The Large-Angle Pendulum Period

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The recent article by Kidd and Fogg[1] should result in the general inclusion of the formula

$$T = 2\pi\sqrt{\frac{\ell}{g\cos(\frac{\theta_m}{2})}} \tag{1}$$

into new editions of physics textbooks, replacing the standard formula

$$T = 2\pi\sqrt{\frac{\ell}{g}}. \tag{2}$$

T is the period of oscillations for a simple pendulum of length ℓ, g is the acceleration caused by gravity, and θ_m is the maximum angle of oscillation. In addition, the standard approximation

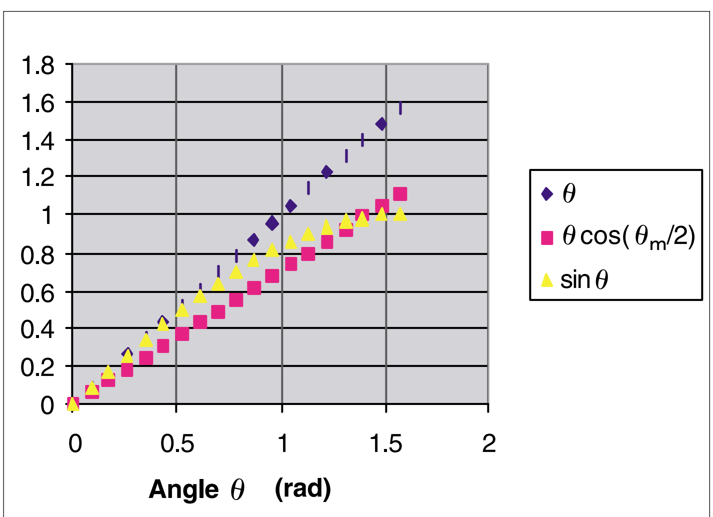

Fig. 1. Graph of the sine function and its two linear approximations.

$$\sin\theta \approx \theta \tag{3}$$

should be improved to

$$\sin\theta \approx \theta\cos(\frac{\theta_m}{2}) \tag{4}$$

in textbooks as well. Substitution of Eq. (4) for Eq. (3) in the usual small-angle derivation leads directly to Eq. (1).

The better formula has been staring us in the face for decades and we haven't noticed. The well-known trigonometric relation for the half-angle formula is

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}. \tag{5}$$

We make the approximations

$$\sin\frac{\theta}{2} \approx \frac{\theta}{2} \tag{6}$$

and

$$\cos\frac{\theta}{2} \approx \cos(\frac{\theta_m}{2}). \tag{7}$$

Sin $\frac{\theta}{2}$ is replaced by $\frac{\theta}{2}$ to give a **linear function**, and cos $\frac{\theta}{2}$ is replaced by cos $(\frac{\theta_m}{2})$ to obtain a **constant.** Thus, we have a constant times the linear function θ, which is what we want in approximating simple harmonic motion. Since $\frac{\theta}{2}$ is always a bit too large compared to sin $\frac{\theta}{2}$, we approximate cos $\frac{\theta}{2}$ by cos$(\frac{\theta_m}{2})$, which is a bit too small, so that their product is about right. Note that we do not approximate cos $\frac{\theta}{2}$

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by 1, which is too large and would multiply the already-too-large $\frac{\theta}{2}$, giving us $\sin\theta \approx \theta$, which is the prior poor approximation and is always too large. Note that the approximations were made over the range of angles 0 to $\frac{\theta_m}{2}$ and not over the range of angles 0 to θ_m because $\sin\theta$ is more linear in the range 0 to $\frac{\theta_m}{2}$ than it is over the entire range 0 to θ_m. We have thus obtained Eq. (4), which is a much better approximation for $\sin\theta$ than is Eq. (3).

Figure 1 shows a graph of $\sin\theta$ versus θ between 0 and θ_{max} for the arbitrary choice $\theta_{max} = 90°$. Also shown are two straight lines: the steeper one is the standard approximation, Eq. (3), with slope of value one ($\theta = \theta$), and the other line, using Eq. (4), has the constant slope equal to the slope of the $\sin\theta$ function at its midpoint angle, in this example 45°. It is obvious that the latter is a better overall linear approximation to $\sin\theta$ than is θ, especially near θ_{max}, where θ is 57% larger than $\sin\theta$. For large-amplitude oscillations, it is important that the straight line be close to the value of $\sin\theta$ near θ_{max} because that is where the pendulum spends more time. It spends little time near $\theta = 0$, where it swings rapidly.

The reader might get the impression, looking at the graph, that for small θ_m the steeper line would give a better approximation. But we must remember that changing θ_m changes the slope also. For any amplitude, Eq. (1) gives better values for the period than Eq. (2). Equation (1) gives periods that are always slightly larger than the exact period, while in comparison, Eq. (2) gives periods that are always smaller than the exact period. At $\theta_{max} = 30°$, Eq. (1) is 0.0075% too large while Eq. (2) is 1.7% too small. At $\theta_{max} = 90°$, Eq. (1) is 0.75% too large while Eq. (2) is 15% too small.

If authors include this new formula in textbooks, students will be able to find more realistic values for some current textbook problems.[2]

References

1. R.B. Kidd and S.L. Fogg, "A simple formula for the large-angle pendulum period," *Phys. Teach*. **40**, 81–83 (Feb. 2002).

2. R.A. Serway and R. J. Beichner, *Physics For Scientists and Engineers*, 5th ed. (Harcourt Brace, Orlando, FL, 2000), Vol. I, p. 416, Problem 28.

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