Flying High, Thinking Low? What Every Aeronaut Needs To Know

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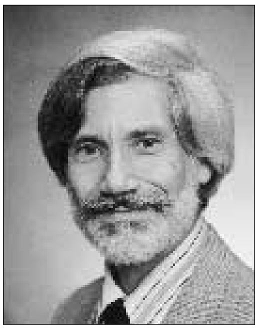

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As a physics teacher, I have often pointed out—to motivate a captive audience who would not likely have been sitting before me had not medical and other professional school requirements loomed over them—that there is survival value to learning physics. To go unarmed into a technologically complex world without the slightest understanding of the universal laws and fundamental principles that make such a world possible is to be as naked and helpless as our paleolithic ancestors must have been before lightning and thunder. That, at least, was how the rhetoric went—and I cannot say with conviction that the majority of students found it convincing. But here at last is an indisputable example—drawn from no less a bastion of journalistic integrity than the *APS NEWS*—that awareness of physics could convey a degree of protection against self-destructive acts of ignorance.[1]

Comedy of Errors

The case at hand is that of the unfortunate Californian who longed to float leisurely some 10 meters above his back yard, eating sandwiches and drinking beer, until such time as he chose to descend. To realize his dream, he purchased 45 weather balloons, which he inflated with helium and attached to his lawnchair, secured by a tether to the bumper of his jeep. Then, having provisioned his lawnchair with the necessary snacks and a pellet gun with which to pop the balloons to effect his decent, the enterprising aeronaut released the tether—whereupon (according to the news report) he streaked like a rocket into the sky, reaching equilibrium, not at 30 feet as intended, but at 11,000 feet!

There he drifted cold and frightened for 14 hours until he was noticed by the pilot of a passing jetliner. (Now the plight of the hapless man is in reality no laughing matter, but can you imagine what must have gone through the mind of the air traffic controller to whom the pilot reported having passed someone with a pellet gun in a lawnchair at 11,000 feet?) Eventually rescued by the crew of a helicopter, the physics-deficient flier was arrested for having flown his lawnchair into the air-approach corridor of Los Angeles International Airport.

The *APS NEWS* report of this adventure reached me at a most propitious moment, my physics class having just completed its study of fluids and begun to examine the properties of ideal gases. There was a lesson—indeed several—to be learned from this adventure and, not being one to waste an opportunity, I promptly made it the focus of the following day's lecture. With the data provided in the news article—plus a modicum of creative modeling—a physics student can predict with adequate accuracy the height at which his or her lawnchair would settle (and would thereby know enough at least to throw in a down jacket and thermos of hot tea along with the sandwiches and beer). There *is* survival value to the study of physics!

Let us examine this vital issue.

The Barometer Story—Model One

I designate by m the mass of the balloons and load and by V the volume of displaced air of density ρ. By Archimedes' principle it follows that the balloons come to rest at an altitude h such that the total weight of the suspended objects is balanced by the buoyant force B, where

$$B = \rho V g = mg \qquad (1a)$$

Thus, the density of the air at h must equal the mean density (total mass/total volume) of the objects:

$$\rho = \frac{m}{V} \qquad (1b)$$

Although the news report does not give the mass and volume explicitly, enough information is furnished to allow a not-unreasonable estimate. First, the total mass. Taking account of all pertinent items, I would assign masses as follows:

aeronaut	85 kg
lawnchair	20 kg
45 balloons	10 kg
six-pack of beer + pellet gun + sandwiches	5 kg

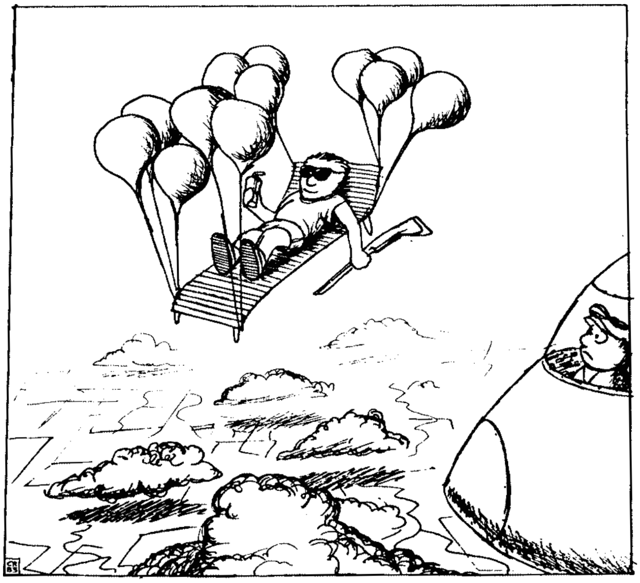
for a total m = 120 kg. The aeronaut may seem a bit portly, but then I inferred from the news report that he drinks a lot of beer. I have also assumed that the lawnchair is of the sturdy wooden variety and not a flimsy aluminum one.

Regarding the displaced volume, the report specifies only that, when fully inflated, the radius of a balloon exceeds two feet. Based on a weather balloon I cherished as a child, and the fact that the lawnchair ascended precipitously, I estimate the maximum radius to be closer to three feet. This leads to a total volume $V = 144\ \text{m}^3$. In arriving at this value, I have assumed that, once filled to capacity at ground level, the balloons do not inflate further upon rising (for to proceed otherwise, I would need information about the elastic properties of the balloon material—and the problem would become virtually intractable to introductory physics students).

From Eq. (lb) and the preceding assumptions, the question then becomes: At what height above ground is the air density $\rho = \dfrac{120\ \text{kg}}{144\ \text{m}^3} = 0.83\ \text{kg/m}^3$? Recall that at ground level, where the pressure is $p_0 = 1\ \text{atm} \sim 10^5\ \text{N/m}^2$, the corresponding density of the air (at room temperature $T \sim 293$ K) is, to good approximation, $\rho_0 = 1.2\ \text{kg/m}^3$. Thus $\rho / \rho_0 \sim 0.69$.

The simplest (albeit approximate) method of attack is to apply what I call the "Barometer-Story formula," named for a delightful essay that I habitually read to my class whenever we study fluids.[2] Written by a physics teacher (who I am quite willing to believe may have actually had the experience related in the essay—but this I do not know), the story describes the response of a bright student asked on an examination to "Show how it is possible to determine the height of a tall building with the aid of a barometer."

Wearied by college instructors trying to tell him what to think, the student came up with numerous methods—all sound but impractical and altogether intentionally irrelevant to the particular point the teacher wanted to test—with the consequence, of course, that he received a zero for that question. For example, tie a barometer to the end of a cord, swing it as a pendulum, determine the value of g at ground level and at the top of the building. "From the difference between the two values of g," said the student, "the height of the building can in principle be calculated." You get the picture. The essay is short, hilarious, and satisfying (at least to me and my class), for in the end the student triumphs. I highly recommend it to teachers; one of my own students confided afterward that he will now go to his grave knowing the barometer formula, whereas, had he encountered it merely as an end-of-chapter exercise, he would have already forgotten it.



Drawing by Chris R. B. Silverman

From the familiar form of the ideal gas equation of state

$$pV = nRT \tag{2a}$$

(with temperature T expressed in degrees Kelvin), the number of moles per volume (n/V) can be readily eliminated in favor of the gas density (ρ) to yield

$$\rho = \frac{Mp}{RT} \tag{2b}$$

in which M is the formula for molar mass (traditionally termed the molecular "weight," although this is a misnomer). For air, with an approximate composition (accurate enough for our purposes) of 75% N_2 and 25% O_2, the gram molecular weight is $M \sim 29$ *g*. *R,* the universal gas constant, is 8.3 J/mole•K.

If we assume for the present that the temperature of the atmosphere is constant (i.e., independent of height), it follows from Eq. (2b) that density is linearly proportional to pressure and therefore

$$\frac{\rho(h)}{\rho_0} = \frac{p(h)}{p_0} \tag{3}$$

The difference in air pressure between ground level and height h is simply the weight of a column of air of length h and unit cross-sectional area, or

$$p(h) = p_0 - \rho_0 g h \tag{4}$$

if, as an additional approximation, I now take the air to be incompressible. [Eq. (4) is the pressure-height relation

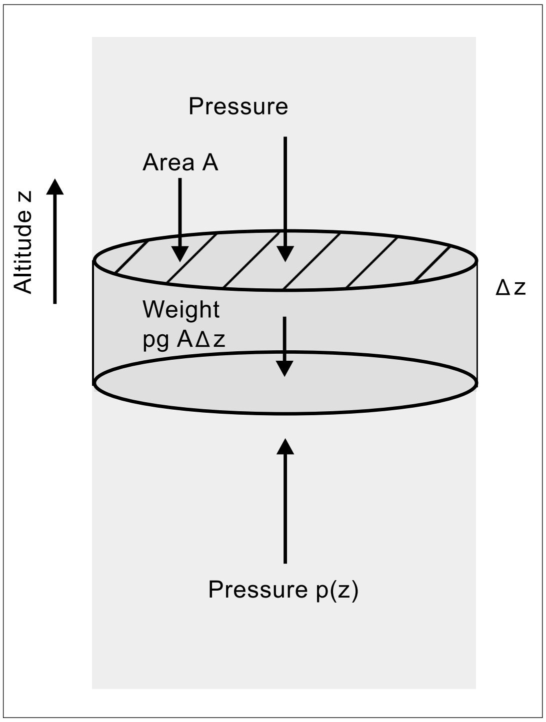


Fig. 1. Diagram of forces on a cylindrical section of air within an isothermal atmosphere in static equilibrium.

that the physics teacher sought from the recalcitrant student in the Barometer Story.]

Strictly speaking, Eqs. (3) and (4) are inconsistent with one another, for the density of the gas cannot both change and be constant in the same problem. However, since the variation in density is already accounted for in Eq. (3), it is not too crude an approximation over a sufficiently small change in altitude to assume constant density for the evaluation of $p(h)$. How small is "sufficiently small"? With insertion of Eq. (4) into Eq. (3), the resulting expression itself suggests an answer:

$$\rho(h) \sim \rho_0\left(1 - \frac{\rho_0 g h}{p_0}\right) = \rho_0\left(1 - \frac{h}{h_0}\right) \qquad (5)$$

The approximation should be valid for altitudes low compared with the characteristic height

$$h_0 \equiv \frac{p_0}{\rho_0 g} \sim 8600 \text{ m} \qquad (6)$$

I mention, in anticipation of the following section, that Eq. (5) is in fact a series expansion to first order (in h / h_0) of the exact expression for the density variation of an isothermal atmosphere. The advantage of this first approach (Model One) is that the use of calculus can be avoided, if necessary, in an algebra-based physics course.

Substitution of Eq. (5) into Eq. (1b) leads to

$$h = h_0\left(1 - \frac{\rho}{\rho_0}\right) \sim 2600 \text{ m} \sim 8600 \text{ ft} \qquad (7)$$

as the equilibrium height of the lawnchair. This is somewhat lower than the reported height, but then we did not have to work too hard to get the answer—and in any event the outcome is orders of magnitude beyond what the aeronaut *thought* his elevation would be (based on no quantitative reasoning at all).

But let us work a little harder and do a little better.

The Isothermal Atmosphere—Model Two

Under the assumption of the previous section, that the temperature of the air remains constant (let us say at room temperature $T = 293$ K), it is not difficult to derive the exact variation of density ρ with altitude z. Figure 1 shows the pertinent dynamical details. A cylindrical plug of gas of cross section A and height Δz remains in static equilibrium if the upward force of the air, $p(z)A$, on the bottom of the plug balances the sum of the downward force of the air, $p(z + \Delta z)A$, on the top of the plug and the force of gravity, $p\,g\,A\,\Delta z$, at the center of mass of the plug, leading to the well-known barometric equation

$$-\rho g = \frac{p(z + \Delta z) - p(z)}{\Delta z} \xrightarrow[\Delta z \to 0]{} \frac{dp}{dz} \qquad (8)$$

Replacing pressure p in Eq. (8) by the expression (2b) for density ρ leads to the equation

$$\frac{d\rho}{dz} = -\left(\frac{Mg}{RT}\right)\rho = -\frac{\rho}{h_0} \qquad (9a)$$

or equivalently

$$\frac{d\rho}{\rho} = d\,\ell n\,\rho = -\frac{dz}{h_0} \qquad (9b)$$

which is readily integrated (between $z = 0$ and $z = h$) to yield the exponential solution

$$\rho(h) = \rho_0 e^{-h/h_0} \qquad (10)$$

Note that the characteristic height $h_0 = \frac{RT}{Mg}$ in Eq. (9a) is precisely the same quantity as the h_0 in Eq. (6); this readily follows from use of Eq. (2b).

Although the solution of a differential equation may lie outside the scope of algebra-based introductory physics, the exponential function is of such overall importance that its origin and basic properties should not, I believe, be omitted.[3] Two points in particular are worth noting to a class. First, the exponential function arises whenever the variation in a quantity is proportional to the remaining quantity, e.g., $d\rho \propto \rho$ in Eq. (9a). Second, from the definition of the transcendental number e as a limiting process[4]

[$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$], we can approximate the function

$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$ by $1 + x$, the first term in the binomial expansion of $\left(1+\frac{x}{n}\right)^n$ for any *n*. Applied to Eq. (10), this first-order approximation generates the earlier result, Eq. (7).

From the exact solution (10), the equilibrium altitude reached by the aeronaut is found to be

$$h = h_0 \,\ell n \left(\frac{V\rho_0}{m}\right) \sim 3100 \text{ m} \sim 10{,}300 \text{ ft} \qquad (11)$$

which lies quite close to the 11,000-ft altitude reported in the news.

However, with yet more effort—although not so much as to bring the problem outside the reach of students in a calculus-based introductory physics course—we can obtain a more reliable answer. And it is *worth* the effort, for we are about to encounter something unexpected and counterintuitive.

The Linear Atmosphere—Model Three

Although the prediction of Eq. (11) is good, the assumption that the temperature of the atmosphere remains the same at all heights is not valid. I can recall a number of transoceanic flights in which the cruising altitude of the aircraft and the outside temperature were simultaneously displayed over the cabin entrance; at roughly five miles high, the air temperature had fallen to approximately –20 °C. If the temperature varied linearly with altitude and the ground was close to +20 °C (room temperature), the preceding observation would imply a rate dT/dh of about –8 °C/mile or –5 °C/km. This is actually very close to the linear variation of –6.5 °C/km recorded by atmospheric scientists over the approximate 12- to 16-km extent of the troposphere, the lowest layer of Earth's envelope of air.[5]

Since the height of the troposphere greatly exceeds the reported equilibrium altitude of the aeronaut, let us adopt the constant rate $dT/dh = -6.5$ °C/km and explore the consequences of a "linear atmosphere" model. It is often useful, I have found, to work with dimensionless ratios when solving a problem. In the present case this entails introducing a second characteristic height, z_0, defined by the temperature-altitude relation

$$T(z) = T_0\left(1 - \frac{z}{z_0}\right) \qquad (12)$$

with T_0 the temperature (293 K) at ground level. From the requirement that $dT/dz = -\frac{T_0}{z_0} = -6.5$ °C/km, it follows that $z_0 \sim 45{,}000$ m.

Substitution of Eq. (12) into the barometric equation (8) leads to a differential equation

$$\frac{d\rho}{\rho} = d\,\ell n\,\rho = -\left(\frac{1}{h_0} - \frac{1}{z_0}\right)\frac{dz}{1-\frac{z}{z_0}} \qquad (13)$$

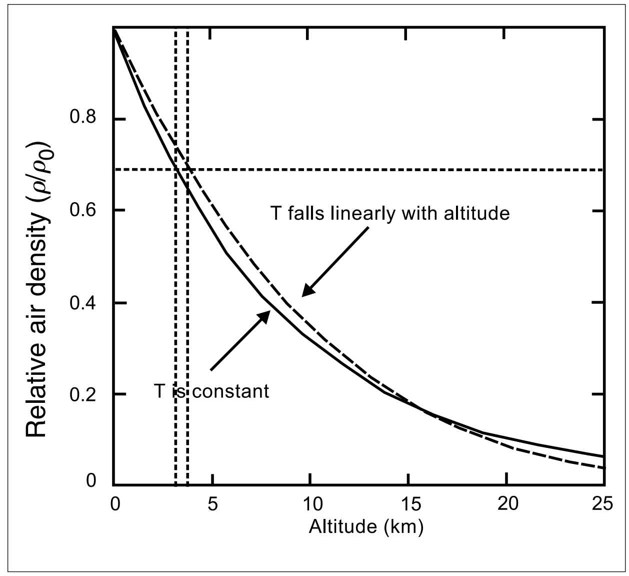


Fig. 2. Variation of air density with altitude for the constant-temperature and linear-temperature atmospheres. Horizontal dashed line marks relative air density $\frac{m/V}{\rho_0} \sim 0.69$ at height at which aeronaut settles. Two vertical dashed lines denote the corresponding altitudes. Note that the aeronaut levels off at a greater altitude in the linear-temperature atmosphere than in an isothermal atmosphere.

which at first glance may seem complicated, but in reality is quite straightforward to integrate, for it involves the exact differential of a natural logarithm on both sides. Note, too, that if we let z_0 increase without bound, the atmosphere again becomes isothermal [see Eq. (12)], and the right-hand side of Eq. (13) reduces to Eq. (9b) of the previous section. For finite z_0, however, integration of Eq. (13) from $z = 0$ to $z = h$ yields a power-law expression:

$$\rho = \rho_0\left(1 - \frac{h}{z_0}\right)^{\frac{z_0}{h_0}-1} \qquad (14)$$

Although the mathematical forms of solutions (14) and (10) are outwardly quite dissimilar, their kinship becomes apparent when the representation of an exponential as a limiting process is again recalled. If the parenthetic expression on the right side of Eq. (14) were recast as $\left(1 - \frac{h_0}{z_0}\frac{h}{h_0}\right)^{\frac{z_0}{h_0}-1}$, then it would have the approximate value of e^{-h/h_0} if $\frac{z_0}{h_0}$ were sufficiently large (so that –1 in the exponent could be neglected). For the parameters pertinent to our problem, the actual value of this ratio is

$$\frac{z_0}{h_0} = \frac{Mg}{R|dT/dz|} \sim 5.3 \qquad (15)$$

and is independent of the choice of ground-level temperature T_0.

As our final estimate of the aeronaut's altitude h, the inversion of Eq. (14) leads to

$$h = z_0\left(1 - \frac{m}{\rho_0 V}\right)^{\frac{1}{(z_0/h_0) - 1}} \sim 3700 \text{ m} \sim 12{,}100 \text{ ft} \quad (16)$$

which also accords well with the reported facts (and is probably closer to the true altitude if our assumptions regarding m and V are accurate).

For purposes of comparison, Fig. 2 illustrates the variation in air density with altitude for both the isothermal and linear-temperature atmospheres.

But something does not seem quite right here—and an astute student may well remark upon it. Look at the numerical outcome in Eq. (16). It is larger—*larger*, mind you—than the estimate derived from Eq. (11) for an isothermal atmosphere. Yet the air temperature is now falling with altitude. Should we not expect the density of colder air to be greater than that of warmer air—and therefore the aeronaut to level off at a lower altitude than if the atmosphere remained at room temperature all the way up? This curious feature is brought out strikingly in Fig. 2. At any fixed value of the relative air density ρ/ρ_0, the linear-temperature curve lies to the right of the constant-temperature curve—i.e., at greater altitude—over the entire extent of the troposphere (~ 0 to 15 km).

There is no calculational error. A cursory examination of the barometric equation of motion shows the resulting behavior to be indeed possible. Since $p \propto \rho T$, the derivative dp/dz in the barometric equation (8a) leads to two terms: one, deriving from dρ/dz, reduces the air density with increasing altitude, but the other term, arising from dT/dz, bears the opposite sign and thereby causes the density to fall off at a slower rate than that of the isothermal atmosphere. It is these two opposing actions that lead to the coefficient $\frac{1}{h_0} - \frac{1}{z_0}$ in Eq. (13).

But how can that be? What went awry?

Concluding Remarks: Winds of Change

Nothing went awry. Rather, we have rediscovered a seminal property of air—indeed any fluid—heated from below: it rises (and sometimes in startling ways). A graphic example of this behavior, first explained by Lord Rayleigh[6] and today still a subject of intensive investigation, is the Rayleigh-Bénard effect, the self-organization of convection cells within a short column of fluid confined between two planar barriers, the lower maintained at the greater temperature. Earth's atmosphere provides another example, less startling perhaps than the phenomenon studied by Bénard and Rayleigh, but no less interesting—and certainly far more significant in its overall impact on all of us. It is this convective flow in the atmosphere that bathes us in sea breezes by day and land breezes by night and rattles us unnervingly with atmospheric turbulence during our air flights.

Were the atmosphere left unperturbed for a sufficiently long time, it would eventually assume the quiescent state of thermal equilibrium, the density of each gas component falling exponentially with height. But such is not the case. Incessantly agitated under a negative temperature gradient, air is continually transferred from one part of the atmosphere to another. However—and this is the crucial feature—since the conduction of heat in gases is very slow, the atmosphere is never permitted to assume the equilibrium distribution we have discussed in the Model Two section. Instead, before an element of gas newly arrived at some location can adjust its temperature to that of its surroundings, it is again moved away. The distribution of the atmosphere, therefore, is determined by the condition that an element of gas, on being moved from one place to another, takes up the requisite pressure and volume in its new position without there being any loss or gain of heat by conduction.[7]

The foregoing process by which a quantity of gas undergoes a change in pressure, volume, and temperature without exchanging heat with the environment is termed adiabatic, and the laws for adiabatic processes

$$pV^{\gamma} = \text{constant} \quad (17a)$$

$$T\rho^{\gamma - 1} = \text{constant} \quad (17b)$$

in which $\gamma = c_p/c_V$ is the ratio of the molar specific heat of a gas at constant pressure (c_p) to the molar specific heat at constant volume (c_V), which can be found in almost any thermodynamics text.[8] For a diatomic gas, γ is expected on the basis of the equipartition theory of classical physics to be 7/5 =1.4.

Had we known to begin with the adiabatic laws (together with the ideal gas equation of state and the barometric equation), we could have deduced the linear dependence of temperature on altitude rather than adopt it as an empirical fact. By casting the resulting expressions into forms comparable to Eqs. (12) and (14), we could then relate the heat capacity ratio γ to our ratio of characteristic heights z_0/h_0 and thereby predict the rate of temperature fall through the chain of connections

$$\frac{z_0}{h_0} = \frac{Mg}{R|\mathrm{d}T/\mathrm{d}z|} = \frac{\gamma}{\gamma - 1} = \frac{c_p}{R} \quad (18a)$$

$$\frac{\mathrm{d}T}{\mathrm{d}z} = -\frac{g}{(c_p/M)} = -\frac{\text{gravitational field strength}}{\text{heat capacity per unit mass}} \quad (18b)$$

Insertion of the classical value $\gamma = 1.4$ into Eq. (18a) gives dT/dz ~ –10 °C/km, which is not too far from the actual rate of –6.5 °C/km.[9] The discrepancy may be attributable to the fact that in reality Earth's atmosphere is an extremely complex system, affected in no small way by the irregularities of the planet's surface and the reflectivity of the clouds.

It is precisely such complexity, however, that makes the physical world so intriguing and therefore the physicist's capacity to interpret it in terms of a few basic laws and simple models so remarkable. The predicament of our aeronaut aside, perhaps it is not so much the "survival value" of physics that is worth emphasizing after all, but the intrinsic pleasure and satisfaction that comes with understanding.

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3. A thorough discussion of the properties and ramifications of the exponential function can be found in A. A. Bartlett, "Forgotten fundamentals of the energy crisis," *Am. J. Phys.* **46**, 876–888 (1978). The first reference lists the author's numerous contributions on the exponential to *TPT*.
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9. A more direct way to arrive at Eq. (18b) is to use the thermodynamic relation $dQ = c_p dT - V dp$ for a differential quantity of heat absorbed by an ideal gas. In the case of an adiabatic process ($dQ = 0$), the preceding equation, together with the barometric equation, yields dT/dz immediately.