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The *RLC* circuit and the determination of inductance

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The damped motion of charges is investigated using a modified *RLC* circuit. These experiments yield three sets of data for inductance measurement and provide exercises for students to use the CRO and relate various concepts in electromagnetism.

In introductory physics courses at advanced level (Duncan 1975, Nelson and Parker 1977) and college level (Resnick and Halliday 1966, Ohanian 1989), pupils learn that there are three types of motion of electric charges associated with an *RLC* circuit, namely overdamped motion, critically damped motion and underdamped oscillation. For fixed *C* and *L*, the occurrence of each type is determined by the value of the resistance, *R*. A simple way to demonstrate this effect is to connect a square-wave generator directly to the *RLC* circuit (Preston 1985) (figure 1). Typically, using a signal generator with zero ground level and $\pm V$ outputs, two charging curves will be produced within each triggering cycle of the square wave (figure 2).

In principle, an analysis of the CRO trace of the *RLC* circuit can be used to measure inductance. However, because of the distortion of the trace produced by the impedance of the signal generator and the initial charging state of the capacitor, a direct connection of the *RLC* circuit to the signal generator is generally unsuitable for quantitative measurement.

Here we employ a modified *RLC* circuit to produce an undistorted CRO trace of damped motion, which yields three independent measurements of inductance down to 1 μH. By changing *R*, a smooth transition from the overdamped motion via the critically damped motion to the underdamped oscillation can be observed. These experiments not only provide ample opportunities for students to exercise a number of basic laboratory skills, such as using the oscilloscope as a measuring instrument and extracting information from a log–linear plot, but they also help beginners to relate and gain a better understanding of various theoretical concepts in electromagnetism, such as the magnetic field produced by a current, *RC* decay, a differential circuit, inductance and electromagnetic oscillation. The circuit also leads naturally to a discussion of the analogy between electrical and mechanical systems and can be used, at the demonstrator's own discretion, to verify the formula for the resonance frequency of an *LC* oscillator.

The modified circuit

Figure 3 shows a simple modification of figure 1 which ensures that the *RLC* loop will perform independently of the signal generator. Part A is an *RC* differential circuit which converts the square wave into periodic sharp pulses, and part B is the *RLC* circuit to be studied. Parts A and B are coupled together via a diode which serves two different purposes:

(i) During each charging cycle, it allows only the positive pulse to charge the capacitor.

(ii) When the capacitor discharges through the *RL* combination, it opens the connection between the *RLC* loop and the pulse generator, provided that the amplitude of oscillation is less than about 0.6 V, the forward conducting voltage of a silicon diode.

A brief analysis of the *RLC* circuit will help us to decide what circuit parameters should be used and what measurements are to be made.

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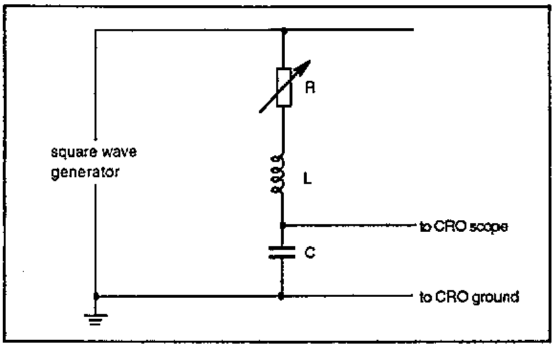


Figure 1. A simple circuit for showing the oscillation of electric charges.

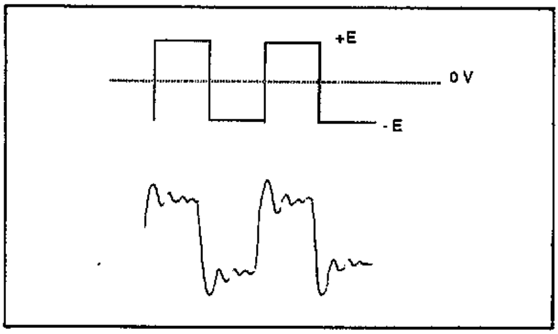


Figure 2. Typical input (top) and output (bottom) waveforms observed in the circuit shown in figure 1.

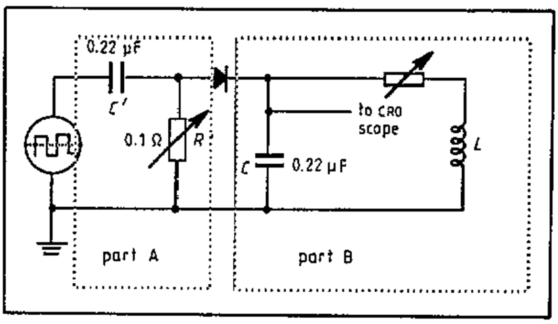
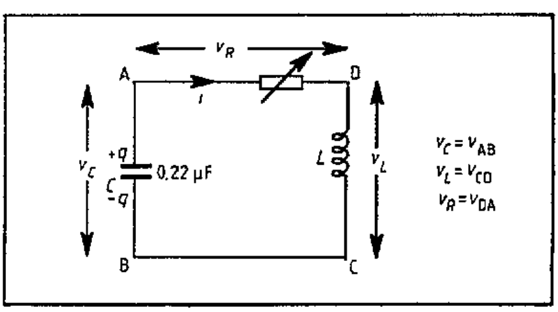


Figure 3. A modified circuit connection for showing oscillation of electric charges.

Figure 4. Sign convention of current and charge in an *RLC* circuit.



Theory

Let q be the charge in the capacitor of an *RLC* circuit, and $i = -\mathrm{d}q/\mathrm{d}t$ (the negative sign implies that i is positive when q is decreasing). Applying Kirchhoff's second rule to the *RLC* loop (figure 4), we have

$$v_C + v_L + v_R = 0$$

$$\frac{q}{C} - L\frac{\mathrm{d}i}{\mathrm{d}t} - iR = 0$$

or

$$L\,\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + R\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{q}{C} = 0.$$

The solution (for a more detailed treatment, see e.g. Ayres 1972) of this differential equation depends on the values of R, L and C and the initial charging state of the capacitor. In general, three different forms may be obtained:

(i) If $R^2 > 4L/C$, overdamped motion is observed (figure 5, top). In particular, if $R^2 \gg 4L/C$, the solution will assume the form of a simple *RC* decay curve:

$$q = q_1 \exp(-t/CR). \quad (1)$$

(ii) If $R^2 = 4L/C$, critically damped motion is observed (figure 5, middle), and the solution will assume the form

$$q = (q'_1 + q'_2 t)\exp[-(R/2L)t]. \quad (2)$$

(iii) If $R^2 < 4L/C$, underdamped oscillation will be observed (figure 5, bottom). For $4L/C \gg R^2$, the solution will become

$$q = q''_1 \exp[-(R/2L)t]\sin(2\pi t/T + \phi) \quad (3)$$

where q_1, q'_1, q'_2, q''_1 and ϕ are constants to be determined by initial conditions and $T = 2\pi(LC)^{-1/2}$ is the period of the *LC* oscillation.

Choices of circuit parameters

In our experiment we employ $C' = 0.22\ \mu\mathrm{F}$ and $R' = 0\text{–}1\ \Omega$ in the charging circuit. A power signal generator (e.g. Unilab model 062.101 with maximum output 1.0 A) is required to generate a short but very strong charging pulse. The capacitance in the *RLC* circuit is $C = 0.23 \pm 0.01\ \mu\mathrm{F}$. In order to obtain an underdamped curve with distinct cycles, the charging frequency, f', should be about 1/10 that of the resonance frequency. Also, the range of variable resistance, R, must be larger than the resistance at which critically damped motion is observed. Recommended combinations of f' and R are shown in table 1.

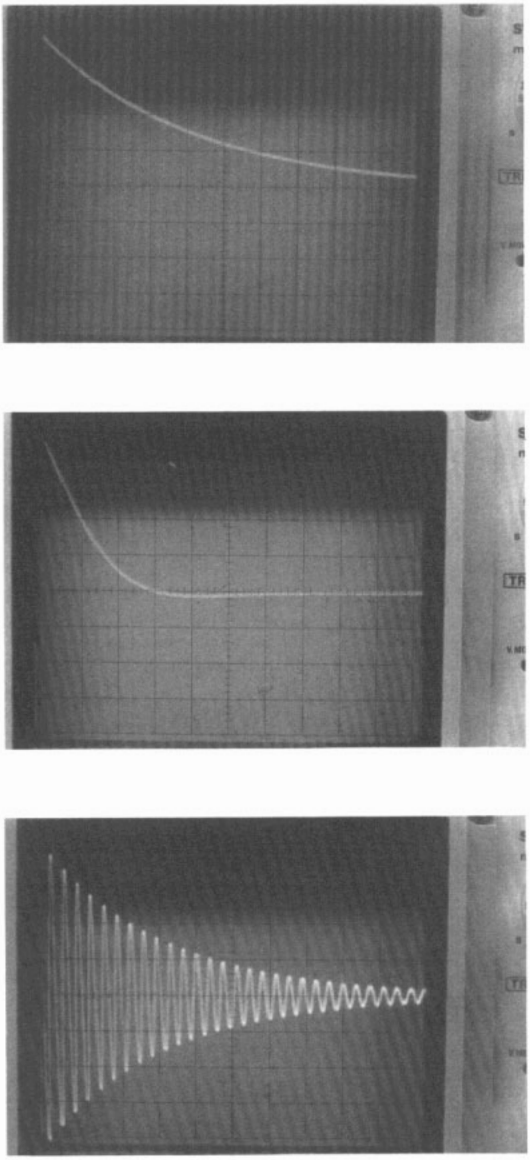


Figure 5. Overdamped motion (top), critically damped motion (middle) and underdamped oscillation (bottom).

Three independent measurements of L

Two values of L can be obtained by considering only the underdamped motion represented by (3). First, by measuring the period of the oscillation, L can be calculated using

$$L = T^2/4\pi^2 C. \quad (4)$$

Second, if we plot the logarithm of the amplitude of oscillation, $\ln V$, against time, t, L can be calculated from

$$L = -R/2m \quad (5)$$

Table 1. Choices of f' and R for $C = 0.23\ \mu F$.

L (H)	f' (Hz)	R (Ω)
1	50	$0\text{–}15 \times 10^3$
10^{-3}	500	0–500
10^{-6}	5000	0–20

where m is the slope of the graph of $\ln V$ against t.

A third reading for L can be obtained by finding the value of R that produces critically damped motion. When this happens,

$$L = R^2 C/4. \quad (6)$$

So L can be calculated if R and C are known.

Results and discussion

Two inductors have been chosen as specimens. The first is a single-layer, 255 turn, air-core solenoid, 0.28 m long, and with a cross-sectional area of 0.0055 m^2. Using the formula $L = \mu_0 A N^2/l$ (Nelkon and Parker 1977), the calculated inductance is 1.60×10^{-3} H. The second is a single circular coil of copper wire, 1.0 m long and 1.0 mm in diameter. Using the formula $L = (\mu_0 l/2\pi)\ \ln(8l/\pi d - 7/4)$ (Harnwell 1949), where l is the length and d is the diameter of the wire, the calculated inductance is 1.22×10^{-6} H.

Measurements for calculating L from (4), (5) and (6) are listed in table 2. The period T is obtained directly from the CRO trace. The slope m is obtained from a ($\ln V$ versus t) plot at $R = 7.7 \pm 0.1\ \Omega$ for the solenoid and $1.0 \pm 0.05\ \Omega$ for the coil. The value of R at critical damping is measured using a digital multimeter.

The inductances measured or calculated from various methods are summarized in table 3. All experimental results listed are accurate to within $\pm 10\%$. They are consistent with one another and with the theoretical value. The result obtained using formula (6) is less accurate than the other two methods because the position of the rheostat at critical damping can be judged only by inspection; that is, we adjust the rheostat until the trace of v_c returns to the zero level in the shortest time without oscillation. The error in determining R at critical damping is further magnified by the dependence of L on R^2. Moreover, because the decay time for critical damping is very short, its trace may be superimposed on the charging curve of the differential circuit. This explains why we fail to obtain an accurate result for the single coil using this method. However, results from (4) and

Table 2. Data for three independent measurements of inductance.

Specimen	Period T when $R \approx 0\,\Omega$	Slope m	Value of R at critical damping
Solenoid	$(5.8 \pm 0.1) \times 20\ \mu s$	$-(2.92 \pm 0.1) \times 10^3$	$(150 \pm 8)\,\Omega$
Coil	$(6.7 \pm 0.1) \times 0.5\ \mu s$	$-(0.42 \pm 0.02) \times 10^6$	Not used due to large error

Table 3. Summary of results.

Specimen	Method			From theoretical formula
	using (4)	using (5)	using (6)	
Solenoid (in mH)	1.45 ± 0.10	1.31 ± 0.10	1.31 ± 0.13	1.60
Coil (in μH)	1.22 ± 0.10	1.19 ± 0.10	not used	1.22

(5) rely on the calibration of the time base of the CRO whereas the result obtained from (6) is independent of the accuracy of the time base. This is a point worthy of class discussion.

In measuring the inductance of the solenoid, all experimental results are found to be smaller than the theoretical value by about 10–20%. This discrepancy is understandable because in most treatments in physics textbooks the formula $L = \mu_0 A N^2 / l$ is obtained by assuming a constant flux density throughout the solenoid, whereas in fact it decreases to only half of its maximum value at the end. However, the experimental results agree well with a more exact formula $L = \mu_0 K A N^2 / l$, where K is called Nagaoka's constant and is a function of the dimensions of the solenoid. In our case, $K = 0.9$ (Smith 1953) and the calculated inductance is $L = 0.9 \times 1.60$ mH $= 1.44$ mH. Alternatively, the experimental values might also be checked using an accurate electronic inductance meter (Philip Harris 1992).

In measuring the inductance of the circular coil, we discover that the measured value is not sensitive to the shape of the coil, provided that different segments of the coil are kept far apart from one another. This sounds reasonable because, by the Biot–Savart law, the magnetic field produced by a small wire segment has an inverse square dependence on distance, and so the flux linkage between different segments can be neglected unless the coil is twisted. Consequently, the inductance formula $L = (\mu_0 l / 2\pi) \ln(8l/\pi d - 7/4)$ is a very good approximation to a coil of arbitrary shape.

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