LETTERS AND COMMENTS

Comment on ‘Families of Keplerian orbits’

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Abstract
A surprising analytic solution based on the historical publication of the famous rocket engineer Tsander is given. The Keplerian orbit is analytically fully defined by the velocity vector at only one given distance from the force centre.

Butikov in [1] discusses Keplerian orbits that are generated by satellites whose initial velocities at a given initial point have arbitrary directions. This comment seeks not to contradict Butikov’s discussion but to add some historical facts and an analytic solution relating to the subject. The Keplerian orbits of satellites not only have interesting properties from a geometric point of view, but the problem also has an interesting history and surprising analytic solution.

The history of studies of how Keplerian orbits depend on launch velocity is at least 80 years old, and perhaps started when the orbits of long-range rockets outside the atmosphere were studied by the famous rocket engineer Friedrich Zander (born into a German family in Riga, Latvia), who later developed the first Soviet rocket, GIRD-X. The militaristic background of Tsander’s research is evident. Tsander solved the problem of reaching any point on Earth by long-range flying rockets.

A number of the basic statements of contemporary astrodynamics were set forth and scientifically established by Tsander during the first half of the 1920s. However, this has not been reflected in the literature. A part of Tsander’s works on astrodynamics was written in German in the shorthand script of the Gabelsberger system. His famous work ‘Problems of flight by jet propulsion’ was first published in Russian in 1932. After World War II, a significant portion of the works of Tsander on astrodynamics was compiled and published in one work [2]. This work was translated into English by NASA in 1964 [3]. We find the discussion of Keplerian orbits for satellites in chapter 15 (Flight of rockets reaching far beyond the atmosphere) of this publication. Tsander starts this chapter with the sentence: ‘I have calculated the range outside the atmosphere of rockets with a given initial velocity for a static Earth’ [4].

1 On leave from: Faculty of Physics and Mathematics, Latvia University, Zellu 8, LV-1002 Riga, Latvia.
Let us follow Tsander’s calculations to get an analytic solution for orbits depending on velocity. I will use only small transformations and improvements to derive scaled equations of orbits.

Analysing Kepler’s laws, Tsander found the following relationships set speed by the initial velocity \( v \) and angle \( \beta \) between the initial velocity \( v \) and \( v_0 \) (orbital velocity) or horizon for an ellipse with given main semiaxis \( a \) and eccentricity \( \varepsilon \) (equations (14) and (12) in [3]):

\[
v = v_0 \sqrt{2 - \frac{r_0}{a}}, \quad \cos \beta = \frac{a^2 (1 - \varepsilon^2)}{(2a - r_0) r_0}
\]

where \( r_0 \) is the initial point of launch. The solution of this system gives expressions for the scaled main semiaxis and eccentricity as functions of velocity \( v \) and direction \( \beta \):

\[
a = \frac{1}{2 - \gamma^2} \quad (1) \\
\varepsilon = \sqrt{1 - (\gamma \cos \beta)^2 (2 - \gamma^2)} \quad (2)
\]

where \( \gamma = \frac{v}{v_0} \) is the scaled velocity. Putting equations (1) and (2) in the polar equation of the ellipse, we get a dimensionless equation for a family of Keplerian orbits:

\[
\rho = \frac{(\gamma \cos \beta)^2}{1 + \sqrt{1 - (\gamma \cos \beta)^2 (2 - \gamma^2)}} \cos \varphi \quad (3)
\]

where \( \rho = \frac{r}{r_0} \).

Solving this equation for \( \rho = 1 \) we get the true anomaly for the initial point

\[
\cos \vartheta = \frac{(\gamma \cos \beta)^2 - 1}{\sqrt{1 - (\gamma \cos \beta)^2 (2 - \gamma^2)}}. \quad (4)
\]

With equations (3) and (4) the family of Keplerian orbits as functions of \( \gamma \) and \( \beta \) can be expressed by

\[
x = \rho \cos(\varphi - \vartheta), \quad y = \rho \sin(\varphi - \vartheta). \quad (5)
\]

Let us note that the Keplerian family of orbits depends only on velocity. This means that the orbit is defined if we know the velocity vector \((\gamma, \beta)\) at only one point \( \rho = 1 \).

Most properties of Keplerian families can be deduced from this analytic solution. For example, the main semiaxis equation (1) is the same for all orbits of satellites launched with equal magnitudes of the initial velocities \( \gamma \). Butikov describes another interesting property of the family of orbits, too [1]: ‘The second focus of each orbit lies on a circle whose centre is located at the common initial position.’ Analysing the expressions for coordinates of the second focus, we can find the radius of this circle as \( \frac{\gamma^2}{2 - \gamma^2} \).

Equations (3)–(5) enable us to generate families of orbits using the simplest computer algebra system, for example Derive, or a graphic calculator, such as TI-92. A script is given for Derive in the appendix.

In addition to the interesting geometric properties shown by Butikov on the base of numerical simulations, the treatment of families of Keplerian orbits has an interesting history and a surprising solution: if we know the velocity and its direction at only one given distance from the force centre, then we can construct the orbit analytically.

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Appendix

Derive script for generating families of orbits according to equations (3)–(5)

#1 \[ \gamma : \varepsilon (0, \sqrt{2}), \beta : \varepsilon (-\pi/2, \pi/2) \]

#2 \[ \varepsilon (\gamma, \beta) := \sqrt{1 - (\gamma \cos \beta)^2(2 - \gamma^2)}, \rho (\gamma, \beta) := (\gamma \cos \beta)^2/(1 + \varepsilon (\gamma, \beta) \cos \phi), \]

\[ \vartheta (\gamma, \beta) := \text{atan}(\gamma^2 \sin \beta \cdot \cos \beta, (\gamma \cos \beta)^2 - 1) \]

#3 \[ \omega (\gamma, \beta) := \rho (\gamma, \beta) \cos (\vartheta (\gamma, \beta)), \rho (\gamma, \beta) \sin (\vartheta (\gamma, \beta)) \]

#4 vector(o(0.9, \beta), \beta, -90^\circ, 90^\circ, 10^\circ)

#5 vector(o(1.1, \beta), \beta, -90^\circ, 90^\circ, 10^\circ)

#6 ‘Plot #4 or #5 for family of 18 orbits with launching velocity \( v = 0.9v_0 \) or \( 1.1v_0 \)’

References