The rate of deflection of light in an accelerated frame and a gravitational field

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We present calculations of the rate of deflection of light per unit central angle ϕ in a set of stationary frames along the light path in the gravitational field of the sun and in an equivalent (except for curvature) set of accelerated frames in flat spacetime in a study designed to further understanding of the equivalence principle in general relativity. The rate of deflection is emphasized in keeping with the local restriction of the equivalence principle in a metric theory of gravitation. In the sequence of stationary frames it is possible to distinguish the contribution from acceleration with respect to local inertial frames (the equivalence principle) from the total rate of deflection which includes the effect of spacetime curvature. Our results indicate that the deflection rate as a function of central angle can be expressed as $d\alpha/d\phi=-(m/R)(1+2q)\cos^3\phi$, where m is the geometric mass of the sun, R is the minimum radius at $\phi=0$, and q is a curvature tagging parameter such that with $q=0$ we have only the effect of acceleration and with $q=1$ we have the full Schwarzschild curvature. © *1999 American Association of Physics Teachers.*

I. INTRODUCTION

In 1911 Einstein[1] published a prediction for the bending of a ray of light from a distant star passing the sun at minimum radius R and suggested that it could be observed during a total solar eclipse. It was in this paper that he introduced the bold assumption of the complete physical equivalence between a stationary frame in a homogeneous gravitational field and an accelerated frame in field-free space, not only for mechanical processes but for all physical processes, including, in particular, the propagation of light. His calculation of the total deflection, based upon the assumption of a time dilation in the radial direction and an associated gradient in the speed of light implied by the equivalence principle, gave a value $\alpha=-2m/R$, where $m=GM/c^2$ is the geometric mass of the sun.[2] This value differed by a factor of 2 from his later prediction, $\alpha=-4m/R$ $(=-1.75'')$ for grazing incidence), calculated from a null geodesic path in the spacetime manifold, a basic postulate for light in general relativity.[3] The subsequent approximate observational verifications of the later prediction by the 1919 expeditions to the total eclipses in Sobral and Principe were a key factor in the early acceptance of the general theory. More recently measurements of the deflection of radio waves, passing the limb of the sun from distant quasars, have confirmed the prediction of general relativity to better than 1%.[4] Today there is widespread interest in gravitational light deflection because of its involvement in gravitational microlensing events in the galactic halo.[5] In these events typically light from a star in a nearby galaxy such as the Large Magellanic Cloud, or from a distant star in our own galaxy, is temporarily enhanced at an observation point on earth as a result of its passing around an object in the galactic halo. The gravitational field of the object deflects the light from the star and focuses more of it on the observation point than would normally fall there. Observation of such events may reveal the existence of a spherical distribution of dark matter about the center of our galaxy.

The failure of the equivalence principle to fully account for the total deflection of electromagnetic waves in the sun's gravitational field has been analyzed and discussed in this journal by many authors from different points of view.[6–17] Although it might seem that after so much attention this subject would be exhausted, we feel that there is still further insight to be gained by concentrating on a purely local quantity, namely the rate of deflection per unit central angle ϕ in a sequence of stationary frames along the light path in the sun's gravitational field and in an ''equivalent'' (except for spacetime curvature) set of accelerated frames in flat spacetime.[18] As Weinberg has pointed out in his book,[19] the equivalence principle is necessarily a local concept in the context of a metric theory of gravity.

In a recent article in this journal,[20] Moreau, Neutze, and Ross showed that in a local, displaced, stationary frame, with the origin situated at some distance r from the center of a gravitating body such as the sun, the equivalence principle becomes manifest, that is the metric near the origin of the displaced frame can be expressed as the metric of an accelerating frame in flat spacetime plus terms associated with spacetime curvature. In the present study we exploit this property to answer the basic question, what fraction of the total deflection rate, at any given value of the central angle ϕ, is due to acceleration with respect to local inertial frames (the equivalence principle), and what fraction is due to spacetime curvature?

The situation is illustrated in Fig. 1 where the straight line running across the figure represents the zeroth-order approximation to the light path, that is no deflection at all, and S is a local, stationary frame with its origin located at a radius $r=R\sec\phi$ from the center of the sun and with its z axis pointing in the radial direction. In order to answer our question we have calculated the deflection rate in the stationary frame S at central angle ϕ in three different ways: (1) by mapping the events along a straight, light-like world line in an inertial frame to an accelerated frame S' in flat spacetime that is ''equivalent'' to S; (2) from a null geodesic based on the standard Schwarzschild metric;[21] and (3) from a null geodesic in a set of displaced rectangular coordinates in S.[20]

The calculation (1) gives a result for the rate of deflection per unit central angle at central angle ϕ,

$$\frac{d\alpha}{d\phi}=-\frac{m}{R}\cos^3\phi, \tag{1.1}$$

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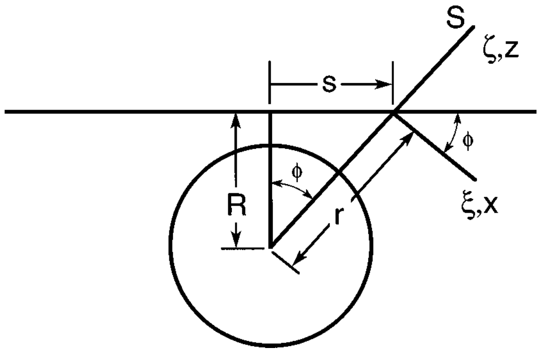

Fig. 1. The zeroth-order light path (no deflection) in the sun's gravitational field with a minimum radius R at central angle $\phi=0$. S is a stationary frame at radius r and central angle ϕ with coordinates ξ, ζ and nondimensional coordinates $x=\xi/r$ and $z=\zeta/r$.

where $m=GM/c^2$ is the geometric mass of the sun and R is the minimum radius along the light path. Calculation (2), based on the standard Schwarzschild metric, gives a result for the total rate of deflection,

$$\frac{d\alpha}{d\phi}=-\frac{3m}{R}\cos^3\phi. \tag{1.2}$$

The deflection rate given by Eq. (1.1) includes the effect of acceleration, but not of curvature, because the accelerated frame S' is in flat spacetime, whereas the rate given by Eq. (1.2) includes the effects of both acceleration and curvature. The purpose of the third calculation is to provide in a single calculation a means of interpreting, and interpolating between, the results of the first two calculations. In the displaced rectangular coordinates of the stationary S frames along the light path in the sun's gravitational field, it is possible to tag the terms in the metric with a parameter q such that with $q=0$ the metric is exactly the flat metric of the corresponding ''equivalent'' accelerating frame S' of calculation (1), and with $q=1$ the metric is the Ricci flat, but Riemann curved Schwarzschild metric of calculation (2).[20] The deflection rate obtained from calculation (3) is given by

$$\frac{d\alpha}{d\phi}=-\frac{m}{R}(1+2q)\cos^3\phi. \tag{1.3}$$

Our results show that while one-third of the deflection rate is due to acceleration with respect to local inertial frames (the equivalence principle), two-thirds is due to spacetime curvature. Thus, even in the limit of an infinitesimal region, the equivalence principle does not account for the total deflection rate, or even the major part of it.

II. DEFLECTION RATE IN AN ACCELERATED FRAME

Referring once again to Fig. 1, we replace the stationary frame S at radius $r=R\sec\phi$ and central angle ϕ in the sun's gravitational field with an ''equivalent'' accelerating frame S' in flat spacetime with rectangular coordinates ξ', η', and ζ'. With respect to an inertial frame $\bar{S}$ with rectangular coordinates $\bar{\xi}$, $\bar{\eta}$, and $\bar{\zeta}$ and axes parallel to S', the origin of S' is accelerating at a rate mc^2/r^2 in the positive $\zeta',\bar{\zeta}$ direction. At time $t'=\bar{t}=0$ the origins of the two frames coincide. We consider a straight, light line lying in the $\bar{\xi},\bar{\zeta}$ plane in $\bar{S}$ and passing through the origin at $\bar{t}=0$. We calculate the rate of deflection in the accelerating S' frame by mapping events along the light line into S'. We want the slope of the light line in S' to be $\tan\phi$ at the origin and choose the slope in $\bar{S}$ accordingly. We note in passing that the light line is a null geodesic in $\bar{S}$, and its mapped image in S' is still a null geodesic, since geometry is not changed by a coordinate transformation. For purposes of discussion and future reference, we calculate the deflection rate in S' in two different ways: (1a) from a coordinate slope of the light line calculated from a ratio of coordinate displacements, and (1b) from a proper slope calculated from a ratio of proper displacements. In the course of discussion of the results of calculations (1a) and (1b), we will raise a central issue, namely matching Schwarzschild radial measure and trigonometry in the ''equivalent'' accelerating frame. We believe that this issue is absolutely crucial in understanding the role of the equivalence principle in the gravitational deflection of light and in unraveling some of the past confusion in this subject.

In terms of nondimensional coordinates, $\theta'=ct'/r$, $x'=\xi'/r$, $y'=\eta'/r$, and $z'=\zeta'/r$, a nondimensional line element in the accelerating S' frame is given by[20]

$$d\sigma^2\equiv\frac{ds^2}{r^2}=-(1-2\mu+2\mu z')d\theta'^2+dx'^2+dy'^2 +(1-2\mu+2\mu z')^{-1}\,dz'^2, \tag{2.1}$$

where $\mu\equiv m/r$. Defining similar nondimensional coordinates in $\bar{S}$, namely $\bar{\theta}=c\bar{t}/r$, $\bar{x}=\bar{\xi}/r$, etc., the coordinate transformation from $\bar{S}$ to S' is given by

$$\theta'=\frac{1}{\mu}\tanh^{-1}\left[\frac{\bar{\theta}}{\bar{z}+\frac{1}{\mu}\sqrt{1-2\mu}}\right], \tag{2.2}$$

$$x'=\bar{x}, \tag{2.3}$$

$$y'=\bar{y}, \tag{2.4}$$

$$z'=\frac{\mu}{2}\left[\left(\bar{z}+\frac{1}{\mu}\sqrt{1-2\mu}\right)^2-\bar{\theta}^2\right]-\frac{1}{2\mu}(1-2\mu), \tag{2.5}$$

where it can be seen that at $\theta'=\bar{\theta}=0$ the origins of S' and $\bar{S}$ coincide. It can be easily shown that Eqs. (2.2)–(2.5) transform the metric of the accelerated S' frame,

$$g_{\mu'\nu'}=\mathrm{diag}[-(1-2\mu+2\mu z'),1,1,(1-2\mu +2\mu z')^{-1}], \tag{2.6}$$

into the Minkowski metric $\eta_{\bar{\mu}\bar{\nu}}=\mathrm{diag}(-1,1,1,1)$ of the inertial frame $\bar{S}$ according to

$$\eta_{\bar{\mu}\bar{\nu}}=\frac{\partial x^{\mu'}}{\partial x^{\bar{\mu}}}\frac{\partial x^{\nu'}}{\partial x^{\bar{\nu}}}g_{\mu'\nu'}, \tag{2.7}$$

where $x^{0'}=\theta'$, $x^{\bar{0}}=\bar{\theta}$, $x^{1'}=x'$, etc.

It can also be readily ascertained that the line element, Eq. (2.1), is appropriate for the ''equivalent'' accelerating frame S'. First of all, the Riemann tensor elements associated with the metric of Eq. (2.1) are all zero, as can be shown by direct calculation or by noting that coordinate transformations do not alter the curvature of the spacetime manifold. So S' is a frame in flat spacetime. Second, we can show that it is an accelerating frame such that a free particle near the origin has exactly the same acceleration with respect to S' as a similar particle in the sun's gravitational field would have with respect to S. If we let $d\sigma^2 = ds^2/r^2 \to -d\pi^2 = -c^2 d\tau^2/r^2$ for time-like intervals in Eq. (2.1) along the world line of the free, massive particle, then we can rewrite Eq. (2.1) as

$$1=(1-2\mu+2\mu z')\left(\frac{d\theta'}{d\pi}\right)^2-\left(\frac{dx'}{d\pi}\right)^2-\left(\frac{dy'}{d\pi}\right)^2 -(1-2\mu+2\mu z')^{-1}\left(\frac{dz'}{d\pi}\right)^2. \tag{2.8}$$

In general, a free particle follows a geodesic given by

$$\frac{d^2x^{\lambda'}}{dp^2}+\Gamma^{\lambda'}_{\mu'\nu'}\frac{dx^{\mu'}}{dp}\frac{dx^{\nu'}}{dp}=0, \tag{2.9}$$

where, for our application, p is a nondimensional affine parameter related linearly to the proper time by $c\tau/r = Ap + B$, and the Christoffel symbols $\Gamma^{\lambda'}_{\mu'\nu'}$ are related to the metric by

$$\Gamma^{\lambda'}_{\mu'\nu'}=\frac{1}{2}g^{\lambda'\rho'}\left(\frac{\partial g_{\mu'\rho'}}{\partial x^{\nu'}}+\frac{\partial g_{\rho'\nu'}}{\partial x^{\mu'}}-\frac{\partial g_{\mu'\nu'}}{\partial x^{\rho'}}\right). \tag{2.10}$$

From Eqs. (2.6), (2.9), and (2.10) with $p=\pi$, the geodesic equations corresponding to the metric in S' are given by

$$\frac{d^2\theta'}{d\pi^2}+\frac{2\mu\dfrac{d\theta'}{d\pi}\dfrac{dz'}{d\pi}}{(1-2\mu+2\mu z')}=0, \tag{2.11}$$

$$\frac{d^2x'}{d\pi^2}=0, \quad \frac{d^2y'}{d\pi^2}=0, \tag{2.12}$$

$$\frac{d^2z'}{d\pi^2}=\frac{\mu\left(\dfrac{dz'}{d\pi}\right)^2}{(1-2\mu+2\mu z')}-\mu(1-2\mu+2\mu z')\left(\frac{d\theta'}{d\pi}\right)^2. \tag{2.13}$$

Now consider motion of a free particle along the z' axis with $dx'/d\pi = dy'/d\pi = 0$ satisfying Eqs. (2.12). Then from Eq. (2.13) and Eq. (2.8) with $dx'/d\pi = dy'/d\pi = 0$, we have

$$\frac{d^2z'}{d\pi^2}=-\mu=-\frac{m}{r}=-\frac{GM}{c^2r}, \tag{2.14}$$

and, converting to dimensional quantities, with $d^2/d\pi^2 = (r^2/c^2)d^2/d\tau^2$ and $z' = \zeta'/r$, the proper acceleration of the free particle is

$$\frac{d^2\zeta'}{d\tau^2}=-\frac{GM}{r^2}. \tag{2.15}$$

Thus S' is an accelerating frame in flat spacetime in which a free particle on the ζ' axis near the origin will have the same acceleration as a free particle on the ζ axis near the origin in S, the stationary frame at radius r in the sun's gravitational field. The S' frame is the closest one can come in flat spacetime to duplicating the physics in the stationary S frame in the sun's gravitational field.

We now begin our calculation (1a). Let us concentrate on a light line in the inertial frame $\bar{S}$ that lies in the $\bar{x}$, $\bar{z}$ plane and passes through the origin at $\bar{\theta}=0$. We will determine the light line from the equations for a null geodesic, and will choose its slope in $\bar{S}$ such that its image in S' has a slope of $\tan\phi$ at the origin. As the metric is constant and the Christoffel symbols are all zero, Eq. (2.9) (with bars instead of primes) gives

$$\frac{d^2\bar{\theta}}{dp^2}=0, \quad \frac{d^2\bar{x}}{dp^2}=0, \quad \frac{d^2\bar{y}}{dp^2}=0, \quad \frac{d^2\bar{z}}{dp^2}=0, \tag{2.16}$$

with solutions

$$\bar{\theta}=C_1p+C_2, \quad \bar{x}=C_3p+C_4, \quad \bar{y}=C_5p+C_6, \quad \bar{z}=C_7p+C_8, \tag{2.17}$$

where C_1 through C_8 are constants of integration. For motion in the $\bar{x}$, $\bar{z}$ plane we set $C_5 = C_6 = 0$, and since we want the light line to pass through the origin at $\bar{\theta}=0$, we set $C_2 = C_4 = C_8 = 0$. The constant C_1 normalizes the affine parameter p and sets the unit of time. We can choose it to be anything we like, but anticipating a comparison we wish to make with the light line in the stationary S frame in the sun's gravitational field, we set $C_1 = 1/\sqrt{1-2\mu}$.

Equations (2.17) are not independent. The solutions must satisfy the line element in $\bar{S}$, and in this case it is null,

$$d\sigma^2=-d\bar{\theta}^2+d\bar{x}^2+d\bar{z}^2=0. \tag{2.18}$$

Dividing through by dp^2, setting $d\bar{\theta}/dp = 1/\sqrt{1-2\mu}$, and rearranging, we have

$$\left(\sqrt{1-2\mu}\,\frac{d\bar{x}}{dp}\right)^2+\left(\sqrt{1-2\mu}\,\frac{d\bar{z}}{dp}\right)^2=1. \tag{2.19}$$

We see that $\sqrt{1-2\mu}\,d\bar{x}/dp$ and $\sqrt{1-2\mu}\,d\bar{z}/dp$ must form two sides of a right triangle of which the hypotenuse equals one. Therefore in Eq. (2.17) $C_3 = d\bar{x}/dp = \cos\beta/\sqrt{1-2\mu}$ and $C_7 = d\bar{z}/dp = \sin\beta/\sqrt{1-2\mu}$ where β can be any angle. But for our case we must have

$$C_3=\frac{\cos\beta}{\sqrt{1-2\mu}}=\frac{\cos\phi}{\sqrt{1-2\mu\cos^2\phi}}, \tag{2.20}$$

$$C_7=\frac{\sin\beta}{\sqrt{1-2\mu}}=\frac{1}{\sqrt{1-2\mu}}\frac{\sin\phi}{\sqrt{1-2\mu\cos^2\phi}}, \tag{2.21}$$

in order for the slope of the light line in S' at the origin to be $\tan\phi$ as required from Fig. 1. That Eqs. (2.20) and (2.21) give the required relationship between β and ϕ will be seen by the result for the slope of the light line in S', Eq. (2.26) below. Substituting Eqs. (2.20) and (2.21) into Eq. (2.17) for C_3 and C_7, we have

$$\bar{x}=\frac{p\cos\phi}{\sqrt{1-2\mu\cos^2\phi}}, \tag{2.22}$$

$$\bar{z}=\frac{1}{\sqrt{1-2\mu}}\frac{p\sin\phi}{\sqrt{1-2\mu\cos^2\phi}}. \tag{2.23}$$

We now transform the light line, Eqs. (2.22) and (2.23), into the accelerated S' frame with the coordinate transformation given by Eqs. (2.3) and (2.5) with $\bar{\theta}=p/\sqrt{1-2\mu}$. The result is

$$x'=\frac{p\cos\phi}{\sqrt{1-2\mu\cos^2\phi}}, \tag{2.24}$$

and, after expanding and canceling terms,

$$z'=-\frac{\mu}{2}\frac{p^2\cos^2\phi}{(1-2\mu\cos^2\phi)}+\frac{p\sin\phi}{\sqrt{1-2\mu\cos^2\phi}}. \tag{2.25}$$

Equations (2.24) and (2.25) represent a null geodesic in S', since they are the result of a coordinate transformation of a null geodesic in $\bar{S}$, and a coordinate transformation neither changes intervals nor geometry. We wish to determine its rate of deflection in S'.

For calculation (1a) we take the coordinate slope of the light line in S' to be given by

$$\tan\alpha=\frac{dz'}{dx'}=\frac{\frac{dz'}{dp}}{\frac{dx'}{dp}}=-\frac{\mu p\cos\phi}{\sqrt{1-2\mu\cos^2\phi}}+\tan\phi$$
$$=-\mu x'+\tan\phi, \tag{2.26}$$

where, from the line element in S', Eq. (2.1), we note that both dz' and dx' are coordinate displacements, but only dx' also measures a proper distance. At the origin the slope $\tan\alpha=\tan\phi$ as promised, and it decreases with increasing x' as the light line deflects in the negative z' direction, that is opposite to the direction of the acceleration of S' with respect to $\bar{S}$. The rate of deflection per unit propagation distance is obtained by differentiating Eq. (2.26) with respect to σ giving

$$\sec^2\alpha\frac{d\alpha}{d\sigma}=-\mu\frac{dx'}{d\sigma}=-\mu\cos\alpha, \tag{2.27}$$

and evaluating the result at the origin where $\alpha=\phi$, we have

$$\left(\frac{d\alpha}{d\sigma}\right)_0=-\mu\cos^3\phi. \tag{2.28}$$

Converting to dimensional quantities, Eq. (2.28) becomes

$$\frac{d\alpha}{ds}=\frac{d\sigma}{ds}\frac{d\alpha}{d\sigma}=\frac{1}{r}\frac{d\alpha}{d\sigma}=-\frac{m}{r^2}\cos^3\phi=-\frac{m}{R^2}\cos^5\phi. \tag{2.29}$$

Finally, since $s=R\tan\phi$ as shown in Fig. 1, the deflection rate per unit central angle is

$$\frac{d\alpha}{d\phi}=\frac{ds}{d\phi}\frac{d\alpha}{ds}=-\frac{m}{R}\cos^3\phi. \tag{2.30}$$

We now do calculation (1b), replacing the coordinate displacement dz' in the slope by the corresponding proper displacement

$$d\sigma_{z'}=\frac{dz'}{\sqrt{1-2\mu+2\mu z'}}, \tag{2.31}$$

which is obtained from Eq. (2.1) by setting $d\theta'=dx'=dy'=0$. Using the proper displacement $d\sigma_{z'}$ in the calculation it is more convenient to choose $C_1=1$ in Eq. (2.17), and in this case we get the correct slope at the origin in S' if we set $C_3=\cos\phi$ and $C_7=\sin\phi$. Then in $\bar{S}$ we have simply $\bar{x}=p\cos\phi$ and $\bar{z}=p\sin\phi$ and correspondingly in S',

$$x'=p\cos\phi, \tag{2.32}$$

and

$$z'=-\tfrac{1}{2}\mu p^2\cos^2\phi+p\sin\phi\sqrt{1-2\mu}. \tag{2.33}$$

This time, defining $\tan\alpha$ to be the proper slope of the light line in S', we write

$$\tan\alpha=\frac{\frac{d\sigma_{z'}}{dp}}{\frac{dx'}{dp}}=\frac{\frac{d\sigma_{z'}}{dz'}\frac{dz'}{dp}}{\frac{dx'}{dp}}=-\frac{\mu p\cos\phi}{\sqrt{1-2\mu+2\mu z'}}+\sqrt{\frac{1-2\mu}{1-2\mu+2\mu z'}}\tan\phi$$
$$=-\frac{\mu x'}{\sqrt{1-2\mu+2\mu z'}}+\sqrt{\frac{1-2\mu}{1-2\mu+2\mu z'}}\tan\phi, \tag{2.34}$$

where $\tan\alpha=\tan\phi$ at the origin. The rate of deflection per unit propagation distance is given by

$$\sec^2\alpha\frac{d\alpha}{d\sigma}=-\frac{\mu}{\sqrt{1-2\mu+2\mu z'}}\frac{dx'}{d\sigma}+\frac{\mu^2x'}{(1-2\mu+2\mu z')^{3/2}}\frac{dz'}{d\sigma}-\frac{\mu\sqrt{1-2\mu}}{(1-2\mu+2\mu z')^{3/2}}\left(\frac{dz'}{d\sigma}\right)\tan\phi. \tag{2.35}$$

From Eq. (2.1), with $d\theta'=dy'=0$, we can write

$$1=\left(\frac{dx'}{d\sigma}\right)^2+\frac{1}{(1-2\mu+2\mu z')}\left(\frac{dz'}{d\sigma}\right)^2, \tag{2.36}$$

and therefore

$$\frac{dx'}{d\sigma}=\cos\alpha,\quad \frac{d\sigma_{z'}}{d\sigma}=\frac{1}{\sqrt{1-2\mu+2\mu z'}}\frac{dz'}{d\sigma}=\sin\alpha. \tag{2.37}$$

Substituting Eqs. (2.37) into Eq. (2.35) and dividing through by $\sec^2\alpha$, we have

$$\frac{d\alpha}{d\sigma}=-\frac{\mu}{\sqrt{1-2\mu+2\mu z'}}\cos^3\alpha+\frac{\mu^2x'}{(1-2\mu+2\mu z')}\sin\alpha\cos^2\alpha-\frac{\mu\sqrt{1-2\mu}}{(1-2\mu+2\mu z')}\sin\alpha\cos^2\alpha\tan\phi. \tag{2.38}$$

We now evaluate $d\alpha/d\sigma$ to first order in μ at the origin of S' where $\alpha=\phi$, as shown by Eq. (2.34),

$$\left(\frac{d\alpha}{d\sigma}\right)_0=-\frac{\mu\cos\phi}{\sqrt{1-2\mu}}=-\mu\cos\phi+O(\mu^2). \tag{2.39}$$

Converting to dimensional quantities with $d\alpha/ds=(d\sigma/ds)\times(d\alpha/d\sigma)=(1/r)(d\alpha/d\sigma)$ and assuming $\mu\ll 1$, we have for the rate of deflection per unit propagation distance in S',

$$\frac{d\alpha}{ds}=-\frac{m}{r^2}\cos\phi=-\frac{m}{R^2}\cos^3\phi. \tag{2.40}$$

Furthermore, again noting that $s=R\tan\phi$ along the zeroth-order light path in Fig. 1, we find that the deflection rate per unit central angle is

$$\frac{d\alpha}{d\phi}=\frac{ds}{d\phi}\frac{d\alpha}{ds}=R\sec^2\phi\frac{d\alpha}{ds}=-\frac{m}{R}\cos\phi, \tag{2.41}$$

which agrees with Einstein's (1911) calculation. Finally, we integrate Eq. (2.41) along the light path to obtain Einstein's result for the overall deflection,

$$\alpha=-\frac{2m}{R}\int_0^{\pi/2}\cos\phi\,d\phi=-\frac{2m}{R}. \tag{2.42}$$

It is very interesting that when we calculate the slope in S' with proper displacements we obtain Einstein's results for the deflection rate and the overall deflection. We will explain why this is so in our conclusions.

From our calculations (1a) and (1b) we have obtained two different expressions for the rate of deflection per unit central angle in the set of ''equivalent'' accelerated S' frames at various central angles ϕ, Eq. (2.30) and Eq. (2.41), the first falling off as $\cos^3\phi$ and the second less rapidly as $\cos\phi$. The (1a) result is based on a coordinate slope in S' calculated as the ratio of coordinate displacements dz'/dx', while the (1b) result is based on a proper slope calculated as the ratio of proper displacements $d\sigma_{z'}/dx'$. Which is correct? One would think that the tangent of an angle should always be the ratio of two proper lengths, and the line element in S', Eq. (2.1), shows that $d\sigma_{z'}\equiv dz'/\sqrt{1-2\mu+2\mu z'}$ and dx' are both proper lengths. But we have not asked the right question. Since at any given ϕ the accelerated frame S' is supposed to be the ''equivalent'' of the corresponding stationary frame S in the sun's gravitational field, a more appropriate question would be which calculation is compatible with radial measure and trigonometry in Schwarzschild space and which is not? We shall see that it is the calculation (1a) and not (1b). In fact, we shall see that the (1b) result, while certainly correct in the limited context of an accelerated frame in flat spacetime, leads to a contradiction when we try to interpret it in terms of the equivalence principle. Specifically, we shall find that for values of central angle $\phi>\cos^{-1}(1/\sqrt{3})$ the rate of deflection due to acceleration given by the (1b) result turns out to be greater than the total rate of deflection which we now proceed to calculate in calculation (2) with the full Schwarzschild metric.

III. DEFLECTION RATE IN SCHWARZSCHILD SPACE

The Schwarzschild line element, which arises from the exterior solution of Einstein's field equations for a spherically symmetric, nonrotating central body of geometric mass m, is given by

$$ds^2=-\left(1-\frac{2m}{r}\right)c^2\,dt^2+\left(1-\frac{2m}{r}\right)^{-1}dr^2+r^2(d\chi+\sin^2\chi\,d\phi^2), \tag{3.1}$$

where r is a radial coordinate and χ and ϕ are the usual spherical angular coordinates. According to general relativity, light follows a null $(ds^2=0)$ geodesic path in the spacetime manifold. Our main purpose in this section is to calculate the rate of deflection in the $\chi=\pi/2$ plane as a function of central angle ϕ using the full Schwarzschild metric. But before proceeding we want to examine radial measure and trigonometry in Schwarzschild coordinates in order to support our rejection of result (1b) for the rate of deflection in the accelerated frame S' at central angle ϕ.

Misner, Thorne, and Wheeler[22] point out that although the radial Schwarzschild coordinate r does not measure proper distance from the origin, nevertheless the proper area of a spherical surface at coordinate radius r is $4\pi r^2$, as can be seen by setting $c^2dt^2=dr^2=0$ in Eq. (3.1) and integrating ds^2 over solid angle. An infinitesimal solid angle in Schwarzschild space (measured in ordinary steradians) is given by

$$d\Omega=\frac{ds^2}{r^2}, \tag{3.2}$$

where ds^2 is an element of proper area on the spherical surface at *coordinate* radius r. Since we are only interested in motion in the plane $\chi=\pi/2$, let us simplify the discussion by setting $\sin\chi=1$ and $d\chi=0$ in Eq. (3.1). Then the proper circumference of the circle at coordinate radius r is $2\pi r$, and an infinitesimal central angle (in radians) is given by

$$d\phi=\frac{ds}{r}, \tag{3.3}$$

where ds is an element of proper length on the circle, and

$$r=\int_0^r dr'\neq\int_0^{r_0}\sqrt{b(r')}dr'+\int_{r_0}^r\frac{dr'}{\sqrt{1-\frac{2m}{r'}}} \tag{3.4}$$

is definitely the coordinate radius and not the proper radius which is given by the final expression in which r_0 is the coordinate radius of the central body of geometric mass m, and $b(r)$ is the radial metric of the interior solution of Einstein's field equations. Thus the radian measure of the central angle ϕ in Schwarzschild space is the ratio of a circumferential proper length and a radial coordinate length. But that is not the whole story. The coordinate radius is an essential part of trigonometry in Schwarzschild space. Thus, referring to Fig. 1, R and r are coordinate radii such that $R/r=\cos\phi$ and $s/R=\tan\phi$. In general, an infinitesimal distance ds along the light path has a radial component that is coordinate and a circumferential component that is proper.

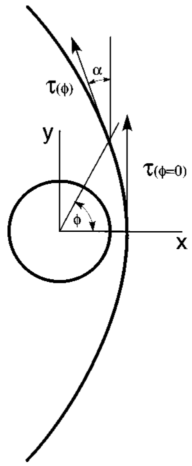

Fig. 2. Deflection of the null geodesic with the standard Schwarzschild metric. At central angle ϕ the angle of deflection α (negative for deflection toward the center) is the angle between the tangent vectors $\boldsymbol{\tau}(\phi)$ and $\boldsymbol{\tau}(\phi=0)$.

Referring back to Fig. 1, we see that the ζ, z coordinates in S are radial, and the ξ, x coordinates are circumferential. Then, since S' is the "equivalent" of S, we should treat ζ',z' as radial and ξ',x' as circumferential. Taking these points about radial measure and trigonometry in Schwarzschild space into consideration and bearing in mind that we want the "equivalent" S' frame to be as equivalent as possible, it seems reasonable to calculate the slope of the light line in S' as $\tan \alpha = dz'/dx'$ rather than $d\sigma_{z'}/dx'$. In fact, it is not only reasonable but also essential. We will show in Sec. IV that the metric of S' is "equivalent" to the metric of S, a stationary frame in Schwarzschild space. Therefore, in order to realize the "equivalence" of S' and S one must employ the same radial measure and trigonometry in S' as that used in Schwarzschild space.

After this preamble on radial measure and trigonometry in Schwarzschild space, we now move on to the main topic of this section, the calculation of the deflection rate in the full Schwarzschild metric. We use the standard dimensional Schwarzschild coordinates of Eq. (3.1) for this calculation with $\chi=\pi/2$. The situation is shown in Fig. 2 where $\boldsymbol{\tau}(\phi)$ is a unit tangent vector to the light line at central angle ϕ which we will use to calculate the rate of deflection.

Instead of solving the null geodesic equations from the beginning, we pick up the calculation from Weinberg,[21] who gives the rate of change of radius with respect to central angle as

$$\frac{dr}{d\phi}=r^2\sqrt{\left(\frac{1}{R^2}-\frac{1}{r^2}\right)+2m\left(\frac{1}{r^3}-\frac{1}{R^3}\right)}, \tag{3.5}$$

where R is the minimum radius at $\phi=0$. Equation (3.5) is exact. Referring again to Fig. 2, the rate of change of the radius vector with respect to ϕ is given by

$$\frac{d\mathbf{r}}{d\phi}=\mathbf{i}\left(\frac{dr}{d\phi}\cos\phi-r\sin\phi\right)+\mathbf{j}\left(\frac{dr}{d\phi}\sin\phi+r\cos\phi\right), \tag{3.6}$$

and the unit tangent vector at ϕ is

$$\boldsymbol{\tau}(\phi)=\frac{d\mathbf{r}}{d\phi}\Big/\left|\frac{d\mathbf{r}}{d\phi}\right| =\frac{\mathbf{i}\left(\frac{dr}{d\phi}\cos\phi-r\sin\phi\right)+\mathbf{j}\left(\frac{dr}{d\phi}\sin\phi+r\cos\phi\right)}{\sqrt{\left(\frac{dr}{d\phi}\right)^2+r^2}}. \tag{3.7}$$

The angle of deflection α (negative for deflection toward the center) is the angle between the tangent vectors $\boldsymbol{\tau}(\phi)$ and $\boldsymbol{\tau}(0)=\mathbf{j}$. The sine of the deflection angle is given by

$$-\mathbf{k}\sin\alpha=\boldsymbol{\tau}(0)\times\boldsymbol{\tau}(\phi)=-\mathbf{k}\frac{\left(\frac{dr}{d\phi}\cos\phi-r\sin\phi\right)}{\sqrt{\left(\frac{dr}{d\phi}\right)^2+r^2}}, \tag{3.8}$$

and the cosine by

$$\cos\alpha=\boldsymbol{\tau}(0)\cdot\boldsymbol{\tau}(\phi)=\frac{\frac{dr}{d\phi}\sin\phi+r\cos\phi}{\sqrt{\left(\frac{dr}{d\phi}\right)^2+r^2}}. \tag{3.9}$$

Dividing Eq. (3.8) by Eq. (3.9) gives

$$\tan\alpha=\frac{\frac{dr}{d\phi}\cos\phi-r\sin\phi}{\frac{dr}{d\phi}\sin\phi+r\cos\phi}. \tag{3.10}$$

The rate of deflection per unit central angle at ϕ is given by the derivative of Eq. (3.10),

$$\frac{d\alpha}{d\phi}=\frac{\left(\frac{dr}{d\phi}\sin\phi+r\cos\phi\right)^2}{\left[\left(\frac{dr}{d\phi}\right)^2+r^2\right]}\frac{d}{d\phi}\left[\frac{\frac{dr}{d\phi}\cos\phi-r\sin\phi}{\frac{dr}{d\phi}\sin\phi+r\cos\phi}\right], \tag{3.11}$$

where we have brought $\sec^2\alpha$ over to the right-hand side as $\cos^2\alpha$ and substituted Eq. (3.9). The result of the differentiation in Eq. (3.11) is a long-winded expression involving r, $dr\,d\phi$, and $d^2r/d\phi^2$. For $dr/d\phi$ and $d^2r/d\phi^2$ we substitute Eq. (3.5) and its derivative,

$$\frac{d^2r}{d\phi^2}=m-r+2(R-2m)\left(\frac{r}{R}\right)^3. \tag{3.12}$$

Carrying out the derivative, making these substitutions, and simplifying, Macsyma[23] gives

$$\frac{d\alpha}{d\phi}=-\frac{3m}{2m+(R-2m)\left(\frac{r}{R}\right)^3}. \tag{3.13}$$

Equation (3.13) is exact. Evaluating it at $\phi=0$ where $r=R$, we have for the maximum rate of deflection,

$$\left(\frac{d\alpha}{d\phi}\right)_{\max}=-\frac{3m}{R}. \tag{3.14}$$

Finally, to obtain the deflection rate as a function of central angle, we substitute the zeroth-order approximation to the light line, $r=R/\cos\phi$ (see Fig. 1), into Eq. (3.13), giving

$$\frac{d\alpha}{d\phi}=-\frac{3m\cos^3\phi}{R\left[1-2\frac{m}{R}(1-\cos^3\phi)\right]}=-\frac{3m}{R}\cos^3\phi+O\left(\left(\frac{m}{R}\right)^2\right). \tag{3.15}$$

We conclude this section by comparing our final results of calculation (2), Eqs. (3.14) and (3.15), for the rate of deflection in the full Schwarzschild metric with our two corresponding results in the ''equivalent'' accelerated frames, Eq. (2.30) from calculation (1a) and Eq. (2.41) from (1b). Equation (3.14), $(d\alpha/d\phi)_{\max}=-3m/R$, is an exact result for the maximum total rate of deflection at minimum radius in the full Schwarzschild metric. At $\phi=0$ both our results in the ''equivalent'' accelerated frame give $d\alpha/d\phi=-m/R$. Thus we can reasonably conclude that of the total maximum rate of deflection, one-third is due to acceleration with respect to local inertial frames (the equivalence principle). For the general case, Eq. (3.15) of calculation (2) gives $d\alpha/d\phi=-(3m/R)\cos^3\phi$ for the total rate of deflection, while Eq. (2.30) of calculation (1a) and Eq. (2.41) of (1b) give $d\alpha/d\phi=-(m/R)\cos^3\phi$ and $d\alpha/d\phi=-(m/R)\cos\phi$, respectively. Since $3\cos^3\phi$ approaches zero faster than $\cos\phi$ as $\phi\to\pi/2$, if we accept the (1b) result, then for $\phi>\cos^{-1}(1/\sqrt{3})\simeq 54.7°$ the deflection rate due to acceleration is greater than the total deflection rate; that is, the part is greater than the whole. For this reason and to be consistent with radial measure and trigonometry in Schwarzschild space, as explained at the start of this section, we must accept the (1a) result and reject the (1b).

With the results of calculations (1a) and (2) we have answered the basic question posed in Sec. I. At any given value of the central angle, one-third of the rate of deflection is due to acceleration (the equivalence principle) and two-thirds are due to spacetime curvature. This (1/3, 2/3) split persists at all points along the light path. However, this conclusion is unsatisfying from the point of view of understanding the equivalence principle because the results were obtained from two separate calculations. Any ''equivalence'' between the flat metric of the accelerated frames and the Schwarzschild metric remains hidden in the latter. Our final calculation (3) of the rate of deflection in a set of stationary frames along the light path in the sun's gravitational field clarifies this issue and serves as a unifying bridge between the first two calculations.

IV. RATE OF DEFLECTION IN THE DISPLACED STATIONARY FRAMES

We consider a set of displaced, stationary frames S at radius r and central angle ϕ along the light path in the sun's gravitational field as shown in Fig. 1. In terms of nondimensional coordinates, $\theta=ct/r$, $x=\xi/r$, $y=\eta/r$, and $z=\zeta/r$, a nondimensional line element in a given S frame at coordinate radius r and central angle ϕ is given by[20]

$$d\sigma^2\equiv\frac{ds^2}{r^2}=-\left[1-\frac{2\mu}{a}\right]d\theta^2+dx^2+dy^2+\frac{[a^3-2\mu(x^2+y^2)]dz^2+4\mu[xy\,dx\,dy+x(1+z)dx\,dz+y(1+z)dy\,dz]}{a^2(a-2\mu)}, \tag{4.1}$$

where $a\equiv\sqrt{1+2z+x^2+y^2+z^2}$ and $\mu\equiv m/r$. The Ricci tensor associated with the metric given by Eq. (4.1) has been calculated and found to be zero, as it should be since the above line element is simply the Schwarzschild line element transformed into the displaced rectangular coordinates.[24] Thus the metric in the displaced, stationary frame satisfies the vacuum Einstein field equations exactly.

The conceptual advantage of the displaced rectangular coordinates is that the effects of acceleration of the displaced stationary frame with respect to falling local inertial frames (the equivalence principle) and spacetime curvature can be distinguished.[20] Consider the following Taylor expansions of the metric tensor elements about the origin.

$$g_{\theta\theta}=-(1-2\mu+2\mu z)+q\mu[-(x^2+y^2)+2z^2+3(x^2+y^2)z-2z^3+\cdots], \tag{4.2}$$

$$g_{zz}^{-1}=1-2\mu+2\mu z+q\mu[(3-4\mu)(x^2+y^2)-2z^2+(16\mu-9)(x^2+y^2)z+2z^3+\cdots], \tag{4.3}$$

$$g_{xy}=q\mu\left[\frac{2xy}{1-2\mu}-\frac{2(3-4\mu)xyz}{1-4\mu+4\mu^2}+\cdots\right], \tag{4.4}$$

$$g_{xz}=q\mu\left[\frac{2x}{1-2\mu}-\frac{4(1-\mu)xz}{1-4\mu+4\mu^2}-\frac{(3-10\mu+8\mu^2)(x^2+y^2)x-(6-12\mu+8\mu^2)xz^2}{1-6\mu+12\mu^2-8\mu^3}+\cdots\right], \tag{4.5}$$

and $g_{yz}=g_{xz}$ $(x\leftrightarrow y)$. We have included a curvature tagging parameter q in these expansions such that with $q=0$ the Riemann tensor is zero, and with $q=1$ we have the full spacetime curvature of the Schwarzschild metric. Referring to Eq. (2.6), it will be noticed that the metric in a displaced stationary frame S at radius r and central angle ϕ, given by Eqs. (4.2)–(4.5) with $q=0$, corresponds exactly to the metric in the corresponding ''equivalent'' accelerating frame S' in flat spacetime. The reason that we expanded g_{zz}^{-1} in Eq. (4.3) instead of g_{zz} was to make this ''equivalence'' manifest.

We now proceed with the calculation of the deflection rate in a given S frame at radius r and central angle ϕ. Light follows a null geodesic path $x^{\lambda}(p)$ in spacetime determined by the geodesic equations,

$$\frac{d^2x^{\lambda}}{dp^2}+\Gamma^{\lambda}_{\mu\nu}\frac{dx^{\mu}}{dp}\frac{dx^{\nu}}{dp}=0, \tag{4.6}$$

where (exactly as in the case of the Minkowski metric in Sec. II) p is a nondimensional affine parameter related linearly to proper time by $c\tau/r=Ap+B$ (for a null geodesic $A=0$), and the Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$ are related to the metric in S by

$$\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}\,g^{\lambda\rho}\left(\frac{\partial g_{\mu\rho}}{\partial x^{\nu}}+\frac{\partial g_{\rho\nu}}{\partial x^{\mu}}-\frac{\partial g_{\mu\nu}}{\partial x^{\rho}}\right), \tag{4.7}$$

where the metric tensor elements are given by Eqs. (4.2)–(4.5). Just as in the case of the inertial frame S, the four Eqs. (4.6) are not independent and the solution must be constrained by the null line element, $d\sigma^2=0$.

In order to calculate the deflection rate at the origin of S, we have to determine the geodesic equations and the null line element to first order in x, y, and z. Being that the geodesic equations involve the first derivatives of the metric with respect to the spacetime coordinates through the Christoffel symbols, we would have to use a metric that is valid to second order. We have attempted this calculation and have not been able to solve the resulting equations. Instead we consider a weak-field approximation in which $\mu=m/r\ll 1$. For light deflection at the limb of the sun μ is of the order of 10^{-6}, and μ is even smaller for $r>R$. Now, in the immediate vicinity of the origin of S where we are calculating the deflection rate, we have four small quantities, $(\mu,x,y,z)\ll 1$, and we truncate the Taylor expansions of the metric elements to second order in these quantities. The resulting geodesic equations are then first order in (μ,x,y,z), since any third-order term, like μx^2, for example, upon differentiation becomes $2\mu x$, which is second order in small quantities.

As can be seen from Eqs. (4.2) to (4.5), all of the terms in $g_{\theta\theta}$ and g_{zz}^{-1} tagged with the parameter q, and all of the terms in g_{xy}, are third order and higher. The metric elements g_{xz} and g_{yz} have leading second-order terms $2q\mu x$ and $2q\mu y$, respectively. We therefore consider the metric tensor,

$$g_{\mu\nu}=\begin{pmatrix} -b(z) & 0 & 0 & 0 \\ 0 & 1 & 0 & 2q\mu x \\ 0 & 0 & 1 & 2q\mu y \\ 0 & 2q\mu x & 2q\mu y & \dfrac{1}{b(z)} \end{pmatrix}, \tag{4.8}$$

where $b(z)\equiv 1-2\mu+2\mu z$, and the corresponding line element,

$$d\sigma^2=-b\,d\theta^2+dx^2+dy^2+\frac{1}{b}\,dz^2 + 4q\mu(x\;dx\;dz+y\;dy\;dz). \tag{4.9}$$

The associated geodesic equations, Eqs. (4.6) with $\lambda=0,1,2,3$, are given by[25]

$$\ddot{\theta}+\frac{1}{b}\frac{db}{dz}\,\dot{z}\,\dot{\theta}=0, \tag{4.10}$$

$$[4q^2\mu^2b(x^2+y^2)-1]\ddot{x}+4q^2\mu^2bx(\dot{x}^2+\dot{y}^2) +q\mu x\,\frac{db}{dz}\left(b\,\dot{\theta}^2-\frac{\dot{z}^2}{b}\right)=0, \tag{4.11}$$

$$[4q^2\mu^2b(x^2+y^2)-1]\ddot{y}+4q^2\mu^2by(\dot{x}^2+\dot{y}^2) +q\mu y\,\frac{db}{dz}\left(b\,\dot{\theta}^2-\frac{\dot{z}^2}{b}\right)=0, \tag{4.12}$$

$$[4q^2\mu^2b(x^2+y^2)-1]\ddot{z}-2q\mu b(\dot{x}^2+\dot{y}^2) -\frac{1}{2}\frac{db}{dz}\left(b\,\dot{\theta}^2-\frac{\dot{z}^2}{b}\right)=0, \tag{4.13}$$

where dots denote derivatives with respect to the affine parameter p. The null constraint is provided by setting $d\sigma^2=0$ in Eq. (4.9) and dividing through by dp^2. The result is

$$-b\,\dot{\theta}^2+\dot{x}^2+\dot{y}^2+\frac{\dot{z}^2}{b}+4q\mu(x\dot{x}+y\dot{y})\dot{z}=0. \tag{4.14}$$

Now we want to solve Eqs. (4.10)–(4.13), subject to the constraint Eq. (4.14), for a null geodesic path in the x,z plane in S that comes in from the negative x,z quadrant, goes through the origin with a slope $\tan\phi$, and goes out in the positive x,z quadrant, all the while being deflected in the negative z direction. Although Eqs. (4.10)–(4.14) contain terms of higher order, we must not forget that they are based upon a metric that is truncated to second order in (μ,x,y,z) and are thus only valid to first order in the vicinity of the origin of S. Therefore, we will solve them to first order in (μ,x,y,z), which is adequate to determine the deflection rate at the origin.

Equation (4.10) can be rewritten as $d/dp\,\ln(b\dot{\theta})=0$ and integrated to give

$$\dot{\theta}=\frac{1}{b}, \tag{4.15}$$

where we have normalized the affine parameter p by setting the integration constant to zero. Furthermore, Eq. (4.12) is trivially satisfied by

$$\ddot{y}=0,\quad \dot{y}=0,\quad y=0, \tag{4.16}$$

for motion in the xz plane. Substituting Eqs. (4.15) and (4.16) into Eqs. (4.11)–(4.14), we have for the geodesic equations in x and z,

$$(4q^2\mu^2bx^2-1)\ddot{x}+4q^2\mu^2bx\dot{x}^2+q\mu\,\frac{1}{b}\,\frac{db}{dz}\,x(1-\dot{z}^2)=0,\tag{4.17}$$

$$(4q^2\mu^2bx^2-1)\ddot{z}-2q\mu b\dot{x}^2-\frac{1}{2}\,\frac{1}{b}\,\frac{db}{dz}\,(1-\dot{z}^2)=0,\tag{4.18}$$

and the null constraint becomes

$$\dot{x}^2+4q\mu x\dot{x}\dot{z}-\frac{1}{b}\,(1-\dot{z}^2)=0.\tag{4.19}$$

Substituting the null constraint, Eq. (4.19), and $db/dz=2\mu$ into Eqs. (4.17) and (4.18), we obtain

$$(4q^2\mu^2bx^2-1)\ddot{x}+4q^2\mu^2bx\dot{x}^2+2q\mu^2x(\dot{x}^2+4q\mu x\dot{x}\dot{z})=0,\tag{4.20}$$

$$(4q^2\mu^2bx^2-1)\ddot{z}-2q\mu b\dot{x}^2-\mu(\dot{x}^2+4q\mu x\dot{x}\dot{z})=0.\tag{4.21}$$

Finally, neglecting terms higher than first order in (μ,x,y,z), Eqs. (4.20) and (4.21) become

$$\ddot{x}=0,\tag{4.22}$$

$$\ddot{z}+\mu(1+2q)\dot{x}^2=0,\tag{4.23}$$

and the null constraint, Eq. (4.19), becomes

$$\dot{x}^2=\frac{1}{b}\,(1-\dot{z}^2).\tag{4.24}$$

As Eqs. (4.22) and (4.23) are second-order differential equations, we need four conditions at the origin corresponding to the particular null geodesic we are seeking. Taking $p=0$ at the origin, the conditions on the coordinates are

$$x(0)=0,\quad z(0)=0.\tag{4.25}$$

The conditions on the coordinate velocities are determined as follows. First of all, we want the slope at the origin to be

$$\frac{dz}{dx}\,(0)=\frac{\dot{z}(0)}{\dot{x}(0)}=\tan\,\phi.\tag{4.26}$$

Next we evaluate the null constraint, Eq. (4.24), at the origin, substitute Eq. (4.26) for $\dot{z}(0)$, and solve for $\dot{x}(0)$ to obtain

$$\dot{x}(0)=\frac{\cos\,\phi}{\sqrt{1-2\mu\,\cos^2\,\phi}}.\tag{4.27}$$

Finally, substituting Eq. (4.27) into Eq. (4.26), we have

$$\dot{z}(0)=\frac{\sin\,\phi}{\sqrt{1-2\mu\,\cos^2\,\phi}}.\tag{4.28}$$

The solution of Eq. (4.22), with the conditions Eqs. (4.25) and (4.27) imposed, is

$$\dot{x}=\frac{\cos\,\phi}{\sqrt{1-2\mu\,\cos^2\,\phi}},\tag{4.29}$$

$$x=\frac{p\,\cos\,\phi}{\sqrt{1-2\mu\,\cos^2\,\phi}}.\tag{4.30}$$

Then, substituting Eq. (4.29) into Eq. (4.23) gives

$$\ddot{z}+\mu(1+2q)\,\frac{\cos^2\,\phi}{1-2\mu\,\cos^2\,\phi}=0.\tag{4.31}$$

The solution of Eq. (4.31), with the conditions Eqs. (4.25) and (4.28) imposed, is

$$\dot{z}=-\mu(1+2q)\,\frac{p\,\cos^2\,\phi}{1-2\mu\,\cos^2\,\phi}+\frac{\sin\,\phi}{\sqrt{1-2\mu\,\cos^2\,\phi}},\tag{4.32}$$

$$z=-\frac{\mu}{2}\,(1+2q)\,\frac{p^2\,\cos^2\,\phi}{(1-2\mu\,\cos^2\,\phi)}+\frac{p\,\sin\,\phi}{\sqrt{1-2\mu\,\cos^2\,\phi}}.\tag{4.33}$$

At this point we note that Eqs. (4.30) and (4.33) with $q=0$ are identical to Eqs. (2.24) and (2.25) which were obtained by a coordinate transformation of the straight line null geodesic in an inertial frame $\bar{S}$ to the accelerated frame S'.

From Eqs. (4.29) and (4.32) the slope near the origin is given by

$$\tan\,\alpha=\frac{\dfrac{dz}{dp}}{\dfrac{dx}{dp}}=-\frac{\mu(1+2q)p\,\cos\,\phi}{\sqrt{1-2\mu\,\cos^2\,\phi}}+\tan\,\phi$$

$$=-\mu(1+2q)p\,\cos\,\phi+\tan\,\phi+O(\mu^2),\tag{4.34}$$

where $\tan\alpha=dz/dx$ is consistent with Schwarzschild radial measure and trigonometry. Upon differentiation with respect to nondimensional propagation distance, Eq. (4.34) gives

$$\sec^2\,\alpha\,\frac{d\alpha}{d\sigma}=-\mu(1+2q)\,\frac{dp}{dx}\,\frac{dx}{d\sigma}\cos\,\phi$$

$$=-\mu(1+2q)\cos\,\alpha+O(\mu^2),\tag{4.35}$$

where we have substituted the inverse of Eq. (4.29) for dp/dx and $dx/d\sigma=\cos\,\alpha$. Converting to dimensional quantities and neglecting $O(\mu^2)$, we have

$$\frac{d\alpha}{ds}=\frac{d\sigma}{ds}\,\frac{d\alpha}{d\sigma}=\frac{1}{r}\,\frac{d\alpha}{d\sigma}=-\frac{m}{r^2}\,(1+2q)\cos^3\,\alpha,\tag{4.36}$$

and evaluating at the origin of S where $\alpha=\phi$,

$$\left(\frac{d\alpha}{ds}\right)_0=-\frac{m}{R^2}\,(1+2q)\cos^5\,\phi,\tag{4.37}$$

since $r=R\,\sec\,\phi$ (see Fig. 1). Finally, the deflection rate per unit central angle is given by

$$\frac{d\alpha}{d\phi}=\frac{ds}{d\phi}\,\frac{d\alpha}{ds}=-\frac{m}{R}\,(1+2q)\cos^3\,\phi,\tag{4.38}$$

since from Fig. 1, $s=R\,\tan\,\phi$ and $ds/d\phi=R\,\sec^2\,\phi$. For $q=1$ Eq. (4.38) agrees with Eq. (3.15), the deflection rate of calculation (2) with the full Schwarzschild metric in the standard Schwarzschild coordinates. Both results are first order in m/R. For $q=0$ it agrees with Eq. (2.30), the deflection rate of calculation (1a) in the ''equivalent'' accelerated frame in flat spacetime.

It only remains to calculate the overall deflection by integrating Eq. (4.38) over the central angle. In this integration it should *not* be imagined that an S frame in Fig. 1 is ''moving with a photon'' and thus rotating as the central angle increases. Rather Eq. (4.38) gives the deflection rate with respect to a set of *stationary* S frames along the light path as a function of central angle. The deflection rate was obtained by considering ''the motion of a photon'' with respect to each stationary S frame. Furthermore, in integrating the deflection rate to obtain the overall deflection, it is immaterial that the S frames vary in orientation. The overall deflection is given by

$$\begin{aligned}\alpha&=-\frac{m}{R}(1+2q)\int_{-\pi/2}^{+\pi/2}\cos^3\phi\,d\phi\\&=-\frac{2m}{R}(1+2q)\int_0^{\pi/2}(1-\sin^2\phi)\cos\phi\,d\phi\\&=-\frac{4m}{3R}(1+2q)\\&=\begin{cases}-\dfrac{4m}{3R} & (q=0)\\[2mm] -\dfrac{4m}{R} & (q=1).\end{cases}\end{aligned}\tag{4.39}$$

While we obtain the standard result for $q=1$, our equivalence principle result with $q=0$ is two-thirds of Einstein's 1911 result as a consequence of our deflection rate falling off as $\cos^3\phi$ instead of $\cos\phi$. We have established the impossibility of the latter and have shown that it results from a failure to match Schwarzschild radial measure and trigonometry in the equivalence principle calculation.

V. CONCLUSION

The results of three separate calculations of the deflection rate of light per unit central angle along the light path in the gravitational field of the sun have been presented in an exercise aimed at further understanding of the equivalence principle in general relativity. The calculations were based upon the following theoretical principles: (1) coordinate transformations of null (light) geodesics from a set of inertial frames to a set of accelerated frames in flat spacetime, with appropriate orientation and acceleration along the light path; (2) a null geodesic associated with the standard Schwarzschild metric; and (3) null geodesics in a set of displaced, stationary frames along the light path in the sun's gravitational field. The rate of deflection rather than the overall deflection has been emphasized, in keeping with the local restriction of the equivalence principle in connection with real gravitational fields. Calculation (1) was performed in two ways: (1a) with a coordinate slope, and (1b) with a proper slope, the latter calculation being rejected because it is inconsistent with Schwarzschild radial measure and trigonometry.

The results of all three calculations are in complete accord and show that at all values of the central angle one-third of the total rate of deflection is due to acceleration with respect to local inertial frames and two-thirds is due to spacetime curvature. The agreement of the calculations (1a) and (3) with $q=0$ is very satisfying because it confirms the local equivalence of the accelerating frames in flat spacetime and the corresponding stationary frames in the gravitational field in the limit of zero curvature. The agreement of calculations (2) and (3) with $q=1$ verifies that the approximations made in the latter are valid in calculating a local quantity like the rate of deflection.

As far as we are aware, we are the first authors to recognize the importance of matching Schwarzschild radial measure and trigonometry in calculations of light deflection based upon the equivalence principle. Matching Schwarzschild measure leads to a deflection rate that falls off as $\cos^3\phi$ while the failure to do so results in a $\cos\phi$ behavior. The latter, when compared to the $3\cos^3\phi$ behavior of the deflection rate obtained in calculation (2) with the standard Schwarzschild metric, leads to the impossible result of the part being greater than the whole for $\phi>\cos^{-1}(1/\sqrt{3})$. While it is certainly valid to measure an angle in an accelerated frame in flat spacetime as the ratio of two proper lengths, when the accelerated frame is being considered as the ''equivalent'' of a stationary frame in Schwarzschild space, then it is important to be consistent in order for the calculations to make any sense. Schwarzschild radial measure and trigonometry in the accelerated frame is a part of the ''equivalence,'' and, judging from our results, a rather important part.

Our conclusions are not in agreement with some of the previously published results that we have referenced in Sec. I. In our notation, a survey of these earlier results reveals a general agreement among the authors on a deflection rate from the equivalence principle (or due to time dilation only) of $d\alpha/d\phi=-(m/R)\cos\phi$ and a total deflection rate of $d\alpha/d\phi=-(2m/R)\cos\phi$. In the respective references, these results are based upon: Newtonian and relativistic treatments;[8,11] rotation of the polarization vector of an electromagnetic field;[12] deflection with respect to an infinitely fast particle;[13] time-like and space-like terms in the metric;[14] and deflection due to three-space curvature.[16] We assess the validity of these results by subjecting them to the same test that we have used to judge our own, namely comparison with the total deflection rate, $d\alpha/d\phi=-(3m/R)\cos^3\phi$, calculated from the standard Schwarzschild metric, our most easily defended result. We have already made the first comparison, because the earlier result from the equivalence principle is identical to our result (1b). At the central angle $\phi=\cos^{-1}(1/\sqrt{3})\simeq 54.7°$ where $\cos\phi=3\cos^3\phi$ the deflection rate due to acceleration with respect to local inertial frames starts to become greater than the total deflection rate. Comparing the two results for total deflection rate, the earlier result is two-thirds of the correct value at $\phi=0$ and becomes greater than the correct value at $\phi=\cos^{-1}(\sqrt{2/3})\simeq 35.3°$. It is remarkable that both results for total deflection rate give the same overall deflection:

$$\alpha=-\frac{2m}{R}\int_{-\pi/2}^{+\pi/2}\cos\phi\,d\phi=-\frac{4m}{R},\tag{5.1}$$

$$\alpha=-\frac{3m}{R}\int_{-\pi/2}^{+\pi/2}\cos^3\phi\,d\phi=-\frac{4m}{R}.\tag{5.2}$$

A final question needs to be answered. Why do Einstein's (1911) results for deflection rate and overall deflection agree with our calculation (1b)? Under the assumption that an accelerated frame is equivalent to a uniform gravitational field, Einstein carried out his calculation in a Euclidean space in which the coordinates all measure proper distance. Our use of the proper displacement $d\sigma_{z'}$ in place of the coordinate displacement dz' in the calculation of the slope effectively canceled out any matching with Schwarzschild and made calculation (1b) a Euclidean calculation as well.

Due to a fluke of mathematics, exemplified by Eqs. (5.1) and (5.2), Einstein's (1911) result for the overall deflection,

$$\alpha = -\frac{m}{R}\int_{-\pi/2}^{+\pi/2}\cos\,\phi\; d\phi = -\frac{2m}{R}, \tag{5.3}$$

has long been considered to be an incomplete part of the total, overall deflection, $\alpha = -4m/R$, given by general relativity. By investigating the associated rates of deflection, we have shown that Einstein's (1911) calculation is inconsistent with general relativity. Therefore, we believe that $\alpha = -(4m/3R)$, given by Eq. (4.39) with $q=0$, should be considered as the valid overall deflection based upon the equivalence principle, since its associated deflection rate is based upon a consistent treatment of radial measure and trigonometry and is consistent consequently with the total deflection rate based upon the standard Schwarzschild metric.

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[2]We employ a sign convention throughout this article that deflection radially inward is negative.

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[18]As a shorthand notation throughout this article, the words ''equivalent'' and ''equivalence'' in quotation marks mean equivalent except for space-time curvature.

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[23]Macsyma is commercial symbolic algebra software.

[24]We are indebted to Dr. Richard Easther of Brown University for this calculation using MAPLE GRTENSOR.

[25]The geodesic equations were generated by Macsyma Component Tensor.

ANOTHER REASON TO REPEAT AN EXPERIMENT

That night I could not sleep. After a delay of five years my idea had paid off with only a few hours' work: I had identified the first gland that contributes to ant communication. More than that, I had discovered what seemed to be a new phenomenon in chemical communication. The pheromone in the gland is not just a guidepost for workers who choose to search for food, but the signal itself—both the command and the instruction during the search for food. The chemical was everything. ... Over the next few days I confirmed the efficiency of the trail pheromone assay over and over. In science there is nothing more pleasant than repeating an experiment that works.

Edward O. Wilson, *Naturalist* (Island Press, Washington, DC, 1994), pp. 291–292.