

A novel solution to Kepler's problem

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Received 18 June 2003

Published 5 September 2003

Online at stacks.iop.org/EJP/24/575

Abstract

The standard (unperturbed) Kepler problem is expressed in Kustaanheimo–Stiefel form and solved utilizing the algebra of quaternions. This provides the necessary background to understand some of the new techniques of celestial mechanics.

1. Introduction

The aim of this paper is to introduce students to a modern and interesting approach to celestial mechanics. In the process, they learn fundamentals of quaternion algebra related to the geometry of three-dimensional space (rotations in particular). With this background, they can follow the subsequent presentation of a solution to Kepler's problem, and understand its basic features. This constitutes a significant stepping stone for studying and understanding dynamics-related features of a single-star solar system (such as our own).

2. Kepler problem

The basic law of gravity states that two spherical bodies attract each other with a force proportional to the product of their masses and inversely proportional to the square of their distance (with the direction along the line connecting their centres).

Usually, one of the two bodies is large (and called the *primary*), and the other small (the *satellite*). Based on the previous law, one can write the following differential equation for the location of the satellite (expressed as a vector in non-rotating Cartesian coordinates, having the origin at the primary's centre):

$$\ddot{\mathbf{r}} + \mu \frac{\mathbf{r}}{r^3} = \mathbf{0} \quad (1)$$

where μ is a sum of the two masses (primary and satellite), multiplied by the gravitational constant. Double dot implies the second derivative with respect to time t , \mathbf{r} is a shorthand for a vector with three components, namely $x(t)$, $y(t)$ and $z(t)$, and $r \equiv |\mathbf{r}|$.

Solving these (effectively three) equations (collectively known as the *Kepler problem*) was accomplished by Newton more than three centuries ago (a monumental achievement, considering that he first had to invent the concept of a differential equation itself), resulting in the following well known answer: the solutions are ellipses, with the primary's centre at

one of the foci. The initial conditions (stipulating the satellite's location and velocity at time $t = 0$) will of course pick out only one such ellipse as the satellite's orbit.

The fact that planets' orbits are elliptical had been discovered empirically—based on Tycho Brahe's data—by Kepler, and it is now called the first Kepler law. The time dependence of the satellite's passage through its orbit is based on the remaining two Kepler's laws.

Several decades later, Newton's theory confirmed Kepler's empirical laws—one of the first scientific triumphs of this kind.

The objective of this paper is to demonstrate a new and interesting way of solving the same problem using quaternions (an option not available to Newton—quaternions became a part of mathematics only during the 19th century).

3. Algebra of quaternions

The algebra has *three* imaginary units (instead of the usual one) denoted i , j and k , each of them squaring to -1 . Any two of these anticommute (e.g. $ij = -ji$), furthermore, $ij = k$, $jk = i$ and $ki = j$. Together with the ordinary (real) unit 1 , which commutes with each of them, they constitute the algebra's basis. An element of this algebra can thus be written as

$$A + a_3i + a_2j + a_1k \equiv A + \mathbf{a} \equiv \mathbb{A} \quad (2)$$

and interpreted geometrically as a *vector* \mathbf{a} (a_1, a_2 and a_3 representing its x , y and z coordinates, respectively—note the deliberate association of i with the z coordinate), appended to a *scalar* A .

Adding and subtracting quaternions is done in the usual component-wise manner:

$$(A + \mathbf{a}) \pm (B + \mathbf{b}) = A \pm B + \mathbf{a} \pm \mathbf{b}. \quad (3)$$

Quaternion *multiplication* is a bit more tricky, it amounts to:

$$(A + \mathbf{a})(B + \mathbf{b}) = AB - \mathbf{a} \cdot \mathbf{b} + \mathbf{A}\mathbf{b} + B\mathbf{a} - \mathbf{a} \times \mathbf{b}. \quad (4)$$

Exercise. Verify the correctness of this formula. Also prove that this multiplication is associative (it is clearly non-commutative).

Most of the usual functions allow quaternion arguments, yielding a quaternion in return, for example:

$$\exp(A + \mathbf{a}) = \exp(A) \exp(a\hat{\mathbf{a}}) = \exp(A)(\cos a + \hat{\mathbf{a}} \sin a) \quad (5)$$

where a and $\hat{\mathbf{a}}$ are the magnitude and unit direction, respectively, of \mathbf{a} .

Another related operation is that of a (quaternionic) *conjugate*, namely reversing the sign of i , j and k , thus:

$$\overline{A + \mathbf{a}} = A - \mathbf{a}. \quad (6)$$

Note that

$$\overline{\mathbb{A}\mathbb{B}} = \overline{\mathbb{B}}\overline{\mathbb{A}}. \quad (7)$$

One can also easily show that

$$\overline{\mathbb{A}\mathbb{A}} = \mathbb{A}\overline{\mathbb{A}} \quad (8)$$

is real and non-negative.

We can thus define a quaternion's *magnitude* by

$$|\mathbb{A}| \equiv \sqrt{\overline{\mathbb{A}\mathbb{A}}} \quad (9)$$

and also construct a quaternion's *inverse* by

$$\mathbb{A}^{-1} = \frac{\overline{\mathbb{A}}}{\overline{\mathbb{A}\mathbb{A}}}. \quad (10)$$

Similar to a matrix inverse, we also have

$$(\mathbb{A}\mathbb{B})^{-1} = \mathbb{B}^{-1}\mathbb{A}^{-1}. \quad (11)$$

Finally, we define a special class of quaternions, called (in this paper) *rotations*, which meet the following condition

$$\mathbb{R}\overline{\mathbb{R}} = 1. \quad (12)$$

It is quite obvious that $\exp(\mathbf{w})$, where \mathbf{w} is any vector, is a rotation. The reverse is also true, i.e. any such \mathbb{R} can be expressed in the $\exp(\mathbf{w})$ form.

Proof. Clearly, any such \mathbb{R} can be written as $\pm\sqrt{1-a^2} + \mathbf{a} = \pm\sqrt{1-a^2} + a\hat{\mathbf{a}}$, where \mathbf{a} is a vector whose magnitude is less than or equal to 1, and \pm implies a *specific* choice of either sign. Now, we find w such that $\cos(w) = \pm\sqrt{1-a^2}$ and $\sin(w) = a$, and take $\mathbf{w} \equiv w\hat{\mathbf{a}}$. \square

4. Three-dimensional rotations

Suppose that we want to rotate points of three-dimensional space (these will now be represented by *pure quaternions*, i.e. quaternions without the real component, thus: $\mathbf{r} = zi + yj + x\mathfrak{k}$) around an axis passing through the origin. Let the direction of a vector \mathbf{u} (also represented by a pure quaternion) specify this axis, and its magnitude u be the angle of rotation (in radians).

One can show that such a rotation is facilitated by

$$\exp\left(-\frac{\mathbf{u}}{2}\right)\mathbf{r}\exp\left(\frac{\mathbf{u}}{2}\right) \equiv \overline{\mathbb{R}}\mathbf{r}\mathbb{R}. \quad (13)$$

Exercise. Supply the proof.

It is also possible to show that any such $\mathbb{R} = \exp(\frac{\mathbf{u}}{2})$ can be written in the seemingly more complicated (but geometrically rather meaningful) manner of

$$\exp\left(i\frac{\psi}{2}\right)\exp\left(\mathfrak{k}\frac{\theta}{2}\right)\exp\left(i\frac{\phi}{2}\right) \quad (14)$$

where ψ , θ and ϕ are called *Euler angles*.

Proof. Expanding and simplifying (14) yields

$$\cos\frac{\theta}{2}\exp\left(i\frac{\psi+\phi}{2}\right) + \mathfrak{k}\sin\frac{\theta}{2}\exp\left(i\frac{\psi-\phi}{2}\right). \quad (15)$$

On the other hand, any quaternion can be evidently written as

$$P\exp(i\alpha) + \mathfrak{k}Q\exp(i\beta) \quad (16)$$

where P , Q , α and β are real.

All we have to show is that, in the case of \mathbb{R} , $P^2 + Q^2 = 1$; after that, the actual parameter matching of (15) and (16) is trivial.

This can be done by

$$\begin{aligned} 1 &= \mathbb{R}\overline{\mathbb{R}} \\ &= [P\exp(i\alpha) + \mathfrak{k}Q\exp(i\beta)][P\exp(-i\alpha) - Q\exp(-i\beta)\mathfrak{k}] \\ &= P^2 + \mathfrak{k}QP\exp[i(\beta - \alpha)] - PQ\exp[i(\alpha - \beta)]\mathfrak{k} + Q^2 = P^2 + Q^2 \end{aligned} \quad (17)$$

since

$$e^{i\xi}\mathfrak{k} = \mathfrak{k}e^{-i\xi} \quad (18)$$

for any real ξ . \square

5. Reformulating the Kepler problem

Let us return to (1), where \mathbf{r} is now seen as a vector in its quaternion representation. This by itself does not change much, until we try to simplify the equation (really a shorthand for a *set* of equations) by a convenient transformation of both \mathbf{r} (the dependent variable) and t (the independent variable).

As we all know, a differential equation often simplifies when a new dependent variable is introduced, usually as some simple combination of the old dependent *and* independent variables (e.g. $v = x^2y$).

Less frequently, we simplify equations by introducing a new independent variable, usually as a function of the old independent variable (e.g. $z = x^2$). But, one should realize that it is quite legitimate for a new independent variable to be also a function of *both* (independent and dependent) original variables, and that is exactly how we will proceed.

We introduce a new dependent variable \mathbb{U} (this time a *full* quaternion) by

$$\mathbf{r} \equiv \overline{\mathbb{U}}\mathfrak{k}\mathbb{U} \quad (19)$$

and a new independent variable s by

$$\frac{dt}{ds} = 2r \sqrt{\frac{a}{\mu}} \quad (20)$$

where a is a positive constant, chosen (at this point) arbitrarily, and r is the magnitude of \mathbf{r} . Note that, in terms of the new \mathbb{U} , $r = \overline{\mathbb{U}}\mathbb{U}$.

Proof.

$$r = \sqrt{\overline{\mathbf{r}}\mathbf{r}} = \sqrt{-\overline{\mathbb{U}}\mathfrak{k}\mathbb{U}\overline{\mathbb{U}}\mathfrak{k}\mathbb{U}} = \sqrt{(\overline{\mathbb{U}}\mathbb{U})^2} = \overline{\mathbb{U}}\mathbb{U}. \quad \square$$

Also note that instead of an explicit formula for t (in terms of s and r), we have an expression for its derivative. This makes the transformation a bit more complicated (or perhaps just unusual), but that is what is needed in this case.

In terms of the new variables, (1) has the following form (known as the Kustaanheimo–Stiefel equation [1, 2]):

$$2\mathbb{U}'' - (2\mathbb{U}'\overline{\mathbb{U}}' - 4a)\frac{\mathbb{U}}{r} + 2\mathfrak{k}\mathbb{U}'\frac{\Gamma}{r} + \mathfrak{k}\mathbb{U}\left(\frac{\Gamma}{r}\right)' = 0 \quad (21)$$

where $'$ denotes differentiation with respect to s , r is a shorthand for $\overline{\mathbb{U}}\mathbb{U}$, and Γ for

$$\overline{\mathbb{U}}\mathfrak{k}\mathbb{U}' - \overline{\mathbb{U}}'\mathfrak{k}\mathbb{U} = 2r \sqrt{\frac{a}{\mu}} (\overline{\mathbb{U}}\mathfrak{k}\dot{\mathbb{U}} - \dot{\overline{\mathbb{U}}}\mathfrak{k}\mathbb{U}) \quad (22)$$

(clearly *real*, since it is equal to its own conjugate).

Proof.

$$\dot{\mathbf{r}} = 2\overline{\mathbb{U}}\mathfrak{k}\dot{\mathbb{U}} - \sqrt{\frac{\mu}{a}} \frac{\Gamma}{2r} \quad (23)$$

premultiplied by $2\sqrt{\frac{a}{\mu}}\mathfrak{k}\mathbb{U}$, implies

$$2\sqrt{\frac{a}{\mu}}\mathfrak{k}\mathbb{U}\dot{\mathbf{r}} = -2\mathbb{U}' - \frac{\mathfrak{k}\mathbb{U}\Gamma}{r}. \quad (24)$$

Applying $2r\sqrt{\frac{a}{\mu}}\frac{d}{dt} \equiv \frac{d}{ds}$ to the last equation yields:

$$2r\sqrt{\frac{a}{\mu}}\left(2\sqrt{\frac{a}{\mu}}\mathfrak{k}\mathbb{U}\dot{\mathbf{r}}\right)' = -2\mathbb{U}'' - \left(\frac{\mathfrak{k}\mathbb{U}\Gamma}{r}\right)'. \quad (25)$$

Expanding the left-hand side results, with the help of (1), in

$$-4a\mathfrak{k}\mathbb{U}\frac{\mathbf{r}}{r^2} + 2\sqrt{\frac{a}{\mu}}\mathfrak{k}\mathbb{U}'\dot{\mathbf{r}} = 4a\frac{\mathbb{U}}{r} + \frac{\mathfrak{k}\mathbb{U}'}{r}(2\overline{\mathbb{U}}'\mathfrak{k}\mathbb{U} + \Gamma) \quad (26)$$

since

$$\mathfrak{k}\mathbb{U}\mathbf{r} \equiv \frac{\mathbb{U}\mathbf{r}}{r} = -\mathbb{U}r \quad (27)$$

and

$$\dot{\mathbf{r}} = 2\overline{\mathbb{U}}'\mathfrak{k}\mathbb{U} + \sqrt{\frac{\mu}{a}}\frac{\Gamma}{2r}. \quad (28)$$

Substituting (26) for the left-hand side of (25) finally gives:

$$4a\frac{\mathbb{U}}{r} + \frac{\mathfrak{k}\mathbb{U}'}{r}(2\overline{\mathbb{U}}'\mathfrak{k}\mathbb{U} + \Gamma) = -2\mathbb{U}'' - \mathfrak{k}\mathbb{U}\left(\frac{\Gamma}{r}\right)' - \mathfrak{k}\mathbb{U}'\frac{\Gamma}{r} \quad (29)$$

(the rest is just a rearrangement of terms). \square

So, where is the simplification? Equation (21) looks more complicated than equation (1), and we have quaternions to worry about on top of it! Well, there is still some hope.

We have replaced the old *three*-component \mathbf{r} by a *four*-component \mathbb{U} . This is not an unusual thing to do; we all remember that to solve $y'' + a(x)y' + b(x)y = 0$ one introduces *two* functions, u and v , in place of y , and removes the resulting redundancy later by introducing a convenient constraint (physicists call it a *gauge*) which effectively makes u a function of v .

Let us do something similar here. A convenient gauge is clearly $\Gamma \equiv 0$ (the important thing is that it is *one*-dimensional), since it removes two unpleasant terms of (21). The equation then simplifies to

$$\mathbb{U}'' - \frac{\mathbb{U}\overline{\mathbb{U}}' - 2a}{r}\mathbb{U} = 0 \quad (30)$$

which seems a lot more manageable, especially if $\frac{\mathbb{U}\overline{\mathbb{U}}' - 2a}{r}$ proves to be a constant, as it does!

Proof. Differentiating $\frac{\mathbb{U}\overline{\mathbb{U}}' - 2a}{r}$ with respect to s yields

$$\frac{\mathbb{U}''\overline{\mathbb{U}} + \mathbb{U}'\overline{\mathbb{U}}'}{r} - \frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r^2}(\mathbb{U}'\overline{\mathbb{U}} + \mathbb{U}\overline{\mathbb{U}}'). \quad (31)$$

Replacing \mathbb{U}'' by $\frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r}\mathbb{U}$ and $\overline{\mathbb{U}}''$ by $\frac{\mathbb{U}\overline{\mathbb{U}}' - 2a}{r}\overline{\mathbb{U}}$ ($\frac{\mathbb{U}\overline{\mathbb{U}}' - 2a}{r}$ is real) results in

$$\frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r}\frac{\mathbb{U}'\overline{\mathbb{U}} + \mathbb{U}\overline{\mathbb{U}}'}{r} - \frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r^2}(\mathbb{U}'\overline{\mathbb{U}} + \mathbb{U}\overline{\mathbb{U}}') = 0. \quad (32)$$

\square

One can also show that

$$\frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r} = 2a\left(\frac{v^2}{2\mu} - \frac{1}{r}\right) \quad (33)$$

where $v \equiv |\dot{\mathbf{r}}|$. This makes it proportional both to a and the total energy of the system (no wonder it is a constant—conservation of total energy is one of the holiest principles of physics).

Proof.

$$\frac{\mathbb{U}'\overline{\mathbb{U}}' - 2a}{r} = \frac{4r^2\frac{a}{\mu}\dot{\mathbb{U}}\overline{\mathbb{U}} - 2a}{r}. \quad (34)$$

Since

$$\dot{\mathbf{r}} = \bar{\mathbb{U}}\mathfrak{k}\dot{\mathbb{U}} + \bar{\mathbb{U}}\mathfrak{k}\dot{\mathbb{U}} = 2\bar{\mathbb{U}}\mathfrak{k}\dot{\mathbb{U}} = 2\bar{\mathbb{U}}\mathfrak{k}\dot{\mathbb{U}} \quad (35)$$

(due to our $\Gamma = 0$ constraint), we get

$$v^2 = -\dot{\mathbf{r}}\dot{\mathbf{r}} = 4\bar{\mathbb{U}}\mathfrak{k}\dot{\mathbb{U}}\bar{\mathbb{U}}\mathfrak{k}\dot{\mathbb{U}} = 4r\dot{\mathbb{U}}\dot{\mathbb{U}}. \quad (36)$$

□

To make things interesting, the total energy must be *negative* (otherwise, the two bodies would just fly apart, rather than orbiting each other). To simplify equation (30) further, we set a (so far arbitrary but positive) to make (33) equal to -1 .

All we have to solve in the end is thus the simple looking

$$\mathbb{U}'' + \mathbb{U} = 0. \quad (37)$$

6. Solving the Kepler problem

The previous equation is still a quaternion (four-component) equation, but linear, fully decoupled and easy to deal with (effectively, it is the good old $y'' + y = 0$ repeated four times, once for each component).

The general solution is thus

$$\mathbb{U} = \tilde{\mathbb{P}} \sin s + \tilde{\mathbb{Q}} \cos s \quad (38)$$

where $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{Q}}$ are two constant, ‘almost’ arbitrary (subject to the $\Gamma = 0$ condition, which reduces the number of free parameters from eight to seven, and consistent with our previous $\frac{\mathbb{U}\bar{\mathbb{U}}-2a}{r} = -1$ choice) quaternions.

Mathematically, we are done. But where are the ellipses? And what is the physical meaning of the seven free parameters of our solution? To answer these questions requires a bit more work.

What we need to do is to express (38) in a more meaningful manner. First we replace $\sin s$ by

$$\frac{\exp(is) - \exp(-is)}{2i} \quad (39)$$

and $\cos s$ by

$$\frac{\exp(is) + \exp(-is)}{2}. \quad (40)$$

Note that we could have chosen any other unit direction in place of i . But, i is the most convenient choice (being orthogonal to both j and k), in view of equation (19).

Once that is done, we can clearly re-write (38) in an equivalent form of

$$\mathbb{U} = \mathbb{P} \exp(is) + \mathbb{Q} \exp(-is). \quad (41)$$

To make the solution truly meaningful, we have to re-parametrize it further:

$$\mathbb{U} = \exp(\mathfrak{k}\delta)\{A(1 + j\gamma) \exp[i(s - s_p)] + B \exp[-i(s - s_p)]\}\mathbb{R} \quad (42)$$

where δ , γ , A , B and s_p are *real* constant parameters, and \mathbb{R} is a (constant) rotation. Note that, altogether, we still have eight free parameters—a clear prerequisite for making (41) and (42) equivalent.

Exercise. Prove that there is a one-to-one correspondence between the right-hand sides of (41) and (42).

Let us now try to interpret the individual parameters of (42).

First, we have to realize that the actual *physical* solution is \mathbf{r} , not \mathbb{U} . When we substitute the right-hand side of (42) into (19), the δ -parameter cancels out. It has, therefore, no physical meaning at all, and can be set to 0 (we are thus fixing our gauge) without affecting \mathbf{r} .

Similarly, substituting the right-hand side of (42) into (22) results in $\Gamma = 4\gamma A^2$.

Proof. First we evaluate

$$\begin{aligned} \overline{\mathbb{U}}\mathfrak{k}\mathbb{U}' &= \overline{\mathbb{R}}\{A \exp[-i(s - s_p)](1 - j\gamma) + B \exp[i(s - s_p)]\} \\ &\quad \times \mathfrak{k}\{A(1 + j\gamma) \exp[i(s - s_p)] - B \exp[-i(s - s_p)]\}i\mathbb{R} \\ &= \overline{\mathbb{R}}\{A^2(1 - \gamma^2) \exp[-2i(s - s_p)] \\ &\quad - B^2 \exp[2i(s - s_p)] - 2\gamma AB \cos[2(s - s_p)]\}j\mathbb{R} + 2\gamma A^2. \end{aligned} \tag{43}$$

Adding the corresponding conjugate yields zero for the first term, and $4\gamma A^2$ for the second one. \square

To ensure that $\Gamma = 0$, we have to set $\gamma = 0$ (the other possible choice, namely $A = 0$, would effectively eliminate both A and γ —the remaining solution would no longer be fully general).

Finally, we want to make sure that $\frac{\mathbb{U}'\overline{\mathbb{U}} - 2a}{r} = -1$. Since $\frac{\mathbb{U}'\overline{\mathbb{U}} - 2a}{r}$ is a constant of motion, making it equal to -1 at one value of s makes it so for all times. We will thus ‘fix’ it at $s = s_p$, where the value of $\frac{\mathbb{U}'\overline{\mathbb{U}} - 2a}{r}$ equals

$$\frac{(A - B)^2 - 2a}{(A + B)^2}. \tag{44}$$

Proof. Differentiating

$$\mathbb{U} = \{A \exp[i(s - s_p)] + B \exp[-i(s - s_p)]\}\mathbb{R} \tag{45}$$

with respect to s yields

$$\mathbb{U}' = \{iA \exp[i(s - s_p)] - iB \exp[-i(s - s_p)]\}\mathbb{R} \tag{46}$$

implying that $\mathbb{U}'\overline{\mathbb{U}}$, evaluated at $s = s_p$, equals $(A - B)^2$.

Similarly $r = \mathbb{U}\overline{\mathbb{U}}$, evaluated at $s = s_p$, equals $(A + B)^2$. \square

Expression (44) will thus equal to -1 only if $A^2 + B^2 = a$ (showing this is quite trivial).

It is now convenient to trade off the A and B parameters for a and $\beta \equiv \frac{B}{A}$, and write the final version of our solution as

$$\mathbb{U} = \sqrt{\frac{a}{1 + \beta^2}} [e^{i(s-s_p)} + \beta e^{-i(s-s_p)}]\mathbb{R} \tag{47}$$

which easily yields (see (19))

$$\begin{aligned} \mathbf{r} &= \overline{\mathbb{R}}[e^{-i(s-s_p)} + \beta e^{i(s-s_p)}] \frac{a\mathfrak{k}}{1 + \beta^2} [e^{i(s-s_p)} + \beta e^{-i(s-s_p)}]\mathbb{R} \\ &= \overline{\mathbb{R}} \frac{a\mathfrak{k}}{1 + \beta^2} [e^{i(s-s_p)} + \beta e^{-i(s-s_p)}]^2\mathbb{R} \\ &= \overline{\mathbb{R}} \frac{a\mathfrak{k}}{1 + \beta^2} [e^{2i(s-s_p)} + \beta^2 e^{-2i(s-s_p)} + 2\beta]\mathbb{R} \end{aligned} \tag{48}$$

due to (18).

The last expression is clearly a curve in the x - y plane (it has only \mathfrak{k} and j components), rotated in the $\overline{\mathbb{R}}\mathbf{r}_0\mathbb{R}$ manner of (13). So, what kind of curve is it?

Its x and y components (before the rotation, which of course does not change its shape) are

$$\frac{a}{1+\beta^2}[(1+\beta^2)\cos\omega+2\beta] = a\cos\omega + \frac{2a\beta}{1+\beta^2} \quad (49)$$

and

$$\frac{a}{1+\beta^2}(1-\beta^2)\sin\omega \quad (50)$$

respectively, where $\omega \equiv 2(s - s_p)$ is the so-called *eccentric anomaly*. The curve thus meets the following equation

$$\left(x - \frac{2a\beta}{1-\beta^2}\right)^2 + \left(\frac{1+\beta^2}{1-\beta^2}y\right)^2 = a^2 \quad (51)$$

which is an ellipse with eccentricity $e = \frac{2\beta}{1+\beta^2}$.

Proof. Firstly, $1 - e^2 = 1 - \frac{4\beta^2}{(1+\beta^2)^2} = \left(\frac{1-\beta^2}{1+\beta^2}\right)^2$. The previous equation can thus be written in a more recognizable form of

$$\frac{(x - ae)^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1. \quad (52)$$

This is an ellipse with (semi) major and minor axes of length a and $a\sqrt{1 - e^2}$, respectively, and an eccentricity of $\frac{\sqrt{a^2 - a^2(1 - e^2)}}{a} = e$. The distance of each focus from the ellipse's centre is $\sqrt{a^2 - a^2(1 - e^2)} = ae$ (the left of the foci is thus located at the origin). \square

As

$$r = \overline{UU} = \overline{U\overline{U}} = \frac{a}{1+\beta^2}(1+\beta^2+2\beta\cos\omega) = a(1+e\cos\omega) \quad (53)$$

we can easily integrate the right-hand side of (20) with respect to $s \equiv \frac{w}{2} + s_p$, to obtain the following relationship between time t and the eccentric anomaly ω :

$$t - t_0 = \frac{a^{3/2}}{\sqrt{\mu}}(\omega + e\sin\omega). \quad (54)$$

This is a version of the so-called *Kepler equation*, which serves to establish the varying speed of the satellite through its elliptical orbit.

7. Further challenges

Suppose now that, in addition to being attracted by the primary, additional (small) forces are acting on the satellite, due to: the gravitational pull of other satellites, the primary and/or the satellite not being perfectly spherical, atmospheric drag, tidal forces, etc. The basic equation (1) then changes accordingly, by acquiring an extra term for each such effect, thus becoming the so-called *perturbed Kepler problem*.

One of the most challenging of these is the *lunar problem*: the moon, perturbed by an extra pull of the (rather distant, but huge) sun—causing a 0.6% distortion of the regular force.

Another intricate situation arises when the perturbing force has a period commensurable (in an integer ratio, such as 2:1) to that of the satellite—a phenomenon called *resonance*. This happens quite often in our solar system, the asteroid belt (perturbed by Jupiter) being a prime example.

The technique just described can be extended to solve, iteratively (i.e. the solution is expanded in terms of a small parameter), all of these possible cases. It is particularly well

suited to deal with resonances, and its utilization of quaternions then becomes indispensable. The interested reader should refer to [3] for a mathematical description of the technique, and to [4] and [5] for some of its applications (understanding the current paper is a necessary prerequisite).

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