Perturbation of a classical oscillator: A variation on a theme of Huygens

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The motion of a particle in different potentials is investigated theoretically and experimentally. The dependence of the period of oscillation on the amplitude is studied for pendula associated with some of these potentials. A technique is proposed to modify the trajectory of a pendulum bob so that it moves along a predetermined curve, and a simple and low cost experiment to study the relation between the period and amplitude for different potentials is discussed. We report on the motion of several pendula whose periods decrease with increasing amplitude. In particular, we study the effects of a perturbation of the form z^4 on the frequency of oscillation of a simple harmonic oscillator. Our results agree with the expectation that any perturbation of a simple harmonic oscillator destroys its isochronism.
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I. INTRODUCTION

Perturbation analysis is used extensively in physics and engineering. However, only a few experiments that illustrate and test the results of perturbation methods, are available in the literature. Thus, it would be useful to consider simple systems that can be analyzed using perturbation methods and that can also be studied experimentally. The systems we will discuss can also be used to establish a connection between classical isochronism and the quantum mechanical equal spacing of the energy levels in a harmonic oscillator.[1] The motion of a particle can be used to investigate the consequences of nonlinear effects in an oscillatory system.

We will first apply the Poincaré-Lindstedt perturbation method[2,3] to the classical harmonic oscillator with a perturbation potential z^4. Next we will review the classical motion of a particle restricted to move along a predefined frictionless trajectory and show how to determine its period of oscillation theoretically.[3,4] We then will discuss a practical method that allows us to carry out experimental studies on the dynamics of a particle restricted to move along a given trajectory or potential. We show that a particle that moves along a frictionless curve is equivalent to the motion of the bob of a pendulum that moves along the same curve. This pendulum can be used to explore the brachistochrone and tautochrone properties of the cycloidal trajectories. We also develop a mathematical formalism to construct a pendulum of this kind. Then we present our experimental apparatus used to measure the variation of the period of a particle in different potentials as a function of the amplitude of oscillation. We report the motion of several pendula whose periods, contrary to the simple pendulum, decrease with amplitude. Finally, we compare our experimental results with the theoretical expectations.

II. THEORETICAL CONSIDERATIONS

A. Classical perturbation on a harmonic oscillator

Consider a one-dimensional oscillator perturbed by a potential of the form z^4. The total energy of that system is

$$E=\frac{1}{2}m\left(\frac{dz}{dt}\right)^2+\frac{1}{2}m\omega_0^2z^2+\frac{1}{4}\lambda\frac{m\omega_0^2}{a_0^2}z^4, \tag{1}$$

where m is the particle mass and z is its position. The quantity $m\omega_0^2$ is the elastic constant of the restoring force, a_0 is a characteristic length parameter, and λ is a dimensionless strength parameter of the perturbation potential. The corresponding equation of motion is

$$\frac{d^2z}{dt^2}+\omega_0^2z+\lambda\frac{\omega_0^2}{a_0^2}z^3=0. \tag{2}$$

Equation (2) is known as the Duffing equation without a driving force and can be solved using the Poincaré-Lindstedt perturbation method.[2,3,5] This method involves expressing the solution $z(t)$ as a series expansion in the parameter λ,

$$z(t)=\sum_{n=0}^{\infty}\lambda^n z_n(t). \tag{3}$$

We introduce a new independent variable $\tau=\omega.t$ and express ω as a series of the form,

$$\omega=\sum_{n=0}^{\infty}\lambda^n\omega_n. \tag{4}$$

We then substitute Eqs. (3) and (4) into Eq. (2) and collect terms of equal powers in λ and obtain a set of second-order differential equations in $z_n(t)$ that can be solved recursively. The first two equations are

$$\frac{d^2z_0}{d\tau^2}+z_0(\tau)=0 \tag{5}$$

and

$$\frac{d^2z_1(\tau)}{d\tau^2}+z_1+2\left(\frac{\omega_1}{\omega_0}\right)\frac{d^2z_0(\tau)}{d\tau^2}+\frac{1}{a_0^2}z_0^3(\tau)=0. \tag{6}$$

The solutions of Eqs. (5) and (6) are[2]

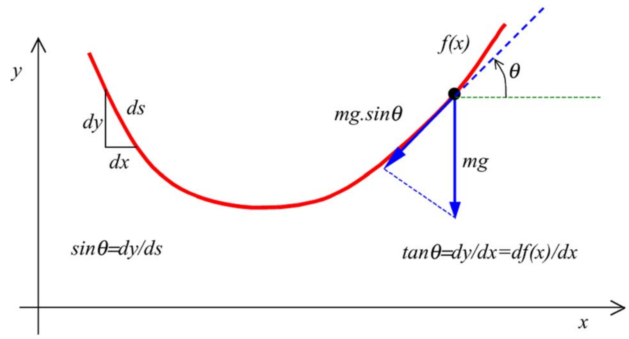

Fig. 1. Motion of a frictionless particle along a wire whose shape is described by $f(x)$. The y axis is chosen in the direction of the local gravitational field, g.

$$z_0(\tau) = A\cos(\tau+\phi) = A\cos(\omega_0 t+\phi), \tag{7}$$

where the constants A and ϕ depend on the initial conditions, and

$$z_1(\tau) = \frac{A^3}{32a_0^2}(\cos(3\tau)-\cos(\tau)) = \frac{A^3}{32a_0^2}(\cos(3\omega t)-\cos(\omega t)). \tag{8}$$

With the initial conditions $z(0)=A$ and $dz(0)/dt=0$, the solution of the equation of motion is to first order in λ

$$z(t) = A\cos(\omega t) + \frac{A^3}{32a_0^2}\lambda(\cos(3\omega t)-\cos(\omega t)), \tag{9}$$

with

$$\omega = \omega_0 + \frac{3\omega_0 A^2}{8a_0^2}\lambda. \tag{10}$$

Therefore, the period of oscillation is

$$T(A) = \frac{T_0}{\left(1+\frac{3}{8}(A^2/a_0^2)\lambda\right)}, \tag{11}$$

where $T_0=2\pi/\omega_0$ and A is the amplitude of oscillation.

Equation (11) shows that to first order in λ the isochronism of the harmonic oscillator is broken and the period becomes dependent on the amplitude. If we can design an oscillator with a perturbation potential of the form in Eq. (1), we could experimentally test the validity of Eqs. (9)–(11) by measuring the dependence of the period on the amplitude. In particular, the equation of motion of the simple pendulum satisfies Eq. (1) with $\lambda=-1/6$ (Refs. 2, 3, 6, and 7) and $a_0 = L$, the length of the pendulum.

B. Motion of a particle along a wire without friction

Consider a particle that moves under the influence of gravity along a flat frictionless wire whose shape is described by the function $f(x)$. This classical problem[3,4,8] is illustrated schematically in Fig. 1. The variable s denotes the length of an arc along the wire with respect to some arbitrary point of reference. From elementary calculus we have

$$ds = \sqrt{1+\left(\frac{df(x)}{dx}\right)^2}dx. \tag{12}$$

The equation of motion for a particle of mass m along the wire under the action of its weight is

$$m\frac{d^2s}{dt^2} = -mg\sin\theta = -mg\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}, \tag{13}$$

where θ is the angle between the tangent and the x axis as indicated in Fig. 1. If we combine Eqs. (13) and (12), we obtain

$$\frac{d^2s}{dt^2} = -g\frac{dy}{ds}. \tag{14}$$

If the shape of the trajectory, $y=f(x)$, is described parametrically by the equations $x=x(\phi)$ and $y=y(\phi)$, conservation of energy yields the relation

$$E_0 = mgy_0 = \frac{1}{2}m\left(\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dy}{d\phi}\right)^2\right)\left(\frac{d\phi}{dt}\right)^2 + mgy(x), \tag{15}$$

with amplitude $x_0=x(\phi_0)$. The period of oscillation can be obtained by integrating in time along a complete cycle. For a symmetric trajectory we obtain

$$T(x_0) = 4\int_0^{\phi_0}\frac{d\phi}{d\phi/dt} = \frac{4}{\sqrt{2g}}\int_0^{\phi_0}\frac{\sqrt{x'(\phi)^2+y'(\phi)^2}}{\sqrt{y(\phi_0)-y(\phi)}}d\phi. \tag{16}$$

The period can also be expressed in terms of the classical action[3,4] $S(\phi_0)=\oint p\,dx$ as

$$T(x_0) = \frac{\partial S}{\partial E} = \frac{\partial}{\partial E}\left[4\sqrt{2g}m \times \int_0^{\phi_0}\sqrt{y(\phi_0)-y(\phi)}\sqrt{x'(\phi)^2+y'(\phi)^2}d\phi\right]. \tag{17}$$

Equation (17) is useful for numerically evaluating the period of oscillation and avoiding the singularities of the integral (16) at the classical points of return. Equations (16) and (17) are exact expressions for the period of oscillation of a particle moving along a curve described parametrically by $(x(\phi), y(\phi))$. Note that Eqs. (10) and (11) are approximate expressions for the period to first order in the parameter λ for the special case that the potential energy has the form given by Eq. (1).

If the particle moves along a cycloidal path, that is, the wire is shaped as a cycloid with parametric equations described by

$$\begin{cases} x(\phi) = a(\phi+\sin\phi) \\ y(\phi) = a(1-\cos\phi) = 2\,a\sin^2(\phi/2) \end{cases} \tag{18}$$

then according to Eq. (12), the arc length along this trajectory is

$$ds_0 = \sqrt{\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dy}{d\phi}\right)^2}d\phi = 2a\cos(\phi/2)d\phi. \tag{19}$$

We introduce a new generalized variable

$$z \equiv \sin(\phi/2), \tag{20}$$

and combine Eqs. (18)–(20) to find

$$ds_0 = 2a\cos(\phi/2)d\phi = 4a\,dz \quad \text{and} \quad dy = z\,ds_0. \tag{21}$$

In terms of the generalized variable z, Eq. (14) becomes

$$\frac{d^2z}{dt^2} = -\frac{g}{4a}z = -\omega_0^2 z, \tag{22}$$

which is the equation of motion of a harmonic oscillator of angular frequency $\omega_0^2 = g/4a$.

As is well known, the period and the frequency of a simple harmonic oscillator are independent of the amplitude, that is, the harmonic oscillator is isochronous. This result is a consequence of two well-known properties[3,4,8] of the motion of a particle along a cycloidal path, namely the brachistochrone and tautochrone properties of the cycloid. The brachistochrone[3,4,8] property means that the time it takes a particle to descend along this trajectory is a minimum. The tautochrone[3,4,8] property means that the time to reach the bottom of a cycloidal trajectory is independent of the initial position. In other words, the oscillation of a particle along a cycloidal path is isochronous so the cycloidal pendulum is the exact analog of the simple harmonic oscillator.[4]

The question we now address is how should we perturb the cycloid so that the particle that moves along it has an equation of motion given by Eq. (2)? We propose the following parametric equation for this trajectory:

$$\left.\begin{aligned} x(\phi) &= f(\phi) = a(\phi + \sin(\phi)) + a\lambda\varphi(\phi) \\ y(\phi) &= g(\phi) = a(1-\cos(\phi)) + a\lambda\psi(\phi) \end{aligned}\right\}. \tag{23}$$

Here $\varphi(\phi)$ and $\psi(\phi)$ are functions that generate the deformation of the path which leads to the equation of motion of the type in Eq. (2), and λ is a dimensionless perturbative parameter. We also require that $x(\phi=0)=0$ and $y(\phi=0)=0$, which leads to $\varphi(\phi=0)=0$ and $\psi(\phi=0)=0$. The arc length along the perturbed trajectory in Eq. (23) is to first order in λ,

$$\begin{aligned} ds^2 &= 4a^2\cos^2(\phi/2)d\phi^2 \\ &\quad + a^2\lambda\{(1+\cos\phi)\varphi'(\phi) \\ &\quad + \sin\phi\,\psi'(\phi)\}d\phi^2. \end{aligned} \tag{24}$$

If we also require that the element of length described by Eq. (19) remains invariant to first order in λ, we have

$$\varphi'(\phi) = -\left\{\frac{\sin\phi}{1+\cos\phi}\right\}\psi'(\phi). \tag{25}$$

The requirement that the equation of motion (14) should have the form (2) leads directly to

$$\psi(\phi) = \sin^4\frac{\phi}{2}. \tag{26}$$

We combine this result with Eq. (25) to obtain

$$\varphi(\phi) = +\sin(\phi) - \frac{1}{8}\sin(2\phi) - \frac{3}{4}\phi. \tag{27}$$

If we find a way to constrain the particle to move along the path described parametrically by Eqs. (23), (26), and (27), the equation of motion of this particle will be described by Eq. (2) to first order in λ. We substitute Eqs. (26) and (27) into Eq. (23) and express the arc length along the perturbed trajectory as

$$\begin{aligned} \left(\frac{ds}{d\phi}\right)^2 &= 4a^2\cos^2(\phi/2) + 4a^2\lambda^2\sin^6(\phi/2) \\ &= \left(\frac{ds_0}{d\phi}\right)^2 + 4a^2\lambda^2\sin^6(\phi/2), \end{aligned} \tag{28}$$

where $(ds_0/d\phi)$ represents the unperturbed arc length along the cycloidal path given by Eq. (19). Note that the modification of the arc length due to the perturbation is second order in λ.

The motion of a particle along a frictionless wire is almost impossible to implement in practice. It is very difficult to precisely shape a wire into a predefined curve. More importantly, it is not easy to prevent the motion of the particle along the wire from being dominated by a strong and complicated friction force.[9,10] Friction would completely obscure the simplicity of the physics of the problem.

In this paper we discuss a simple experimental technique that allows us to build a pendulum whose bob is restricted to move along a predefined trajectory, equivalent to an imaginary wire. Consequently, the motion of this pendulum is equivalent to the proposed problem and avoids the sliding friction of the wire.

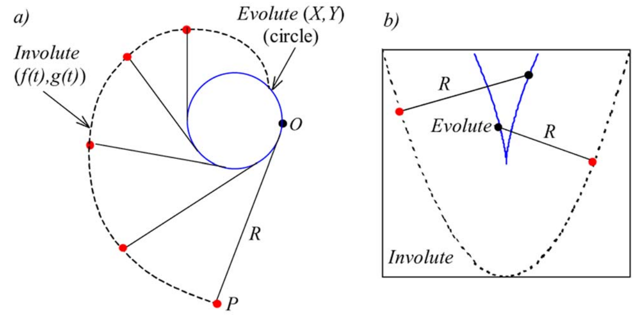

Fig. 2. Examples of evolutes and involutes. (a) String attached to the evolute (circle) at point O; the other end describes the involute (spiral). (b) The radius of curvature of the involute (dashed line) spans the evolute (solid line), but it is not possible to attach a string to the evolute that describes the involute. Note that the radius of curvature of the involute (R) is never tangent to the evolute in this case.

C. Evolutes and involutes: Special pendula

Attach a string of length L to point O on a curve, for example, a circle. Then extend the string so it is always taut and tangent to the curve at the point of contact. The curve is called the *evolute*,[11–13] a circle in this example. The locus drawn by a pen attached to the other end of the string P is called the *involute* of the original curve. This path is illustrated schematically in Fig. 2(a). Strictly speaking, the evolute is the curve determined by the centers of curvature of the involute.[11]

There is a close connection between evolutes and involutes.[11,12] Although an involute curve has an unique evolute, for every evolute there can be an infinite number of involutes depending on the initial point chosen or where the string is attached, as in the examples of Fig. 2. If the parametric equations of the involute are given by

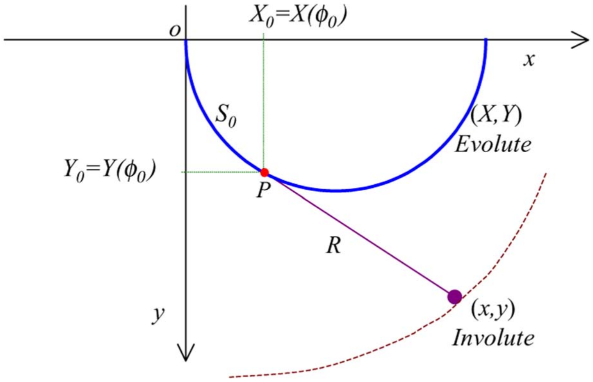

Fig. 3. Relation between pendular involute (dashed line) and evolute (full line). The string is attached at point o, and P is the point where the string begins to detach from the evolute.

$$\begin{cases} x = f(\phi) \\ y = g(\phi) \end{cases}, \qquad (29)$$

assuming that $f(\phi)$ and $g(\phi)$ are differentiable functions of ϕ, then the locus of the center of curvature (evolute) can be written as

$$\begin{cases} X(\phi) = f(\phi) - R\sin\theta \\ Y(\phi) = g(\phi) + R\cos\theta, \end{cases} \qquad (30)$$

where $(X(\phi), Y(\phi))$ are the parametric equations of the evolute, R is the radius of curvature of the involute given by[12]

$$R = \frac{(f'^2 + g'^2)^{3/2}}{f'g'' - f''g'}, \qquad (31)$$

and θ is the angle between the tangent to the involute curve and the x axis. The primes denote differentiation with respect to the parameter ϕ, that is, $f' = df/d\phi$. In vector notation

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \frac{1}{\sqrt{f'^2 + g'^2}} \begin{bmatrix} f' \\ g' \end{bmatrix}. \qquad (32)$$

From these expressions it can be shown that the parametric equations of the corresponding evolute are given by[11,12]

$$\begin{cases} X(\phi) = f(\phi) - \dfrac{(f'^2 + g'^2) g'(\phi)}{f'g'' - f''g'} \\ Y(\phi) = g(\phi) + \dfrac{(f'^2 + g'^2) f'(\phi)}{f'g'' - f''g'}. \end{cases} \qquad (33)$$

If we want a particle (bob) to move along a predetermined trajectory (involute) described by $(f(\phi), g(\phi))$, it might appear that all we need to do is to build a pair of evolutes as illustrated schematically in Fig. 3 However, the equation of the evolute, Eq. (33), does not guarantee that the radius of curvature is always tangent to the evolute, as illustrated in Fig. 2(b). Therefore, Eq. (33) represents a necessary but insufficient condition for a pendulum whose bob describes a predefined involute curve. Thus, many evolute-involute pairs may not allow us to build a useful pendulum.

We need to find a more restrictive condition for a pendulum whose bob describes a given trajectory (involute). If we start from a given evolute, we require two conditions to be fulfilled: (1) The string attached from a given point of the evolute must have a constant length L, and (2) the string must be tangent to the evolute at the point where it begins to detach from it. These conditions are illustrated in Fig. 3, where o denotes the point of attachment of the pendulum, and S_0 is the arc length of the evolute between the origin $o(\phi=0)$ and the point where the string is tangent to the evolute $P(\phi=\phi_0)$. Further details on these connections are discussed in the Appendix. The first condition leads to the relation

$$L = S_0 + R = \int_{\phi=0}^{\phi_0} \sqrt{X'(\phi)^2 + Y'(\phi)^2}\, d\phi + R. \qquad (34)$$

Here R is the radius of curvature of the involute, (X_0, Y_0) are the coordinates of the point P on the evolute, and S_0 is the arc length of the evolute between the origin o and the point of contact P. The second condition, that is, that the radius of curvature R must be tangent to the evolute in P, leads to

$$\frac{y(\phi_0) - Y_0}{x(\phi_0) - X_0} = \frac{Y'(\phi_0)}{X'(\phi_0)} \qquad (35)$$

with $R^2 = (x - X_0)^2 + (y - Y_0)^2$. By combining Eqs. (34) and (35) we obtain the parametric equation of the involute that sweeps the bob associated with a given evolute of the pendulum (see Fig. 3):

$$\begin{cases} x(\phi) = X(\phi) + \dfrac{X'(\phi)}{\sqrt{X'(\phi)^2 + Y'(\phi)^2}}[L - S_0] \\ y(\phi) = Y(\phi) + \dfrac{Y'(\phi)}{\sqrt{X'(\phi)^2 + Y'(\phi)^2}}[L - S_0]. \end{cases} \qquad (36)$$

The involute generated by Eq. (36) will be referred to as the *pendular involute* to distinguish it from the normal involute described by Eqs. (29) and (33). Equation (33) provides a necessary condition for obtaining the evolute from a known involute. On the other hand, Eq. (36) gives the involute corresponding to an evolute with a fixed point of attachment. Of course, the shape of the evolute has to be such that the string spontaneously follows the shape of the evolute from the point of attachment up to where it detaches from the evolute tangent to it. It is noteworthy that an evolute that has "waves" would not be a good candidate. This condition is equivalent to requiring that the second derivative of the evolute always have the same sign.

If we want a particle to move along a given involute, we can use Eq. (33) to obtain the corresponding evolute. Then we must verify that this evolute satisfies the more restrictive conditions given by Eq. (36). If the involute obtained using Eq. (36) coincides with the original involute, we have found the desired pair of curves to build the pendulum. In any case, Eq. (36) provides the pendular involute that the bob of the pendulum will describe for any evolute.

If we want a particle to move along a cycloidal curve of the form described by Eq. (18), we need to build the corresponding evolute. For a cycloidal involute, it can be readily shown using Eqs. (33) and (36) that the corresponding evolute is also a cycloid with a parametric equation given by[11,12,14]

$$\begin{cases} X(\phi) = a(\phi - \sin\phi) \\ Y(\phi) = L - a + a\cos\phi. \end{cases} \qquad (37)$$

The length L of the string can be found by calculating the radius of curvature of the involute. For the cycloidal pendulum whose trajectory is described by Eq. (18), the length must be $L=4a$. Christian Huygens,[15] the putative inventor of the pendulum clock, obtained this result. Huygens developed the idea of evolute-involute to build a cycloidal pendulum. This idea is a good example of the combination of technical ingenuity and a clear theoretical understanding of the problem, qualities that characterized Huygens' work.

One of the aims of the present investigation is to study the oscillatory motion of a particle in different types of potentials. Its period can be estimated using the perturbative approximation described by Eq. (11) or using the nonperturbative approach described by Eqs. (16) and (17). With these objectives in mind, we will study the motion of pendula that use the following evolutes:

(a) *Cycloidal evolute*. If we attach a string of length L to the cusp ($\phi=0$) of the cycloidal evolute described by Eq. (37), the equations of the resulting involute obtained using Eq. (36) are

$$\left.\begin{aligned} x(\phi) &= f(\phi) = a\cdot(\phi+\sin(\phi)) + a\cdot\left(\frac{L-4a}{a}\right)\sin(\phi/2) \\ y(\phi) &= g(\phi) = a\cdot(1-\cos(\phi)) + 2a\cdot\left(\frac{L-4a}{a}\right)\sin^2(\phi/4) \end{aligned}\right\}. \quad (38)$$

If the length of the pendulum is $L=4a$, Eq. (38) becomes identical to a cycloid, Eq. (18). Note that the trajectory described by Eq. (38) does not have the form given by Eqs. (23), (26), and (27). Therefore the nonperturbative expressions, Eqs. (16) or (17), must be used to obtain the period.

(b) *Perturbed cycloidal evolute*. One of the goals of this paper is the construction of an oscillator whose equation of motion is described by Eq. (2). In this case the trajectory of the bob (pendular involute) should be described by Eqs. (23), (26), and (27). One way to do so is to find the equation of the corresponding evolute using Eq. (33) and construct a pair of evolutes with this shape. Some caution must be exercised when implementing this approach, as we discuss in the following. Three types of trajectories of the form (23) were chosen using $\lambda=0.08$, $\lambda=0$ (unperturbed), and $\lambda=-0.08$. These choices of λ were based on a compromise between a small perturbation that could be treated successfully by first-order perturbation theory but that could still produce an observable effect on the period of the pendulum. The choice $\lambda=0$ results in a pair of evolute and involute that satisfies Eqs. (33) and (36). The cases of $\lambda\neq 0$ leads to evolutes that do not satisfy Eqs. (33) and (36) simultaneously. Another problem with the evolutes obtained for $\lambda\neq 0$ is that the point of suspension of the pendulum becomes a loop, destroying the possibility of building a useful pendulum. Nevertheless, if we construct pairs of evolutes using Eq. (33), attach the string to the cusp of the evolute, and then use Eq. (36) to find the corresponding pendular involute, these trajectories have the desired forms; that is, the trajectories are well described by Eqs. (23), (26), and (27) but with effective values of λ different from that of the evolute.
The effective value, $\lambda_{\rm eff}$, is found by fitting the actual pendular involute obtained from the evolute using Eq. (36) with a curve of the form (23). Using this procedure, the effective value of the parameter $\lambda_{\rm eff}=0.12$ is obtained for the involute corresponding to the evolute of $\lambda=0.08$. Similarly, the evolute corresponding to $\lambda=-0.08$ yields a pendular involute with $\lambda_{\rm eff}=-0.048$. The results of this approach are shown in Fig. 4.

(c) *Semicubical parabola evolute*. It is expected that when two particles that are restricted to move along a monotonous curve are released from two different positions, the one closer to the equilibrium position will reach this point first. This expectation is in agreement with the well-known property of the simple pendulum. Are there trajectories where this expectation is violated? The answer is yes and we discuss here an interesting case that displays this behavior.

If we build a pendulum using the evolute:

$$Y(X) = L - \tfrac{3}{2}L^{1/3}X^{2/3}, \quad (39)$$

or in parametric form

$$\left.\begin{aligned} X(\phi) &= \frac{\phi^3}{L^2} \\ Y(\phi) &= L - \frac{3}{2}\frac{\phi^2}{L} \end{aligned}\right\}, \quad (40)$$

we can find the corresponding pendular involute using Eqs. (36):

$$\left.\begin{aligned} x(\phi) &= \frac{2\phi}{\sqrt{1+(\phi/L)^2}} - \phi \\ y(\phi) &= 2L - \frac{2L}{\sqrt{1+(\phi/L)^2}} - \frac{1}{2}\frac{\phi^2}{L} \end{aligned}\right\}. \quad (41)$$

The evolute described by Eqs. (39) or (40) is known as the semicubical or Neile's parabola.[11,12] The regular involute for this evolute obtained by Eq. (33) is just a simple parabola. This pair of curves is illustrated schematically in Fig. 2(b). The pendular involute that Eq. (41) represents is quite different from a simple parabola. Notice that the trajectory represented by Eq. (41) is not of the form given by Eqs. (23), (26), and (27), and therefore, its period cannot be obtained perturbatively using Eq. (11). Nonetheless, its period can be found using Eqs. (16) or (17).

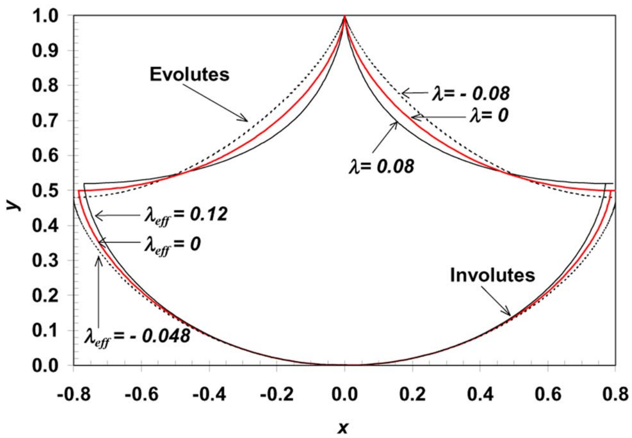

Fig. 4. Unperturbed and perturbed cycloidal involutes and their corresponding evolutes.

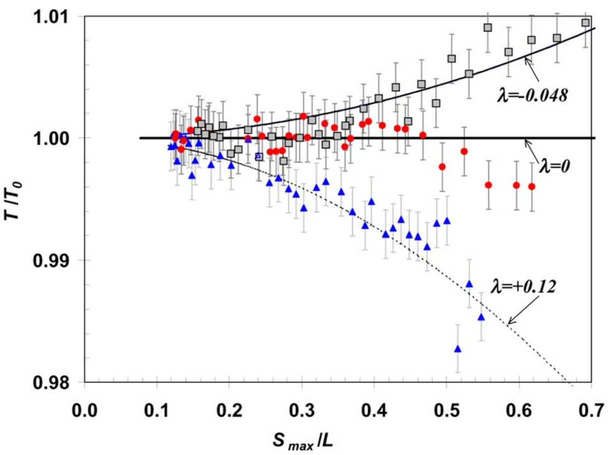

Fig. 6. Experimental results for the ratio of the measured period T to the asymptotic value of the period for small amplitude T_0 as a function of the ratio of the amplitude of oscillation $S_{\max}$ to the length of the pendulum L. The solid circles correspond to the results for the cycloidal pendulum whose trajectory is described by Eq. (18). The solid triangles represent the results for a perturbed pendulum whose trajectory is described by Eq. (23) with $\lambda_{\text{eff}}=0.12$. The open squares are the corresponding results for a perturbed pendulum with $\lambda_{\text{eff}}=-0.048$. The continuous lines are the theoretical expectations for each case using Eq. (11).

III. EXPERIMENTAL SETUP

A schematic diagram of the experimental setup is illustrated in Fig. 5. A photogate is connected to a personal computer and placed at the lowest position of the bob. This photogate determines the period of oscillation of the pendulum with an uncertainty of about 2 ms. The bob (mas=0.2 kg) is suspended from a 1 mm diameter cotton string (cotton twisted twine, 4 kg tensile strength). The total length of the pendulum (from the point of suspension to the center of mass of the bob) is $L=(100.0\pm0.5)$ cm. A 2-m flexible measuring tape was placed just below the trajectory of the bob. The tape was glued onto a 2 cm×0.2 cm flexible plastic strip. By adjusting the position of its ends and its central point, it is possible to bend the tape into a curve that is parallel to the trajectory of the bob. This device allowed measurements of the amplitude of the pendulum with an uncertainty of about 1 cm. By visual inspection, the maximum amplitude $S_{\max}$ of each oscillation was read. The maximum amplitude was characterized in each case by the dimensionless parameter $S_{\max}/L$. Note that the ratio S/L represents the angle between the string and the vertical only for the simple pendulum. Nonetheless, it can always be used as a generalized coordinate of the system. The evolutes were made of a 0.5 in. thick wooden plate. First, a plot of each evolute was made to scale. The plot was glued onto the wooden plate and then cut along the evolute. The evolutes were attached to a wooden bar that supported the system, as indicated in Fig. 5. Four sets of evolutes were constructed for the present study: (a) a pure cycloidal evolute of the form (37) with $L=1$ m and $a=0.25$ m, (b) two sets of perturbed cycloidal evolutes of the form given by Eq. (23) with $\lambda=0.08$ and $\lambda=-0.08$, and (c) a set of semicubical evolutes of the form (39) or (40) with $L=1$ m.

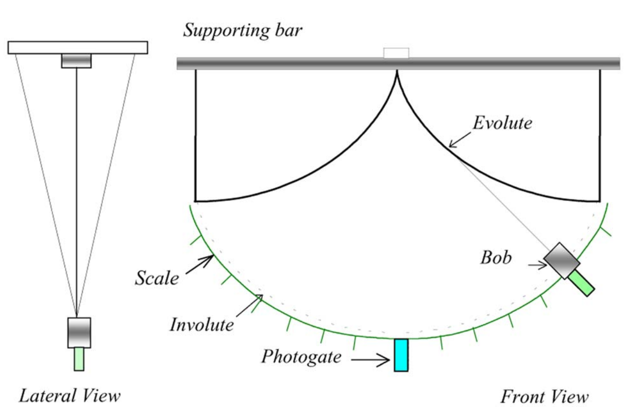

Fig. 5. Schematic diagram of the experimental aparatus. The evolute is made of a 0.5 in. thick wooden plate. The bob hangs from three strings that restrict the motion to a plane, as illustrated on the left-hand side. The bob has an opaque plastic attachment that activates the photogate. The amplitude was measured using a flexible metric tape placed below the trajectory.

The main sources of uncertainty come from the determination of the amplitude, the measurement of the periods, the stretching of the string, the fluctuations due to air gusts, and the vibrations and imperfections in the cuts of the evolutes. The uncertainty in the measurement of the amplitude $S_{\max}(\approx 1$ cm) results in a statistical error in $S_{\max}/L$ of 1%. Because this error would be about the same size as the symbols used to represent the data, they were not shown in Figs. 6–8. To estimate the effect of string stretching, the procedure of Ref. 5 was followed. If we assume that the relative variation in length of the string, $\Delta L/L$, is proportional to the string tension, F_T, we have

$$K\frac{\Delta L}{L}=F_T, \tag{42}$$

where K is an empirical constant. There is a systematic increase in the period of the pendulum due to the variation of the centripetal force given by[5]

$$\frac{\Delta T}{T}\approx\frac{11}{16}\left(\frac{mg}{K}\right)\theta_{\max}^2\approx\frac{11}{16}\left(\frac{mg}{K}\right)\left(\frac{S_{\max}}{L}\right)^2. \tag{43}$$

The stretching of the string was estimated to be approximately 2 mm for a weight of 1 kg, and thus $K\approx 5000$ N. According to Eq. (43), there is a systematic error in the determination of the period of about 0.04%. This error is smaller than that due to the time resolution of the photogate, $(\Delta T/T)_{ap}\approx 0.1\%$. The imperfections in the cuts of the evo-

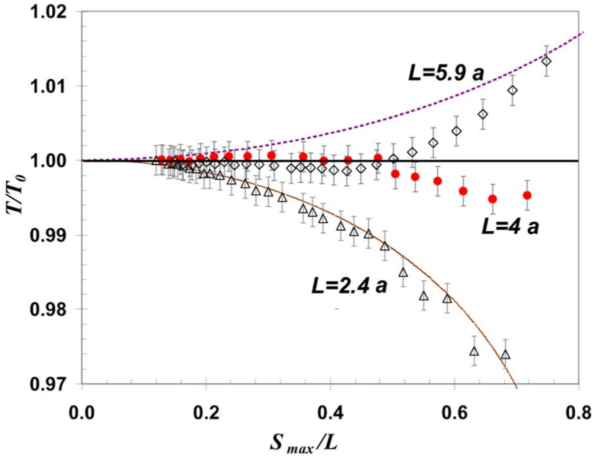

Fig. 7. Experimental results for T/T_0 as a function of S_{max}/L. The lengths of the string were $L=2.4a$ (open triangles), $L=4a$ (solid circles), and $L=5.9a$ (open rhombuses). The theoretical expectations were obtained using the nonperturbative expressions (16) and (17).

lute were about 1 mm. These random variations in the shape of the evolute result in variations in the length of the pendulum which change with amplitude. We follow Ref. 5 to estimate the variation of the period with length variations; the imperfections in cutting the evolutes result in a random variation of the period of about 0.05%. Small gusts and vibrations also affect the measurement of the period. The fluctuation in the period due to this effect was estimated to be around 0.2%. This random fluctuation is the largest source of error in the experimental determination of the period. The vertical error bars in Fig. 6–8 are mainly from this contribution.

IV. RESULTS AND DISCUSSION

Figure 6 displays the experimental results of the ratio of the measured period T to the asymptotic value for small amplitude T_0 as a function of the generalized coordinate S_{max}/L. The results shown in Fig. 6 correspond to $\lambda \neq 0$ and the unperturbed ($\lambda=0$) cycloidal pendula, with $L=1$ m and $a=0.25$ m. For the unperturbed cycloidal pendulum, the trajectory of the bob is described by Eq. (18). For the perturbed cycloidal pendula, the trajectories of the bob are described by Eqs. (23), (25), and (26) with $\lambda_{eff}=+0.12$ and $\lambda_{eff}=-0.048$. In Fig. 6 the theoretical expectation derived from Eq. (11) with $A=S_{max}$, and $a_0=L$ is also included. As expected, the unperturbed cycloidal pendulum has a period that is almost independent of the amplitude (solid circles). We observe that for $\lambda>0$ the period decreases with amplitude. This nonintuitive result implies that if two particles on this type of trajectory are released, the particle that is further away from the equilibrium position (the origin) will reach the origin sooner than the one that is released closer to the origin. At first sight, this property might seem to contradict the brachistochrone property (minimum time of descent) of the cycloidal trajectory. This apparent contradiction is solved by observing Fig. 4 carefully. If any point (different from the origin) along the trajectory for $\lambda>0$ is considered, then it is always possible to find a pure cycloidal trajectory ($\lambda=0$) that joins this point with the origin. Furthermore, the pure cycloidal trajectory must have a smaller radius of curvature, a smaller value of a in Eq. (15), than the perturbed trajectory; that is, the pure cycloidal trajectory corresponds to a pendulum of shorter length and therefore, a smaller period than the original ($\lambda>0$). Consequently, the pure cycloidal trajectory between two points is always the trajectory that joins the two points in the least time, as expected from the brachistochrone property of the cycloid.

Figure 7 shows the theoretical and experimental data corresponding to a pure cycloidal evolute for $L=2.4a$, $L=4a$, and $L=5.9a$, respectively. According to Eq. (38), only the condition $L=4a$ (solid circles) leads to a pure cycloidal trajectory. Our model reproduces the overall features of the experimental results. For $L=5.9a$ a deviation was observed at intermediate angles in several repetitions of the experiment. However, a convincing explanation has not been found. Nonetheless, the deviations are less than 1%, which are close to the uncertainties of the measurements which were about 0.3%. We also notice that for $L=2.4a$ the period of oscillation decreases with amplitude.

Figure 8 presents the experimental results of T/T_0 as a function of S_{max}/L for a simple pendulum and a pendulum of the same length that moves along a trajectory described by Eq. (41). The period of the simple pendulum increases with amplitude and can be compared to a perturbed cycloidal pendulum with $\lambda=-1/6$. The property that the period decreases with amplitude for a particle that moves along the trajectory described by Eq. (41) presents the largest enhancement observed in our study. This nonintuitive behavior was found to be present in three cases, the pendulum with a cycloidal evolute and $L<4a$, the pendulum with a perturbed cycloidal trajectory for $\lambda>0$, and the pendulum whose bob follows the trajectory is described by Eq. (41).

Note that the motion along the trajectory represented by Eq. (38) with $L<4a$ is similar to that of the perturbed cycloidal trajectory for $\lambda>0$ and to that of the pendulum whose trajectory is described by Eq. (41). In this latter case it is also observed that the particle farther away from the equilibrium position will reach the origin sooner than the one released closer to this point.

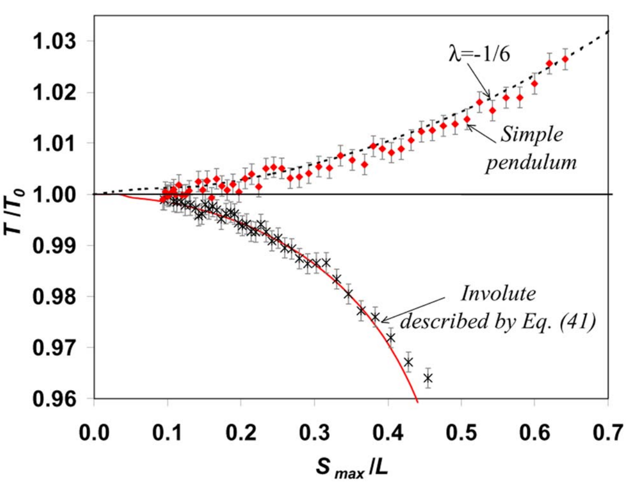

Fig. 8. Experimental results for T/T_0 as a function of S_{max}/L. The solid rhombuses correspond to the results for a simple pendulum. The asterisks represent the results for a perturbed pendulum whose trajectory is described by Eq. (41). The continuous lines are the theoretical expectations for each case. The continuous line that reproduces the data of the simple pendulum was obtained using Eq. (11) with $\lambda=-1/6$. The theoretical line that fits the asterisk data was obtained using Eqs. (16) and (17) together with Eq. (41).

V. SUMMARY

The motion of a particle in different types of potentials was investigated theoretically and experimentally. The experimental method used the properties of the evolute and involute curves. We developed a simple and inexpensive experimental technique that allows us to study the dependence of the period of oscillation on the amplitude for pendula associated with some of these potentials. Our experimental technique takes advantage of photogates to measure the period of oscillation with great precision. The experiment can be used to test the implications of a perturbation potential of the form z^4 on a simple harmonic oscillator as well as other types of potentials for which the period of oscillation cannot be obtained perturbatively.

We obtained oscillators whose periods increase with amplitude as for the simple pendulum and oscillators whose periods decrease with amplitude. Both types of behavior can be explained using the classical model discussed here. Our study constitutes a simple and practical application of perturbation theory to understand the motion of an easily observable system. We found that any perturbation on the simple harmonic oscillator destroys its isochronism. We also developed a procedure to relate the shapes of the evolute with the shape of the trajectory of the bob of a pendulum (involute). Once the form of the involutes is known, it is possible to calculate the period of oscillation of the resulting pendulum.

This experiment can be used to explore the implications of the calculus of variation formalism[3,4,8] regarding the brachistochrone and tautochrone properties of a cycloidal trajectory. We have found several trajectories that have the interesting property that the time it takes for a particle to reach its equilibrium position decreases as we release it farther away from this point. In all the cases studied the experimental results for the period of the pendulum as a function of the amplitude can be explained within the theoretical approach proposed here.

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APPENDIX: TRAJECTORY OF THE BOB OF A GENERALIZED PENDULUM

We further discuss the implications of the more restrictive condition (36) between the evolute and the pendular involute that are relevant to the construction of a pendulum whose bob moves along a predetermined trajectory (involute).

Figure 9 illustrates the geometry of our system. Here α denotes the angle between the tangent to the evolute (string) and the x axis, and θ is the corresponding angle of the tangent of the involute. For the bob to move along the involute, the string must be tangent to the evolute at the point where the string begins to detach from it. This condition is equivalent to requiring that the lines tangent to the evolute and involute are orthogonal, that is,

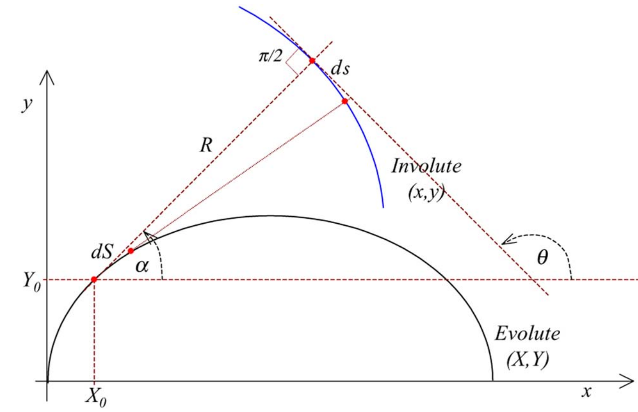

Fig. 9. Geometrical restriction on the evolute and the involute useful for constructing a generalized pendulum whose bob moves along a predefined trajectory (involute).

$$\tan\theta\tan\alpha = -1. \tag{A1}$$

By using the parametric equation of the evolute and the involute, Eq. (A1) can be expressed as

$$\frac{Y'(\phi)}{X'(\phi)}\frac{g'(\phi)}{f'(\phi)} = -1. \tag{A2}$$

Note that this condition is not automatically implied by Eq. (33). Therefore, it needs to be verified in each case for each pair of evolute/involute. On the other hand, Eq. (A2) is included in the expression of the pendular involute, Eq. (36).

For the cycloidal evolute described by Eq. (37) and the corresponding involute, Eq. (38), Eq. (A2) is satisfied. It is also satisfied for the evolute-involute pair described by Eqs. (40) and (41).

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