Algebraic approach to the radioactive decay equations

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The radioactive decay equations are solved using an algebraic approach that is simple and easily extensible to branching systems. Two examples are included to show the nature of the approach. © *2003 American Association of Physics Teachers.*
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I. INTRODUCTION

A radioactive decay chain, $X_1 \rightarrow X_2 \rightarrow \cdots \rightarrow X_i \rightarrow \cdots \rightarrow X_n$, with decay constants λ_i ($i=1$ to n) is described by the following set of differential equations:

$$N_1' = -\lambda_1 N_1, \tag{1a}$$

$$N_2' = \lambda_1 N_1 - \lambda_2 N_2, \tag{1b}$$

$$\vdots$$

$$N_n' = \lambda_{n-1} N_{n-1} - \lambda_n N_n, \tag{1c}$$

where $N_i = N_i(t)$ represents the amount of the ith nuclide at time t, and $N_i' = dN_i(t)/dt$ is its time derivative. The solution of Eq. (1) is fixed by specifying the initial amount of the nuclides in the sample,

$$N_{i,0} = N_i(0) \quad (i=1,\ldots,n). \tag{2}$$

The general solution of Eqs. (1) and (2) was obtained by Bateman[1] by performing a Laplace transform and its subsequent inverse transform using a path integral in the complex plane. This procedure has been recently shortened in Ref. 2.

In this paper, we solve Eqs. (1) and (2) by a direct algebraic approach. This approach is also applicable to branching systems. Section II contains the basic equations of the method. In Sec. III we discuss two examples to illustrate the nature of this approach; the first example deals with a chain and the second with a branching system. Finally, in Sec. IV we summarize the method.

II. BASIC EQUATIONS

Equation (1) can be expressed compactly using matrices,

$$[\mathbf{N}'] = [A][\mathbf{N}], \tag{3}$$

where $[\mathbf{N}]$, $[\mathbf{N}']$, and $[\mathbf{N}_0]$ are vectors with n components:

$$[\mathbf{N}] = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_n \end{bmatrix}, \quad [\mathbf{N}'] = \begin{bmatrix} N_1' \\ N_2' \\ \vdots \\ N_n' \end{bmatrix}, \quad [\mathbf{N}_0] = \begin{bmatrix} N_{1,0} \\ N_{2,0} \\ \vdots \\ N_{n,0} \end{bmatrix}. \tag{4}$$

$[A]$ is the $n \times n$ matrix defined by

$$[A] = \begin{pmatrix} -\lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ \lambda_1 & -\lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & -\lambda_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & -\lambda_n \end{pmatrix}. \tag{5}$$

If we follow the standard method (see, for example, Ref. 3) for solving a homogeneous system of linear differential equations with constant coefficients, we obtain the following compact solution:

$$[\mathbf{N}] = [V][\Lambda][V]^{-1}[\mathbf{N}_0], \tag{6}$$

where $[\Lambda]$ is the $n \times n$ diagonal matrix,

$$[\Lambda] = \mathrm{Diag}[e^{\Lambda_1 t}, e^{\Lambda_2 t}, \ldots, e^{\Lambda_n t}], \tag{7}$$

$\Lambda_1, \Lambda_2, \ldots, \Lambda_n$ are the eigenvalues of the matrix $[A]$, and

$$[V] = (\mathbf{v}_1 | \mathbf{v}_2 | \ldots | \mathbf{v}_n). \tag{8}$$

That is, the n columns $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ of the matrix $[V]$ are the eigenvectors of $[A]$:

$$[A]\mathbf{v}_i = \Lambda_i \mathbf{v}_i \quad (i=1,\ldots,n). \tag{9}$$

Finally, $[V]^{-1}$ is the inverse matrix of $[V]$.

We have assumed that no pair of decay constants are equal. (This condition is also implicitly assumed in Bateman's solution.[1]) This assumption can be verified by examining a table of nuclides.[4]

Note that, in Eq. (6), $[V][\Lambda][V]^{-1}$ acts as the time evolution matrix of this system. In other words, when this matrix acts on the vector $[\mathbf{N}_0]$, which represents the set of initial values, it produces the vector $[\mathbf{N}]$, which represents the number of nuclei in the sample at an arbitrary time.

For radioactive chain without branching, the ingredients of Eq. (6) can be symbolically calculated, and we obtain

$$\Lambda_i = -\lambda_i \quad (i=1,\ldots,n) \tag{10}$$

$$[V] = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ S_{2,1} & 1 & 0 & \cdots & 0 & 0 \\ S_{3,1} & S_{3,2} & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ S_{n,1} & S_{n,2} & S_{n,3} & \cdots & S_{n,n-1} & 1 \end{pmatrix}, \tag{11}$$

$$[V]^{-1}=\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ T_{2,1} & 1 & 0 & \cdots & 0 & 0 \\ T_{3,1} & T_{3,2} & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ T_{n,1} & T_{n,2} & T_{n,3} & \cdots & T_{n,n-1} & 1 \end{pmatrix},$$
(12)

where the symbols $S_{i,j}$ and $T_{i,j}$ may be represented as finite products of fractions of the form

$$F^p_{q,r}=\frac{\lambda_r}{\lambda_q-\lambda_p}.$$
(13)

In particular,

$$S_{i,j}=F^j_{i,i-1}F^j_{i-1,i-2}\cdots F^j_{j+1,j},$$
(14)

and

$$T_{i,j}=F^i_{j,j}F^i_{j+1,j+1}\cdots F^i_{i-1,i-1}.$$
(15)

The result of inserting Eqs. (13)–(15) into Eq. (6) is the general solution of the chain system, which coincides with Bateman's solution.

Bateman's approach cannot be extended to branching systems. However, the algebraic approach used here and expressed in Eq. (6) is applicable to any radioactive system. The only difference with the simple chain is that to calculate $[V]$ and its inverse matrix $[V]^{-1}$, it might be better to proceed numerically using, for example, MATHEMATICA.

III. EXPLICIT EXAMPLES

Let us first consider an example of a radioactive chain. Radon 222 ($T_{1/2}=3.8235$ days) is an α-emitting, chemically inert gas that seeps out of the soil and can sometimes accumulate in houses. It is an intermediate product in the radioactive series of uranium 238. Radon and its radioactive daughters are known carcinogens. In this radioactive series, radon 222 decays into polonium 218 ($T_{1/2}=3.10$ min), which decays into lead 214 ($T_{1/2}=26.8$ min), which in turn decays into bismuth 214 ($T_{1/2}=19.9$ min). We denote the number of Rn 222 (X_1), Po 218 (X_2), Pb 214 (X_3), and Bi 214 (X_4) as N_1, N_2, N_3, and N_4, respectively.

We assume that at $t=0$ there are $N_{1,0}$ nuclei of X_1 and no nuclei of its daughters, and we want to calculate how N_1, N_2, N_3, and N_4 change with time. From the relation $\lambda=\ln 2/T_{1/2}$ and the values of the half-lives just given, we obtain the respective decay constants λ_i, $i=1,\ldots,4$. Thus, the matrix $[A]$ in Eq. (5) is expressed as

$$[A]=\begin{pmatrix} -\lambda_1 & 0 & 0 & 0 \\ \lambda_1 & -\lambda_2 & 0 & 0 \\ 0 & \lambda_2 & -\lambda_3 & 0 \\ 0 & 0 & \lambda_3 & -\lambda_4 \end{pmatrix},$$
(16)

and Eq. (6) becomes

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ S_{2,1} & 1 & 0 & 0 \\ S_{3,1} & S_{3,2} & 1 & 0 \\ S_{4,1} & S_{4,2} & S_{4,3} & 1 \end{pmatrix}$$
$$\times\begin{pmatrix} e^{-\lambda_1 t} & 0 & 0 & 0 \\ 0 & e^{-\lambda_2 t} & 0 & 0 \\ 0 & 0 & e^{-\lambda_3 t} & 0 \\ 0 & 0 & 0 & e^{-\lambda_4 t} \end{pmatrix}$$
$$\times\begin{pmatrix} 1 & 0 & 0 & 0 \\ T_{2,1} & 1 & 0 & 0 \\ T_{3,1} & T_{3,2} & 1 & 0 \\ T_{4,1} & T_{4,2} & T_{4,3} & 1 \end{pmatrix}\begin{pmatrix} N_{1,0} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$
(17)

The matrix elements $S_{i,j}$ and $T_{i,j}$ are calculated using Eqs. (13), (14), and (5). Several examples are

$$S_{2,1}=\frac{\lambda_1}{\lambda_2-\lambda_1},$$
(18a)

$$S_{3,1}=\frac{\lambda_1}{\lambda_2-\lambda_1}\frac{\lambda_2}{\lambda_3-\lambda_1},$$
(18b)

$$T_{2,1}=\frac{\lambda_1}{\lambda_1-\lambda_2},$$
(19a)

$$T_{4,1}=\frac{\lambda_1}{\lambda_1-\lambda_4}\frac{\lambda_2}{\lambda_2-\lambda_4}\frac{\lambda_3}{\lambda_3-\lambda_4}.$$
(19b)

Thus, we finally obtain:

$$N_1(t)=e^{-\lambda_1 t}N_{1,0},$$
(20a)

$$N_2(t)=[e^{-\lambda_1 t}-e^{-\lambda_2 t}]\frac{\lambda_1}{\lambda_2-\lambda_1}N_{1,0},$$
(20b)

$$N_3(t)=\lambda_1\lambda_2\left[\frac{e^{-\lambda_1 t}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)}+\frac{e^{-\lambda_2 t}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)}\right.$$
$$\left.+\frac{e^{-\lambda_3 t}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}\right]N_{1,0},$$
(20c)

$$N_4(t)=\lambda_1\lambda_2\lambda_3\left[\frac{e^{-\lambda_1 t}}{(\lambda_2-\lambda_1)(\lambda_3-\lambda_1)(\lambda_4-\lambda_1)}\right.$$
$$+\frac{e^{-\lambda_2 t}}{(\lambda_1-\lambda_2)(\lambda_3-\lambda_2)(\lambda_4-\lambda_2)}$$
$$+\frac{e^{-\lambda_3 t}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)(\lambda_4-\lambda_3)}$$
$$\left.+\frac{e^{-\lambda_4 t}}{(\lambda_1-\lambda_4)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}\right]N_{1,0}.$$
(20d)

If we substitute the values of λ_i into Eq. (20), the result is

$$N_1(t)=e^{-0.007\,554t}N_{1,0},$$
(21a)

$$N_2(t)=0.000\,563\,4(e^{-0.007\,554t}+e^{-13.416t})N_{1,0},$$
(21b)

$$N_3(t)=(0.004\,894e^{-0.007\,554t}+0.000\,637\,0e^{-13.416t}$$
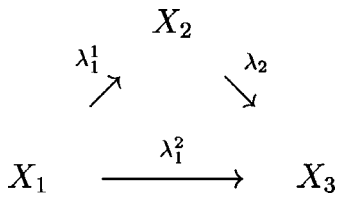$$-0.005\,531e^{-1.5518t})N_{1,0},$$
(21c)

Fig. 1. Branching scheme of Example 2.

$$N_4(t) = (0.003\,647 e^{-0.007\,554t} - 0.000\,087\,29 e^{-13.416t}$$
$$- 0.015\,95 e^{-1.5518t} + 0.012\,39 e^{-2.0899t}) N_{1,0},$$
$$(21d)$$

where the time t is measured in hours. Thus, for a specific value of $N_{1,0}$, we know the evolution of the nuclide population at any time.

Our second example starts with the decay of Bi 212 ($T_{1/2} = 60.55$ min). This nuclide is an intermediate product that is almost at the end of the radioactive series of Th 232. In 64.06% of the time, Bi 212 decays via beta emission into Po 212 ($T_{1/2} = 0.299 \times 10^{-6}$ s), which decays into Pb 208 via alpha emission and is stable. For 35.94% of the time, Bi 212 decays via alpha emission into Tl 208 ($T_{1/2} = 3.053$ min), which subsequently decays into Pb 208.

Because the half-life of Po 212 is extremely short, we can assume that 64.06% of the time, Bi 212 decays directly into Pb 208, and 35.94% of the time, Bi 212 decays into Tl 208, which decays into Pb 208. And hence, this example will effectively consider only the population of three nuclei: Bi 212 (X_1), Tl 208 (X_2), and Pb 208 (X_3). The decay constant of X_1 is written as follows: $\lambda_1 = \lambda_1^1 + \lambda_1^2$, $\lambda_1^1 = 0.3594\lambda_1$, $\lambda_1^2 = 0.6406\lambda_1$. This branching system is illustrated in Fig. 1.

Again, we will assume that at $t=0$ only $N_{1,0}$ is nonzero. For this system, the set of differential equations is

$$\begin{bmatrix} N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{pmatrix} -\lambda_1 & 0 & 0 \\ \lambda_1^1 & -\lambda_2 & 0 \\ \lambda_1^2 & \lambda_2 & 0 \end{pmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}, \tag{22}$$

and its solution is expressed as $[\mathbf{N}] = [V][\Lambda][V]^{-1}[\mathbf{N}_0]$. Now,

$$[\Lambda] = \text{Diag}[e^{-\lambda_1 t}, e^{-\lambda_2 t}, 1], \tag{23}$$

$$[V] = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{\lambda_1^1}{\lambda_2 - \lambda_1} & 1 & 0 \\ \dfrac{\lambda_1^2 - \lambda_2}{\lambda_2 - \lambda_1} & -1 & 1 \end{pmatrix}, \tag{24}$$

$$[V]^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \dfrac{\lambda_1^1}{\lambda_1 - \lambda_2} & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \tag{25}$$

Thus the result is

$$N_1(t) = e^{-0.011\,45t} N_{1,0}, \tag{26a}$$

$$N_2(t) = 0.019\,08(e^{-0.011\,45t} - e^{-0.2270t}) N_{1,0}, \tag{26b}$$

$$N_3(t) = (1 - 1.019\,08 e^{-0.011\,45t} + 0.019\,08 e^{-0.2270t}) N_{1,0}, \tag{26c}$$

where the time t is in minutes.

IV. SUMMARY

As explained in Sec. II and illustrated in Sec. III, the three steps of the method are as follows. The decay constants of the nuclides are used to construct the matrix $[A]$ and the initial conditions are used to construct the vector $[\mathbf{N}_0]$. By solving the equation of eigenvalues and eigenvectors of $[A]$ one obtains matrix $[\Lambda]$ and matrix $[V]$, and inverting matrix $[V]$, one obtains $[V]^{-1}$.

The matrix $[\Lambda]$ can be written by merely inspecting $[A]$, because $[A]$ always maintains the lower triangular form with $-\lambda_1, ..., -\lambda_n$ in the diagonal positions. This fact implies that Eq. (10) is always satisfied. As discussed at the end of Sec. II, there are cases where these algebraic tasks can be done symbolically, which may simplify the calculations. In any case, these tasks can always be carried out numerically by means of a standard algebraic routine.

If we substitute $[V]$, $[\Lambda]$, $[V]^{-1}$, and $[\mathbf{N}_0]$ into Eq. (6), the solution of the problem is obtained. Thus students can solve the equations of any radioactive system if they know how to diagonalize and invert matrices.

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