A Simple Formula for the Large-Angle Pendulum Period

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Measurement of the period of a simple pendulum for small angles, often coupled with a calculation of g, is a standard exercise in lower-division physics laboratory sessions. We recently decided to extend the measurements to large-angle oscillations. We believe that such an experiment should include (1) a simple theoretical formula for the larger-angle period, which should be related in a straightforward way to the familiar small-angle-period equation, and (2) an experimental setup capable of meaningful measurements.

The elliptic integral derivation[1,2] of the large-angle pendulum period in terms of the angular half-amplitude $\alpha/2$ is the standard approach, but it is fairly involved and leads to values that must be looked up in a table. Some excellent introductory textbooks[3,4] simply cite a series approximation, but we feel that some justification is desirable and a series derivation may be a bit advanced for first-semester students. The ingenious strategies of Santarelli et al.,[5] Molina,[6] Ganley,[7] and Caldwell and Boyco[8] are a little too involved for our purposes, and we believe we have contrived a simpler intuitive approach.

Development of the Correction Formula

Our starting point is the analogy between the period $T_0 = 2\pi\sqrt{\ell/g}$ of a pendulum in the small-angle approximation and the period of a simple harmonic oscillator (SHO) $T = 2\pi\sqrt{m/k}$. The formula for the simple harmonic oscillator period can be found by making the usual substitution of $x_{(t)} = A\cos\omega t$ and its second derivative into Newton's second law and solving for ω; a similar procedure leads to the formula for the simple pendulum in the small-angle approximation where $\sin\theta$ is replaced by θ. Although k is a constant for the SHO, it may be defined as $k = |dF/dx|$. We seek the analog of k for the simple pendulum in the form $f(\theta) = |dF/ds|$, where $F = -mg\sin\theta$ and $s = \ell\theta$. [It may be pedagogically useful to digress at this point to demonstrate to students that $-\int_0^\alpha F\ell d\theta = mg\ell(1 - \cos\alpha)$, the energy of the system, where α is again the angular amplitude.]

Clearly, $f_{(\theta)} = mg\cos\theta/\ell$, but the problem, of course, is that $f_{(\theta)}$ is not constant, since the absolute value of θ varies between 0 and α over each quarter-cycle. Nonetheless, we make the assumption that the period T for a considerable range of α can be given in terms of $f_{(\beta)}$ for some fixed representative angle β, i.e., that $T = 2\pi\sqrt{m/f_{(\theta)}}$, where $f_{(\theta)}$ is approximated by $\dfrac{mg\cos\beta}{\ell}$ so that:

$$T = 2\pi\sqrt{\ell/g\cos\beta}. \tag{1}$$

It is reassuring that T goes to T_0 as β approaches 0, but T becomes undefined as β goes to $\pi/2$, so if we want to find T for α up to $\pi/2$, we must use a value of β in the range $0 \leq \beta < \alpha$. It is convenient at this point to consider $T = MT_0$, where multiplier $M = 1/\sqrt{\cos\beta}$. Exact values of M can be found from tables[9] as $K_{(\alpha/2)}/(\pi/2)$ where $K_{(\alpha/2)}$ is a complete elliptic integral of the first kind, and a suitable value of β can be determined from M. It turns out that for $\beta = \alpha/2$, the value of $1/\sqrt{\cos(\alpha/2)}$ agrees to within 1% with the exact value of M for amplitude α up to $\pi/2$ (see Table I for a comparison). We may therefore write the approximation as:

Table I. Comparison of correction formula with elliptic integral.

$\alpha/2$	$K_{(\alpha/2)}$	$\frac{K_{(\alpha/2)}}{\pi/2}$	$\frac{1}{\sqrt{\cos(\alpha/2)}}$	% difference
0°	1.570796	1	1	0
15°	1.598142	1.017409	1.017485	0.0075%
30°	1.685750	1.073182	1.074570	0.1294%
45°	1.854075	1.180341	1.189207	0.7512%
60°	2.156516	1.372881	1.414214	3.0107%
75°	2.768063	1.762204	1.965631	11.5439%
90°	∞	∞	∞	--

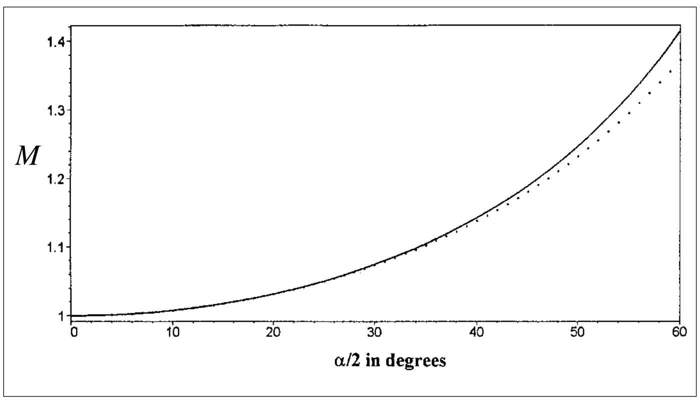

Fig. 1. Exact value of $M = T/T_0$ (dotted curve) and our cosine approximation (solid curve) plotted against angular half-amplitude $\alpha/2$. (Note that graph is truncated near $M = 1$ rather than extending to $M = 0$ in order to maximize area of interest.) Since approximate M is within 1% of exact value for $\alpha/2 = 45°$, it is evident that our approximation is useful up to an amplitude of 90°.

$$T \cong 2\pi\sqrt{\ell/g\cos(\alpha/2)} \text{ for } \alpha \leq \pi/2. \qquad (2)$$

Insight as to the behavior of the approximation may be gleaned from a term-by-term comparison of the series expansion for $1/\sqrt{\cos(\alpha/2)}$ with the corresponding series for the elliptic integral, $K_{(k)}$,[10] after substitution of the series expansion for $k = \sin(\alpha/2)$:

$$\frac{1}{\sqrt{\cos(\alpha/2)}} = 1 + \frac{\alpha^2}{16} + \frac{7\alpha^4}{1536} + \frac{139\alpha^6}{368640} + \ldots^{11} \qquad (3)$$

$$\frac{K(\alpha/2)}{\pi/2} = 1 + \frac{\alpha^2}{16} + \frac{11\alpha^4}{3072} + \frac{173\alpha^6}{737280} + \ldots \qquad (4)$$

Since the first two terms of these two series are identical, series (3) tracks series (4) closely for small α, but as the coefficients of the third and subsequent terms of series (3) are larger than their counterparts in series (4), our formula diverges increasingly from the exact expression as α approaches $\pi/2$.

The Experiment

Our first attempt to devise a large-angle pendulum experiment, using a drilled ball and a light thread, met with difficulty. The increase of period with amplitude is a rather delicate effect, amounting to only about 18% over the range of α from 0 to 90°, and the stretch of the string from being pulled taut destroyed the accuracy of our measurements. A second problem was the rapid decay of amplitude due to air resistance.

We decided to proceed using a physical pendulum having a fairly long period, so as to minimize uncertainty in T, and being massive enough to render air resistance insignificant. We finally settled on a massive flat metal rectangle over a meter long and about 5 cm long, weighing more than 12 pounds. The rod rotated on a steel shaft and ball-bearing assembly inserted a third of the way from its top. The apparatus held its amplitude quite well, taking, for example, more than two minutes for α to decay from 90° to 75°. We realized that a pendulum this massive could be hazardous to a careless student, so the shaft's support was clamped close to the edge of a lab table and the bar allowed to swing below the level of the table, both for safety reasons and to minimize sway of the apparatus.

The duration of one period was measured by a photogate timer in pendulum mode placed on the floor, which featured accuracy within 1% of measured $T \pm 0.001$s. As a precaution against collision, the photogate was triggered by a small cardboard tab extending from the bottom of the pendulum. The value of α was measured using a reference line on the upper end of the pendulum and a large cardboard protractor mounted immediately behind the bar. Small errors in determination of α were not a problem so long as they were not systematic, since the period changes so slowly with amplitude.

After some discussion, we concluded that g was tolerably familiar to students already and that it would be more valuable to concentrate on the relation of T to I, the moment of inertia of the bar, which had been discussed in the lectures. Previous to the lab session, students were instructed to calculate the moment of inertia of the bar in terms of its given dimensions and of its mass (which was not measured), and to bring their results to class as Part A of the experiment. Part B of the experiment consisted of successive teams of students making measurements of T at 15° intervals and posting five consistent measurements on the blackboard for each value of amplitude. Class members then individually found the average value of T for each angle and converted it to the corresponding value of T_0 via multiplication by $\sqrt{\cos(\alpha/2)}$. The six values of T_0 were then averaged to produce a final value. In their spare time, while the teams were taking measurements, other students completed Part A of the experiment by calculating the theoretical value of T_0 from the relation[4]

$$T_0 = 2\pi\sqrt{I/mgd}. \tag{6}$$

(Note that m cancels out in this step.) In Part C of the experiment, students found the percent difference between the theoretical and the experimental values of T_0, which in this instance, perhaps fortuitously, was about 1%.

Conclusion

The small-angle simple-pendulum formula commonly exhibited to first-year students is only the limiting case of a much wider range of real-world behavior. We found that it is feasible to extend the theory to the case of larger amplitudes and to employ it in a fairly involved laboratory experiment. (Less elaborate experiments may be appropriate.) In any case, as has been noted elsewhere,[12] the plane pendulum is particularly rich in physics implications, and an understanding of its behavior over a more realistic range of phenomena is a worthwhile goal.

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