The recent article by Kidd and Fogg¹ should result in the general inclusion of the formula

\[ T = 2\pi \sqrt{\frac{\ell}{g \cos\left(\frac{\theta_m}{2}\right)}} \]  

(1)

into new editions of physics textbooks, replacing the standard formula

\[ T = 2\pi \sqrt{\frac{\ell}{g}}. \]  

(2)

\( T \) is the period of oscillations for a simple pendulum of length \( \ell \), \( g \) is the acceleration caused by gravity, and \( \theta_m \) is the maximum angle of oscillation. In addition, the standard approximation of Eq. (3) in the usual small-angle derivation leads directly to Eq. (1).

The better formula has been staring us in the face for decades and we haven’t noticed. The well-known trigonometric relation for the half-angle formula is

\[ \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}, \]  

(3)

should be improved to

\[ \sin \theta \approx \theta \cos \frac{\theta}{2}. \]  

(4)

in textbooks as well. Substitution of Eq. (4) for Eq. (3) in the usual small-angle derivation leads directly to Eq. (1).

The better formula has been staring us in the face for decades and we haven’t noticed. The well-known trigonometric relation for the half-angle formula is

\[ \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}. \]  

(5)

We make the approximations

\[ \sin \frac{\theta}{2} \approx \frac{\theta}{2} \]  

(6)

and

\[ \cos \frac{\theta}{2} \approx \cos \left(\frac{\theta_m}{2}\right). \]  

(7)

Sin \( \frac{\theta}{2} \) is replaced by \( \frac{\theta}{2} \) to give a linear function, and cos \( \frac{\theta}{2} \) is replaced by \( \cos \left(\frac{\theta_m}{2}\right) \) to obtain a constant. Thus, we have a constant times the linear function \( \theta \), which is what we want in approximating simple harmonic motion. Since \( \frac{\theta}{2} \) is always a bit too large compared to \( \sin \frac{\theta}{2} \), we approximate \( \cos \frac{\theta}{2} \) by \( \cos \left(\frac{\theta_m}{2}\right) \), which is a bit too small, so that their product is about right. Note that we do not approximate \( \cos \frac{\theta}{2} \) the large-angle pendulum period

L. Edward Millet, California State University, Chico, CA

Fig. 1. Graph of the sine function and its two linear approximations.
by 1, which is too large and would multiply the already-too-large \( \frac{\theta}{2} \), giving us \( \sin \theta \approx \theta \), which is the prior poor approximation and is always too large. Note that the approximations were made over the range of angles 0 to \( \frac{\theta}{2} \) and not over the range of angles 0 to \( \theta_m \), because \( \sin \theta \) is more linear in the range 0 to \( \frac{\theta}{2} \) than it is over the entire range 0 to \( \theta_m \).

We have thus obtained Eq. (4), which is a much better approximation for \( \sin \theta \) than is Eq. (3).

Figure 1 shows a graph of \( \sin \theta \) versus \( \theta \) between 0 and \( \theta_{\text{max}} \), for the arbitrary choice \( \theta_{\text{max}} = 90^\circ \). Also shown are two straight lines: the steeper one is the standard approximation, Eq. (3), with slope of value one \( (\theta = \theta) \), and the other line, using Eq. (4), has the constant slope equal to the slope of the \( \sin \theta \) function at its midpoint angle, in this example 45\(^\circ\). It is obvious that the latter is a better overall linear approximation to \( \sin \theta \) than is \( \theta \), especially near \( \theta_{\text{max}} \), where \( \theta_m \) is 57% larger than \( \sin \theta \). For large-amplitude oscillations, it is important that the straight line be close to the value of \( \sin \theta \) near \( \theta_{\text{max}} \) because that is where the pendulum spends more time. It spends little time near \( \theta = 0 \), where it swings rapidly.

The reader might get the impression, looking at the graph, that for small \( \theta_m \) the steeper line would give a better approximation. But we must remember that changing \( \theta_m \) changes the slope also. For any amplitude, Eq. (1) gives better values for the period than Eq. (2). Equation (1) gives periods that are always slightly larger than the exact period, while in comparison, Eq. (2) gives periods that are always smaller than the exact period. At \( \theta_{\text{max}} = 30^\circ \), Eq. (1) is 0.0075% too large while Eq. (2) is 1.7% too small. At \( \theta_{\text{max}} = 90^\circ \), Eq. (1) is 0.75% too large while Eq. (2) is 15% too small.

If authors include this new formula in textbooks, students will be able to find more realistic values for some current textbook problems.\(^2\)

References

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L. Edward Millet is emeritus professor of physics at CSU, Chico, where he taught from 1967 to 2002. He earned a bachelor’s degree in physics from Brigham Young University in 1962 and a Ph.D. in solid state physics from Brigham Young in 1968. At CSU, he proposed and taught an upper-division general education course entitled Relativity and Albert Einstein.

Department of Physics, California State University, Chico, CA 95929-0202; Lemillet@aol.com