alpha_k(0) - \frac{cp_k}{H})e^{-2iHt/\hbar} \qquad (\nu \sim 1.5\ 10^{21} s^{-1})$$

$$x_k(t) = x_k(0) + \frac{c^2 p_k}{H}t + \frac{ic\hbar}{2H}(\alpha_k(0) - \frac{cp_k}{H})e^{-2iHt/\hbar} \qquad (\Delta x \sim 3.9\ 10^{-11} cm)$$

Gargamelle bubble chamber: leptonic neutral current