Projectile motion with air resistance quadratic in the speed

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We consider two-dimensional motion of a projectile experiencing a constant gravitational force and a fluid drag force which is quadratic in the projectile's speed. The equations of motions are coupled nonlinear equations. Their solutions have general properties which are easily visualized, although much different from those obtained when a drag force is neglected. Because of these features a study of these equations would provide an interesting counterpoint to the already familiar case of no drag. In this paper we derive simple approximate solutions to the equations of motion for both short and long times. A numerical example is used to compare these approximate solutions with accurate results obtained by numerical integration from an exact but implicit solution. Finally, the origin of the quadratic drag force is discussed.

I. INTRODUCTION

Physics students are certainly familar with the solution of projectile problems when air resistance is neglected. Since, in practice, air resistance is usually not negligible, it would be of some interest to develop solutions of the equations of motion including an appropriate drag force.

If one assumes a drag force which is linear in the speed, then it is straightforward to solve the equations of motion. Such a drag force actually occurs for low Reynolds numbers, $\mathcal{R} < 1$, and is dependent upon the viscosity of the fluid. To determine whether or not it applies, one must calculate the Reynolds number. For a sphere of radius r moving in a fluid of density ρ and viscosity η, the viscous drag force and the Reynolds number are[1,2]

$$F = 6\pi\eta r v \quad \text{for } \mathcal{R} < 1,$$
$$\mathcal{R} = 2r\rho V\eta^{-1}. \qquad (1)$$

The terminal speed U is then given by

$$6\pi\eta r U = mg = 4\pi r^3\rho_s g/3,$$

where ρ_s is the sphere's density. In order that Eq. (1) apply to the whole of a trajectory, we require $\mathcal{R} \lesssim 1$ for speeds up to the terminal speed. This can be considered a minimum requirement since projectiles are often started off with speeds greater than this. Assuming a sphere which has the density of water, 1 g/cm^3, and using the value $\eta/\rho = 0.15$ cm^2/sec for air, one finds that Eq. (1) applies when[1]

$$r \lesssim 4 \times 10^{-3}\ \text{cm}. \qquad (2)$$

Spheres this small would seem to be of little interest. In fact, these small radii are not that far from the mean free path λ of air molecules, which is typically of the order of 10^{-5} cm. A second criterion for the validity of Eq. (1) is[3]

$$r \gg \lambda. \qquad (3)$$

We conclude from Eqs. (2) and (3) that Eq. (1) almost never applies exclusively.[4]

It is known that for spheres with radii and speeds of practical interest a reasonable approximation to the drag force is[5,6]

$$F = C_D\rho A v^2/2 \quad \text{for } 1 \ll \mathcal{R} \lesssim 10^5. \qquad (4)$$

The drag coefficient is

$$C_D \approx \tfrac{1}{2},$$

and $A = \pi r^2$ is the cross-sectional area. This law, Eq. (4), holds for large Reynolds numbers and is associated with turbulence produced in the sphere's wake as described in Sec. VIII. As an example, it has been shown that a stone of 1-cm radius would have a terminal speed of about 30 m/sec in air.[7] Using the value of ρ/η given above, we see that this speed corresponds to a Reynolds number of 4×10^4. It is true that, if the stone were dropped from rest, Eq. (1) would hold initially, but it would only be for very short times.[7] Thus, it is Eq. (4) rather than Eq. (1) that is of practical interest.

The equations of motion including the drag force of Eq. (4) are more difficult to solve, however. An exact and explicit solution for vertical free fall, or one-dimensional motion, can be obtained rather easily.[8] For two-dimensional motion, however, no such solution appears to exist. This offers an opportunity to develop approximate solutions to a nonlinear problem of some pedagogical interest. Thus, in this paper, we derive some simple approximate solutions for both long and short times. In addition, an exact but implicit solution is developed which enables accurate numerical solutions to be calculated for comparison. Finally, we present a description of the quadratic drag force which focuses attention on the kind of turbulence that is known to be associated with this force law.

II. EQUATIONS OF MOTION AND SOME GENERAL PROPERTIES OF THEIR SOLUTIONS

The equations of motion are

$$m\, dv_x/dt = -bv_x(v_x{}^2 + v_y{}^2)^{1/2},$$
$$m\, dv_y/dt = -mg - bv_y(v_x{}^2 + v_y{}^2)^{1/2}.$$

We have chosen the positive y direction to be vertically upward and the constant b to represent the coefficient in Eq. (4). The natural scale of velocities is set by the terminal speed V, which corresponds to the solution $v_x = 0$, $v_y = -V$. Thus,

$$V = (mg/b)^{1/2}.$$

This is the only combination of the available parameters which has velocity units. Given this speed and the acceleration of gravity, the natural time unit is

$$T = V/g.$$

This is a measure of the time required to reach terminal speed. With these two units all subsequent equations may be expressed in dimensionless form. To this end we define scaled time, velocity, and displacement variables as follows:

$$\tau = t/T,$$
$$u_x = v_x/V, \quad u_y = v_y/V,$$
$$X = x/VT, \quad Y = y/VT.$$

In terms of these new variables, the equations of motion are

$$\dot{u}_x = -u_x(u_x{}^2 + u_y{}^2)^{1/2},$$
$$\dot{u}_y = -1 - u_y(u_x{}^2 + u_y{}^2)^{1/2}. \quad (5)$$

Here we have used dots to denote derivatives of $u_x(\tau)$ and $u_y(\tau)$ with respect to τ. The new displacements are

$$X = X_0 + \int_0^\tau u_x\, d\tau, \quad Y = Y_0 + \int_0^\tau u_y\, d\tau. \quad (6)$$

It will be seen that for times τ, or speeds $u = (u_x{}^2 + u_y{}^2)^{1/2}$, or displacements X or Y small compared to unity the effects of air resistance will be small.

We describe here some general properties of the solutions of Eq. (5): (1) The most often noted feature of Eq. (5) is the existence of the terminal speed V. Hail stones are a good example to use. Their speed with no drag would make them lethal. (2) No matter what the velocity is initially, the final velocity is always the same, namely $u_x = 0$, $u_y = -1$. (3) Every path in the x,y plane corresponding to different initial velocities has a different shape. In the absence of drag every path has the same parabolic shape. Thus, the higher the initial speed of a projectile for a given upward projection angle the more blunted the forward end of its path becomes, i.e., the less symmetrical it is about its peak. (4) The maximum horizontal range for a given initial speed occurs at angles less than 45°, and the greater the initial speed, the lower the projection angle that is required. For example, we find that the maximum horizontal range occurs at 36° for an initial speed $u_0 = 2$. (5) Since u_x approaches zero from its initial value u_{x0}, there is a maximum horizontal displacement which will be of the order of u_{x0} in terms of scaled variables (i.e., $v_{x0}T = u_{x0}VT$). Similarly, if the projectile is fired upward, it will reach a maximum y displacement of the order of u_{y0}, where u_{y0} is the initial value of u_y.

III. LOW TRAJECTORY—SHORT-TIME APPROXIMATION

An initial approach to the solution of Eq. (5) may be made by simplifying these equations from the outset. Consider projections made at low angles such that $u_y{}^2/u_x{}^2 \ll 1$ over a portion of a path. Then Eq. (5) becomes

$$\dot{u}_x \approx -u_x{}^2, \quad \dot{u}_y \approx -1 - u_x u_y.$$

The u_x equation may be integrated to give

$$u_x = u_{x0}(1 + u_{x0}\tau)^{-1}.$$

To simplify the u_y equation, put

$$u_y = h u_x \quad (7)$$

so that $u_x \dot{h} = -1$. Then

$$h = \frac{u_{y0}}{u_{x0}} - \int_0^\tau u_x{}^{-1}\, d\tau. \quad (8)$$

Using Eqs. (7) and (8) and the result for u_x, we obtain

$$u_x = u_{x0}(1 + u_{x0}\tau)^{-1},$$
$$u_y = [u_{y0} - \tau(1 + u_{x0}\tau/2)](1 + u_{x0}\tau)^{-1}. \quad (9)$$

The displacements calculated from Eq. (9) for $X_0 = Y_0 = 0$ are

$$X = \ln(1 + u_{x0}\tau),$$
$$Y = -\tau/2u_{x0} - \tau^2/4 + (u_{y0}/u_{x0} + 1/2u_{x0}{}^2)\ln(1 + u_{x0}\tau). \quad (10)$$

Equations (9) and (10) are both valid as long as $u_y{}^2/u_x{}^2 \ll 1$. Substituting these approximate solutions into this inequality shows that it is satisfied as long as

$$u_{y0}{}^2/u_{x0}{}^2 \ll 1 \quad \text{and} \quad \tau^2 \ll 1.$$

Both of these conditions are necessary since eventually the projectile's velocity turns downward. We may, however, remove the restriction to small initial projection angles as shown in Sec. IV.

If we eliminate the time between X and Y in Eq. (10), we obtain a relation valid for low trajectories, which is

$$Y = X(u_{y0}/u_{x0} + 1/2u_{x0}{}^2) + (1 - e^{2X})/4u_{x0}{}^2.$$

This result has been derived by Lamb[9] using a different approach.

IV. SHORT-TIME APPROXIMATION

By reducing the equations for u_x and u_y to a single equation we will be able to obtain useful approximations for both short and long times. Note that Eq. (5) requires that

$$(\dot{u}_y + 1)/\dot{u}_x = u_y/u_x.$$

This has the solution, already encountered in Eqs. (7) and (8),

$$u_y = u_x\left(\frac{u_{y0}}{u_{x0}} - \int_0^\tau u_x{}^{-1}\, d\tau\right). \quad (11)$$

Substituting Eq. (11) into Eq. (5) and simplifying the result with the substitution

$$z = 1/u_x, \quad (12)$$

we obtain

$$\dot{z} = \left[1 + \left(\frac{u_{y0}}{u_{x0}} - \int_0^\tau z\, d\tau\right)^2\right]^{1/2}. \quad (13)$$

Now expand z about $\tau = 0$ using the preceding equation to evaluate derivatives. This gives

$$z = 1/u_{x0} + u_0\tau/u_{x0} - u_{y0}\tau^2/2u_{x0}u_0 + \cdots, \quad (14)$$

where $u_0 = (u_{x0}{}^2 + u_{y0}{}^2)^{1/2}$. In order that the first two terms suffice, we require

$$\tau \ll u_0{}^2/u_{y0}. \quad (15)$$

Then the approximate solution is, from Eqs. (11), (12), and (14),

$$u_x = u_{x0}(1 + u_0\tau)^{-1},$$
$$u_y = [u_{y0} - \tau(1 + u_0\tau/2)](1 + u_0\tau)^{-1}. \quad (16)$$

These solutions reduce to those given by Eq. (9) when $u_{y0}^2/u_{x0}^2 \ll 1$. The only criterion for the validity of Eq. (16) is the short-time condition, Eq. (15). For very short times Eq. (16) reduces to

$$u_x = u_{x0}, \quad u_y = u_{y0} - \tau,$$

which are the solutions for no drag. The displacements corresponding to the velocities in Eq. (16) are obtained from Eq. (10) by replacing u_{x0} with u_0.

To complement the short-time approximation given here, we will obtain in Sec. V a solution valid for $\tau \gg 1$.

V. LONG-TIME APPROXIMATION

Since u_x ultimately approaches zero, the variable z defined in Eq. (12) will approach infinity at long times. We thus expand Eq. (13) in inverse powers of $\int_0^\tau z\, d\tau$, giving

$$\dot{z} \approx \int_0^\tau z\, d\tau - \frac{u_{y0}}{u_{x0}} + \left(2 \int_0^\tau z\, d\tau\right)^{-1} + \cdots.$$

We assume

$$\int_0^\tau z\, d\tau \gg 1, \quad (17)$$

so that

$$\dot{z} \approx \int_0^\tau z\, d\tau - \frac{u_{y0}}{u_{x0}}. \quad (18)$$

Since $\ddot{z} \approx z$, Eq. (18) is solved by a combination of exponentials. Substituting $z = Ae^\tau + Be^{-\tau}$ into Eq. (18) and requiring A and B to give the initial value of z leads to

$$z = u_{x0}^{-1} \cosh\tau - u_{y0}u_{x0}^{-1} \sinh\tau.$$

Combined with Eq. (11), this gives

$$u_x = u_{x0}(\cosh\tau - u_{y0} \sinh\tau)^{-1},$$
$$u_y = (u_{y0} \cosh\tau - \sinh\tau)(\cosh\tau - u_{y0} \sinh\tau)^{-1}. \quad (19)$$

These results are generally valid when $\tau \gg 1$. Evaluating $\int_0^\tau z\, d\tau$ to see when Eq. (17) is satisfied, we find

$$\int_0^\tau z\, d\tau \approx \frac{u_{y0}}{u_{x0}} + (1 - u_{y0}) \frac{e^\tau}{2u_{x0}} - (1 + u_{y0}) \frac{e^{-\tau}}{2u_{x0}}. \quad (20)$$

Thus, for $u_{x0} \neq 0$, Eq. (17) will ultimately be satisfied provided the coefficient of e^τ in Eq. (20) is positive or

$$u_{y0} < 1.$$

Equation (20) then shows that the long-time solution will apply for earlier times when u_{x0} or u_{y0} or both u_{x0} and u_{y0} are reduced. When $u_{x0} = 0$, Eq. (19) is the exact solution of the equation for *downward* motion,

$$\dot{u}_y = -1 + u_y^2.$$

VI. EXACT IMPLICIT SOLUTION

It is known that Eq. (5) can be reduced to quadratures.[10–12] One way of doing that is described here.

The original equations of motion have been reduced to the single first-order equation, Eq. (13). This is simplified further by defining

$$w = \int_0^\tau z\, d\tau - \frac{u_{y0}}{u_{x0}}, \quad \dot{w} = z.$$

Thus, Eq. (13) becomes

$$\ddot{w} = (1 + w^2)^{1/2}.$$

Multiplying this equation by $\dot{w}$, we have

$$(1/2)(d\dot{w}^2/d\tau) = \dot{w}(1 + w^2)^{1/2} = (df/dw)(dw/d\tau) = df/d\tau, \quad (21)$$

where

$$df/dw = (1 + w^2)^{1/2}. \quad (22)$$

Integration of Eq. (21) gives

$$\dot{w}^2 = \dot{w}_0^2 + 2f(w) - 2f(w_0), \quad (23)$$

and from Eq. (22),

$$f(w) = (1/2)w(1 + w^2)^{1/2} + (1/2) \ln[w + (1 + w^2)^{1/2}].$$

Since $\dot{w}_0 = z_0 = 1/u_{x0}$ and $w_0 = -u_{y0}/u_{x0}$, we obtain upon integration of Eq. (23),

$$\int_{-u_{y0}/u_{x0}}^{w} \frac{d\alpha}{s(\alpha)} = \tau, \quad (24)$$

where

$$s(\alpha) = [1/u_{x0}^2 + 2f(\alpha) - 2f(-u_{y0}/u_{x0})]^{1/2}. \quad (25)$$

Equations (24) and (25) give w implicitly as a function of τ. Even if the indefinite integral could be evaluated, we would probably not be able to invert the resulting expression to obtain an explicit solution. Actually, the numerical integration of Eq. (24) is straightforward. Once w is obtained, $\dot{w}$ is given by Eq. (25); i.e.,

$$\dot{w} = s(w). \quad (26)$$

The velocity components are then

$$u_x = 1/\dot{w}, \quad u_y = -w/\dot{w}. \quad (27)$$

The displacements may be obtained from Eq. (6) or by using the transformations $u_x = \dot{w}\, dX/dw$ and $u_y = \dot{w}\, dY/dw$ to obtain

$$X = X_0 + \int_{-u_{y0}/u_{x0}}^{w} \frac{d\alpha}{s^2(\alpha)},$$

$$Y = Y_0 - \int_{-u_{y0}/u_{x0}}^{w} \alpha \frac{d\alpha}{s^2(\alpha)}.$$

VII. NUMERICAL EXAMPLE

We choose the following initial conditions

$$u_{x0} = 1, \quad u_{y0} = 1/5.$$

These correspond to a large enough speed for air resistance to produce significant effects over the initial trajectory while still being in the range of validity of the short-time solution over most of the horizontal range. The short- and long-time solutions were evaluated and compared with the accurate solution obtained from Eqs. (24)–(27).[13] The results are presented in Figs. 1–3.

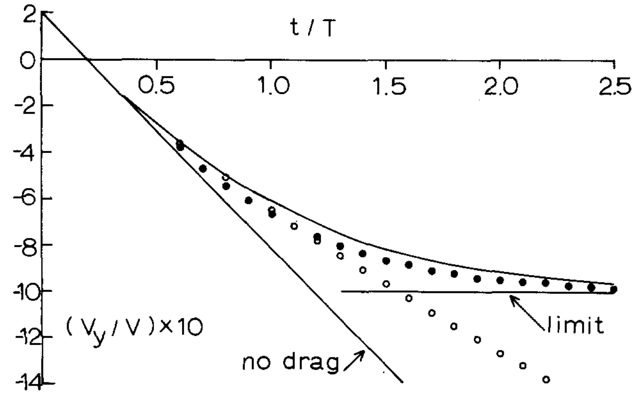

Fig. 1. Vertical component of velocity. Accurate values of the y component of velocity as a function of time are plotted as the solid line which approaches $-V$ as a limit. The short-time approximation is indicated by open circles, while the closed circles show the long-time approximation. For no drag force this component of velocity is the straight line shown.

Figure 1 shows the y component of the velocity as a function of time. The zero drag solution is $u_y = u_{y0} - \tau$, and it starts to deviate from the accurate solution near $\tau = 0.5$. The short-time and long-time solutions are in agreement up to about $\tau = 1.2$, and they are not too much different from the accurate solution. Beyond $\tau = 1.2$ the short-time solution deviates significantly from the accurate solution. The long-time approximation remains close to the accurate solution, which it ultimately approaches near the limiting speed asymptote.

Figure 2 shows the x component of the velocity. In the limit of no drag we have $u_x = u_{x0} = 1$, as indicated there. The short-time approximation shows a significant decrease in u_x that is in good agreement with the accurate solution up to about $\tau = 0.5$. The long-time approximation is substantially in error at these times. For this reason we did not quote formulas for displacements obtained in the long-time approximation. Clearly, the error in integrating this velocity component will be cumulative and relatively large in this case. The long-time solution does approach the accurate solution beyond $\tau = 4$.

The path in the x,y plane is shown in Fig. 3. For the portion of the path shown the no drag limit is already significantly different from the actual path. The short-time approximation is in good agreement with the accurate solution over this region. At longer times the accurate solution is found to approach the limit

$$X \rightarrow 1.15,$$

which is approximately equal to u_{x0} as anticipated.

Another series of calculations were made for an initial speed $u_0 = 2$ and a number of different projection angles to determine the maximum horizontal range. This maximum range was found to be $1.22VT$, and it occurred for an angle of 36° above the horizontal.

VIII. TURBULENCE AND THE QUADRATIC DRAG FORCE

Experiments measuring the drag force often make use of a cylinder whose axis is perpendicular to the flow. This tends to produce a two-dimensional flow pattern which is more easily studied. At the large Reynolds numbers at which a quadratic drag force is observed there is a characteristic periodic pattern of turbulence behind a cylinder. This turbulence consists almost exclusively of vortices which are "ripped off" the sides of the cylinder in a periodic fashion.[14] These vortices are shed first from one side and then the other, giving oppositely rotating elements of fluid downstream whose radial dimensions are comparble to the cylinder's radius. Given this result, one can derive the quadratic force law by applying conservation of energy.

Two other simple arguments may be used. One is based on dimensional analysis and the independence of the drag force on viscosity.[6] The conclusion is that $F = (\text{const})\rho A V^2$, where the constant depends only on the body's shape and A is the cross-sectional area of the body. The second uses Bernoulli's theorem together with an assumption about the decreased pressure in the turbulent wake behind a body.[5] It may be concluded that the pressure drop from front to rear is roughly $\rho V^2/2$; hence, the force is about $\rho A V^2/2$.

An alternative is to apply conservation of energy. This requires that the work done by an external agent in moving a body at speed V through a fluid equals the rate at which energy is produced in the fluid. It is evident in this case that this energy is initially concentrated in vortices whose kinetic

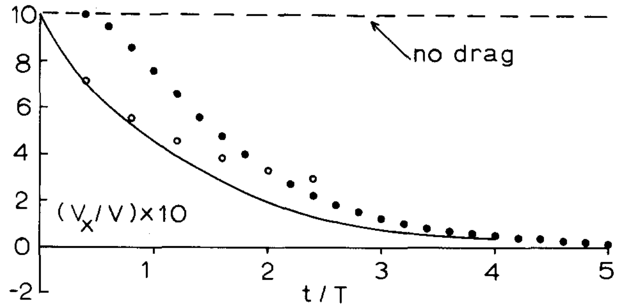

Fig. 2. Horizontal component of velocity. Accurate values of the x component of velocity as a function of time are plotted as the solid line. The short-time approximation is indicated by open circles, while the closed circles show the long-time approximation. For no drag force this component of velocity is constant, as shown.

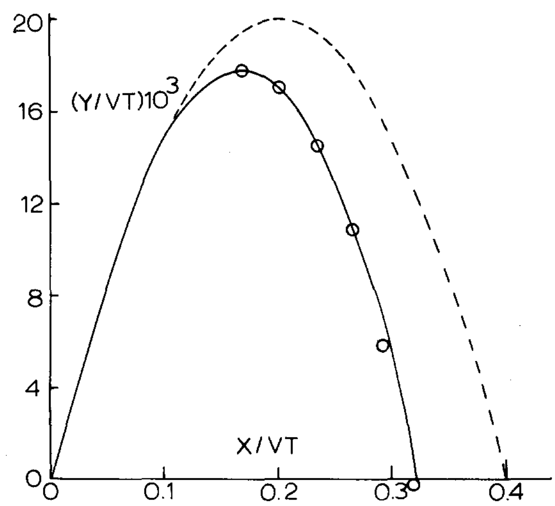

Fig. 3. Trajectories. The solid curve shows the short-time approximation, while the dashed curve gives the path for no drag force. The circles are some points on the trajectory obtained by accurate numerical integration.

energy can be estimated to be roughly that of a rigidly rotating fluid cylinder; i.e.,

$$\mathrm{KE} \approx I\omega^2/2.$$

The moment of inertia is then

$$I \approx Mr^2/2,$$

where M is the mass of fluid rotating at angular speed ω. The radius of a vortex is taken to be that of the cylinder which produced it. The angular speed is then of the order of

$$\omega \approx V/r.$$

On the time scale of their creation, i.e., $\Delta t \approx r/V$, the rate of kinetic energy production is then

$$\Delta(\mathrm{KE})/\Delta t \approx r^2\omega^2 M(\Delta t)^{-1}/4,$$

where

$$M \approx A\rho V\Delta t$$

is the mass of fluid encountered by the cylinder in the time Δt. The power supplied is FV, so

$$FV \approx A\rho V r^2\omega^2/4 \approx \rho A V^3/4$$

and

$$F \approx \rho A V^2/4.$$

The definition of the drag coefficient C_{D} is

$$C_{\mathrm{D}} = F/(\rho A V^2/2),$$

where A is the cross-sectional area of the body that is presented to the fluid stream. Our calculation of F gives $C_{\mathrm{D}} \approx \frac{1}{2}$. From experiment, the drag coefficient of a cylinder is $C_{\mathrm{D}} \approx 1.2$, which is in agreement with our crude estimate.

A sphere generates a three-dimensional flow pattern, but it is known that a somewhat similar vortex shedding process occurs for it.[15]

To conclude this section, we contrast the turbulent situation just described, in which the quadratic force law is valid and $1 \ll \mathcal{R} \lesssim 10^5$, with the nonturbulent regime, in which $\mathcal{R} < 1$. In the latter case the flow is said to be laminar or "layered" rather than turbulent. Streamlines curve around a sphere (say) indicating a disturbance of the fluid which extends a distance of the order of r, the sphere's radius, into the stream from the surface of the sphere.[16] In other words, if the speed of flow of undisturbed fluid relative to the sphere is V, there is a

$$\text{velocity gradient} \approx V/r.$$

This result depends on the fact that the velocity of the fluid relative to the sphere's surface goes to zero at this surface. The viscosity of the fluid determines how one "layer" drags on another. The magnitude of this force depends on the viscosity coefficient η and the velocity gradient as follows[17]:

$$\text{force per unit area} = \eta(\text{velocity gradient}).$$

For a sphere we have

$$F/4\pi r^2 \approx \eta(V/r),$$

or

$$F \approx 4\pi\eta r V \quad \text{for } \mathcal{R} < 1.$$

We thus obtain a reasonable estimate of Stokes's law, Eq. (1), which is $F = 6\pi\eta r V$.

[1]G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge U.P., London, 1967), pp. 233–234.

[2]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, MA, 1959), p. 66.

[3]R. D. Present, *Kinetic Theory of Gases* (McGraw-Hill, New York, 1958), p. 162.

[4]Equation (1) does not even apply in the famous oil drop experiment. See Ref. 3. Even so, a drag force linear in the speed is expected to result from elastic collisions with air molecules.

[5]Reference 1, pp. 339–341.

[6]Reference 2, pp. 169–171.

[7]A. P. French, *Newtonian Mechanics* (Norton, New York, 1971), pp. 213–214.

[8]For example, see J. Lindemutt, Am. J. Phys. **39**, 757 (1971), or G. Feinberg, *ibid.* **33**, 501 (1965).

[9]H. Lamb, *Dynamics* (Cambridge U.P., London, 1960), p. 297.

[10]E. T. Whittaker, *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Dover, New York, 1944), p. 229.

[11]E. J. Routh, *A Treatise on Dynamics of a Particle* (Dover, New York, 1960), pp. 95–96.

[12]J. L. Synge and B. A. Griffith, *Principles of Mechanics* (McGraw-Hill, New York, 1942), pp. 155–157.

[13]Numerical integration of Eq. (24) was carried out using a subroutine based on Simpson's rule, which is supplied by IBM in their Scientific Subroutine Package (SSP). The components of the velocity were obtained from Eqs. (26) and (27). The displacements were then calculated by applying the trapezoidal rule to Eq. (6). A copy of our FORTRAN program will be sent to anyone requesting it.

[14]Reference 1, plate 11, pp. 259–261 and 338. Plate 11 shows a "vortex street" for $\mathcal{R} \approx 2 \times 10^3$. At Reynolds numbers higher than this but still less than about 10^5, a similar pattern is maintained but there is more small-scale turbulence. For a general discussion of turbulence, see Ref. 2, Chap. 3, especially pp. 103, 106, and 116–123. Sketches of the turbulence behind a cylinder at several Reynolds numbers are shown in Vol. II, Chap. 41 of *The Feynman Lectures on Physics* [R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1964)].

[15]We must remember that spin has been assumed to be zero. A spinning sphere will have a lift force in addition to drag. Spin has a noticeable effect on some golf ball trajectories. We should note, however, that well-struck golf balls may have Reynolds numbers $\gtrsim 10^5$, in which case the quadratic law does not apply. In fact, there is an actual decrease in the drag force with increasing speed for a narrow range of Reynolds numbers $\mathcal{R} \gtrsim 10^5$. According to Ref. 1, this "drag crisis" is associated with the onset of turbulence in some part of the boundary layer. The dimples on a ball's surface are able to induce boundary layer turbulence at lower speeds and thus reduce the drag (Ref. 1, pp. 341–342).

[16]Reference 1, plate 1 shows photographs of streamlines around a cylinder for several Reynolds numbers, including $\mathcal{R} = 0.25$.

[17]This is a simplified statement of the equation defining η. See, for example, Ref. 3, p. 37.