# A comparison between the Doppler and cosmological redshifts

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We compare the Doppler effect of special relativity with the cosmological redshift of general relativity in order to clarify the difference between them. Some basic concepts of observational cosmology, such as the definitions of distance and cosmological parameters, are also presented.

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## I. INTRODUCTION

Most students are familiar with the special relativistic Doppler shift. Less familiar is the general relativistic spectral shift, which is very important for observational cosmology. The aim of this article is to clarify the distinction between the cosmological and Doppler shifts and compare their magnitudes.

There are two distinct causes for the spectral shift of the light emitted (or absorbed) by a galaxy: the kinematical Doppler effect of special relativity (SR) and the redshift caused by the expansion of the universe, governed by general relativity (GR). These two effects cannot be distinguished from one another by observing the spectrum of the galaxy or other light source. The Doppler shift of SR is due to the relative velocity between source and observer, and can be negative (blueshift) or positive (redshift), depending on whether the galaxy moves radially toward or away from us. The general relativistic effect is always positive, because the universe is expanding. The redshift $z$ of GR is given by[1]

$$1+z=\frac{\lambda_o}{\lambda_e}=\frac{a(t_o)}{a(t_e)}, \tag{1}$$

where the index $o$ denotes the observed quantity and $e$ denotes the emitted one; $a(t_e)$ is the scale factor of the universe at the event of emission and $a(t_o)$ the scale factor at the event of observation. The ratio $a(t_o)/a(t_e)$ measures the growth in the "size" of the universe.

In this article we compare the relation of the velocity and redshift predicted by SR and the corresponding relation of GR. The two expressions are quite similar for small redshifts, but differ substantially for objects at cosmologically significant distances.

## II. RELATIVISTIC DOPPLER SHIFT

The kinematical Doppler effect of SR is the variation of frequency $\nu$ (or wavelength $\lambda=c/\nu$, where $c$ is the light speed) of an electromagnetic wave due to the relative motion between source and observer. The redshift $z$ is defined as

$$z\equiv\frac{\Delta\lambda}{\lambda}=\frac{\lambda_o-\lambda_e}{\lambda_e}, \tag{2}$$

where the indices have similar meanings ($\lambda_e$ is measured in the rest frame of the source). The standard special relativistic expression for the Doppler shift is

$$1+z=\frac{\lambda_o}{\lambda_e}=\frac{\nu_e}{\nu_o}=\sqrt{\frac{1+v/c}{1-v/c}}, \tag{3}$$

which can be inverted to give the relative velocity as a function of $z$:

$$\frac{v(z)}{c}=\frac{2z+z^2}{2+2z+z^2}. \tag{4}$$

By expanding Eq. (4) in a Taylor series around $z=0$, we obtain

$$\frac{v}{c}=z-\frac{z^2}{2}+\frac{z^4}{4}-\cdots, \tag{5}$$

justifying the approximation $v\approx cz$ for small $z$.

Using the apparent magnitude of galaxies to obtain their distances, the astronomer Edwin Hubble discovered the approximate relation[2]

$$cz=H_0D, \tag{6}$$

for relatively nearby ($z\ll 1$) galaxies, where $H_0$ ($\approx 70$ km s$^{-1}$ Mpc$^{-1}$) is the Hubble "constant" and $D$ is the distance to the galaxy.[2] (A pc is an abbreviation of parallax-second. An observer at 1 pc from the Sun would see the Earth–Sun distance through an angle of 1 arcsec; 1 Mpc$=3.26\times10^6$ light years.)

Initially the redshift was interpreted as a recession velocity of the galaxies, just like the Doppler effect. In fact, most astronomy textbooks give the Hubble law as $v=H_0D$ and galaxy redshift surveys present redshifts as radial velocities, using the nonrelativistic approximation $v=cz$. Harrison[3] points out that the redshift–distance relation $cz=H_0D$ is linear only for small redshifts $z\ll 1$, while the velocity–distance law $v=H_0D$ is valid for all distances. The latter relation is a consequence of the assumptions of a homogeneous and isotropic expanding space–time; expansion must be linear if the universe is homogeneous.

## III. COSMOLOGICAL REDSHIFT

Hubble's discovery (1929) of the expansion of the Universe gave observational support to the Friedmann model (1922), subsequently developed by Lemaître, Robertson, Walker and others.[1] The Friedmann–Lemaître–Robertson–Walker (FLRW) model describes a homogeneous and isotropic expanding universe; the Robertson–Walker line element is given by[1]

$$ds^2=c^2\,dt^2-a^2(t)\left[\frac{dr^2}{1-kr^2}+r^2\,d\theta^2+r^2\sin^2\theta\,d\phi^2\right], \tag{7}$$

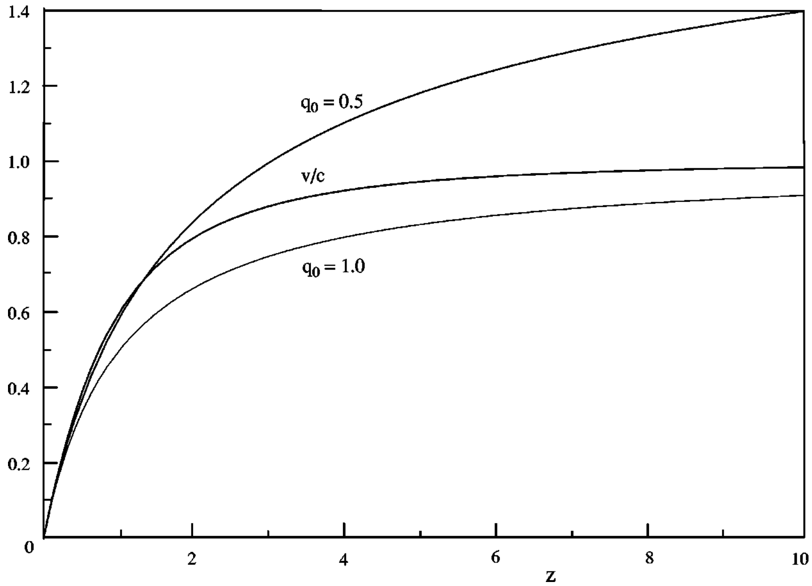


Fig. 1. The upper and lower curves are plots of $H_0 a_0 r/c$ predicted by general relativity; see Eq. (13), for $q_0 = 0.5$ and $q_0 = 1.0$, respectively. The intermediate curve is the prediction of special relativity for $v/c$ given by Eq. (4).

where $t$ is the cosmic time, $r$ is the dimensionless radial coordinate, and $\theta,\phi$ are the usual spherical coordinates; $k$ denotes the curvature of spatial sections and can take the values $0,+1,-1$; $a(t)$ is the scale factor that multiplies all lengths in the universe, including the wavelength.

It is straightforward to show that the cosmological redshift is directly related to the scale factor $a(t)$[1] and is given by

$$1+z=\frac{\lambda_o}{\lambda_e}=\frac{a(t_o)}{a(t_e)}, \tag{8}$$

where $a(t_o)$ is the scale factor at the time of observation and $a(t_e)$ the scale at the time of emission. In general the redshift of a galaxy has a cosmological component and a kinematical one, due to the peculiar velocity of the galaxy (out of the Hubble flow).[4]

The dynamics of the Friedmann model is described by Einstein's field equations, which relate the geometry of space–time to the energy content of the universe. There are two quantities to be determined: the rate of the expansion $\dot{a}$ and the acceleration of the expansion $\ddot{a}$. If the acceleration is positive or zero, the universe will expand forever; if it is negative (deceleration), the universe can either expand forever or collapse, depending on whether it is above or below its escape velocity. The determination of these quantities is crucial for determining the kind and quantity of matter in the universe.

If we define the Hubble parameter $H(t)$ as

$$H(t)=\frac{1}{a(t)}\frac{da}{dt}=\frac{\dot{a}}{a}, \tag{9}$$

the dimensionless deceleration parameter by

$$q=-\frac{\ddot{a}}{H^2 a}, \tag{10}$$

and the density parameter

$$\Omega=\frac{8\pi G}{3H^2}\rho, \tag{11}$$

the field equations for a perfect fluid of density $\rho$ and pressure $p$ can be written as[5]

$$kc^2=(\Omega_0-1)H_0^2 a_0^2, \quad 2q_0=(1+3p/\rho c^2)\Omega_0, \tag{12}$$

where the subscript 0 denotes the present values of the quantities. These parameters will appear in expressions for calculating proper distances in curved space–time.

The exact relation for the proper distance $r_{\mathrm{prop}}=ra(t_0)=ra_0$ in terms of observational parameters was derived by Mattig[6] in 1958 for a Robertson–Walker space–time filled with dust $(p=0)$:

$$\frac{a_0 r}{c}=\frac{q_0 z+(q_0-1)[\sqrt{1+2q_0 z}-1]}{H_0 q_0^2(1+z)}. \tag{13}$$

Although there is no unique definition of the recessional velocity of a distant galaxy, one reasonable definition is the rate of change of the proper distance with respect to cosmic time, that is, $v=r\dot{a}_0=ra_0H_0$. Expanding Eq. (13) in a Taylor series around $z=0$, we obtain

$$\frac{r\dot{a}_0}{c}=z-\frac{(1+q_0)z^2}{2}+\frac{(1+q_0^2)z^3}{2}-\frac{(4-q_0^2+5q_0^3)z^4}{8}+\cdots. \tag{14}$$

Note that there is no value of $q_0$ for which Eq. (14) agrees with the special relativistic formula—Eq. (5). If we compare these equations, we see that the effect of curvature (the $q_0$ terms) becomes important for $z$ approaching 1. Of course, for higher $z$, the exact relation (13) should be used. For $z=1$ and $q_0=0.5$, for example, Eq. (4) of SR gives $v/c=0.6$, while the exact relation of GR, Eq. (13), yields $r\dot{a}_0/c=0.586$ (a discrepancy of 2.4%).

For $q_0=0$, Eq. (13) is undetermined, but we can calculate its limit using l'Hôpital's rule:

$$\lim_{q\to 0}\frac{r\dot{a}_0}{c}=\frac{2z+z^2}{2+2z}, \tag{15}$$

which is bigger than $v/c$ for all $z$ [see Eq. (4)].

Figure 1 compares the formula of SR with the prediction of GR for two values of the parameter $q_0$. When $q_0<1$, the recession velocity exceeds $c$ for large $z$.

## IV. DISCUSSION

Astronomers observe the sky and measure positions, apparent magnitudes, angular diameters, redshifts and number counts of galaxies, nebulae, quasars, etc. These data must be compared to the predictions of theoretical cosmology, in order to confirm or discard a cosmological model. The parameters of the cosmological models not discarded should then be determined with the best precision possible. As we have seen, the redshift $z$ appears in all formulas regarding distances of stellar objects (for example, angular diameter distance, luminosity distance). On the other hand, the interpretation of redshift as distance is made difficult by the kinematical Doppler shift superimposed on the cosmological one. For a cluster of galaxies, statistical methods can be used to eliminate the kinematical contributions; in the mean value of $z$ the kinematical redshifts are canceled by blueshifts. In spite of all difficulties the redshift is a very useful tool of observational cosmology. In theoretical cosmology, the redshift $z$ can be used as a coordinate in place of the radial coordinate $r$.[7] With this choice of coordinates, the comparison of theoretical predictions with observational data is simplified.

Recent observations of type Ia supernovae at redshifts ranging between 0.18 and 1.2 indicate a nonvanishing cosmological constant for a flat $(k=0)$ universe.[8] In this case the deceleration parameter $q_0$ is negative, which implies that the expansion is accelerating. For a nonzero cosmological constant Mattig's relation (13) is not valid, and another expression for the luminosity distance in a FLRW universe is required (see Ref. 8).

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