Nonisochronism in the interrupted pendulum

Salvador Gil[a)]
Escuela de Ciencia y Tecnología, Universidad Nacional de San Martín, Provincia de Buenos Aires, Argentina, Departamento de Física, Universidad de Buenos Aires, Argentina, and Facultad de Ingeniería y Ciencias Exactas y Naturales, Universidad Favaloro, Buenos Aires, Argentina

Daniel E. Di Gregorio
Escuela de Ciencia y Tecnología, Universidad Nacional de San Martín, Provincia de Buenos Aires, Argentina and Laboratorio Tandar, Departamento de Física, Comisión Nacional de Energía Atómica, Buenos Aires, Argentina

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We experimentally studied the dependence of the period of the interrupted pendulum as a function of the amplitude for small angles of oscillation. We found a new kind of dependence of the period with the amplitude of the pendulum that indicates that if the interruption is not located on the main vertical axis that contains the point of suspension, the period of the interrupted pendulum is highly nonisochronous and does not converge to a definite value as the maximum amplitude approaches zero. We have developed a simple model that satisfactorily explains the experimental data with no adjustable parameters. This property of the interrupted pendulum is a general property of the parabolic potential consisting of two quadratic forms with different curvatures that join at a point different from the apex or the vertex.
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I. INTRODUCTION

The interrupted pendulum, invented by Galileo,[1] is a simple pendulum of length l_0, that is interrupted in its motion by a horizontal bar placed on the vertical line that passes through the point of suspension at a distance y_0 from it. Figure 1 shows a schematic diagram of this pendulum. Traditionally, this device has been used to illustrate the conservation of energy because the bob reaches the same height on both sides.[2,3] The motion of this system is fascinating because it can be considered as having two periods: a period associated with the motion on the left side that corresponds to a pendulum of length l_0, and a second period associated with the motion on the right side which corresponds to a pendulum of length l_0-y_0.

As for the case of a simple pendulum, we expect that the interrupted pendulum will become isochronous for small angles of oscillation. Therefore, it is expected that, for a small amplitude of oscillation, the period of the interrupted pendulum will converge to the average value of the periods of the two simple pendula described above, namely,[3,4]

$$T_{12}^0=\frac{1}{2}(T_1+T_2)=\pi\left(\sqrt{\frac{l_0}{g}}+\sqrt{\frac{(l_0-y_0)}{g}}\right), \tag{1}$$

where T_1 denotes the period of the pendulum of length l_0, T_2 refers to the corresponding period of the pendulum of length l_0-y_0, and g is the local acceleration of gravity.

The isochronism of the simple pendulum is related to the fact that, for small amplitudes, the effective potential of a simple pendulum is well approximated by a parabolic potential. It is well known that, for such a potential, the period of oscillation of a particle is constant and independent of the total energy. This property has important consequences, even in the quantum mechanical behavior of the particle. According to the semiclassical correspondence principle,[5–8] at large quantum numbers, the energy difference between neighboring levels is equal to the classical frequency, $f=1/T$, times the Plank's constant, $2\pi\hbar$:

$$\frac{dE_n}{dn}\approx\hbar\,\omega_{\text{Classical}}=\hbar\,\frac{2\pi}{T_{\text{Classical}}}. \tag{2}$$

Consequently, the isochronism is related to the equal spacing of the energy levels of the harmonic oscillator. This semiclassical association is particularly relevant because for quadratic potentials most semiclassical results coincide with the exact quantum mechanical relations.[5–8] Therefore, any peculiar property of the period of the pendulum, for small amplitudes, is bound to have far-reaching consequences.

The property of the interrupted pendulum, that we will discuss, was observed in the course of a laboratory session with first year students. The objective was to verify that, for small angles of oscillation, the period of the interrupted pendulum could be described by Eq. (1). We found that for the experiments performed by most of the students, the period *did* converge to the result expressed by Eq. (1), while for some other experiments, the period *did not* converge to a definite value. In fact, some students found that the period increased as the amplitude decreased due to damping, while other students found that the period constantly decreased. Examining the experimental setup, we found that the cause of this peculiar behavior was related to the fact that the interruption was not located on the vertical axis that intercepts the point of suspension.[3] Unfortunately, at the time the students carried out the experiment, we were not able to provide a satisfactory explanation of this behavior, which we only found two months after the quarter was over. Nonetheless, this experience had a nice moral for both teachers and students, namely the importance of examining and reporting experimental information that at first sight seems to contradict the accepted explanation. These anomalies are the seeds that motivate us to search for more general and better theories that drive the progress of science.

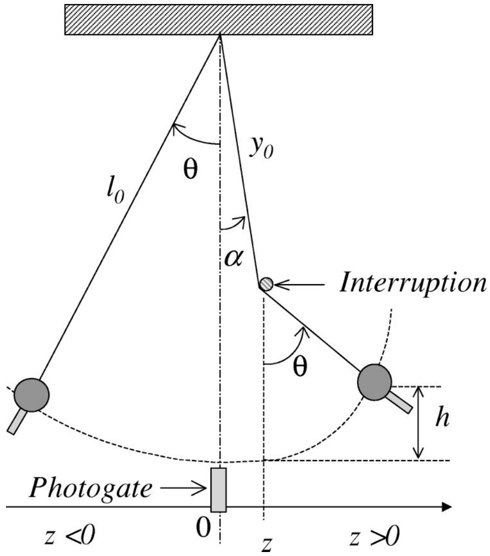

Fig. 1. A schematic diagram of the interrupted pendulum. l_0 is the total length of the pendulum. The interruption is located at a distance y_0 from the point of suspension. z is the horizontal distance from the interruption to the main vertical axis that intercepts the point of suspension. We take z as positive if the interruption is to the right of the vertical axis that intercepts the point of suspension; otherwise z is negative. Similarly, θ, measured relative to the main vertical axis, is positive when the bob is on the right side of this axis.

II. THEORETICAL CONSIDERATIONS

In this section, we will calculate the period of oscillation of the interrupted pendulum, illustrated schematically in Fig. 1, assuming that the interruption is displaced horizontally a distance z from the main vertical axis of the pendulum, i.e., the vertical line that intercepts the point of suspension. The angle α is such that $\sin\alpha = z/y_0$, and θ denotes the angular amplitude. The angle θ is measured relative to the main vertical axis and is considered positive if it is on the right side of this axis. Throughout this study, we will consider the case of small amplitudes of oscillation. Therefore, we will assume that $z \ll y_0 < l_0$, $\sin\alpha = z/y_0 \approx \alpha$, and $\sin\theta \approx \theta$. For angular amplitudes $|\theta| \leqslant 20°$, this approximation is met with an error $\leqslant 2\%$. Taking into account these approximations, the height of the bob relative to its lowest position at any angular amplitude θ can be expressed as follows:

$$h = l_0(1-\cos\theta) \approx l_0\frac{\theta^2}{2}, \quad \text{for} \quad \theta \leqslant \alpha, \tag{3}$$

while for $\theta > \alpha$, the height of the bob is given by

$$h = l_0 - y_0\cos\alpha - (l_0 - y_0)\cos\theta \approx l_0\frac{\theta^2}{2} - \frac{y_0}{2}(\theta^2 - \alpha^2). \tag{4}$$

In the present analysis we will focus on the underdamped situation, i.e., the energy loss per cycle is small compared to the total energy of the system. The total mechanical energy of the pendulum can be written as follows:

$$E_0 = \frac{1}{2}m(l_0\dot{\theta})^2 + mgh \approx \frac{1}{2}ml_0^2\left(\dot{\theta}^2 + \frac{g}{l_0}\theta^2\right), \quad \text{for} \quad \theta \leqslant \alpha, \tag{5}$$

and

$$E_0 = \tfrac{1}{2}m[(l_0 - y_0)\dot{\theta}]^2 + mgh \approx \frac{1}{2}m\left[(l_0-y_0)^2\dot{\theta}^2 + g(l_0-y_0)\theta^2 + g\frac{z^2}{y_0}\right], \quad \text{for} \quad \theta > \alpha, \tag{6}$$

where $\dot{\theta} = d\theta/dt$. We define the frequencies

$$\omega_1^2 = \frac{g}{l_0} = \frac{4\pi^2}{T_1^2} \quad \text{and} \quad \omega_2^2 = \frac{g}{l_0 - y_0} = \frac{\omega_1^2}{(1 - y_0/l_0)}. \tag{7}$$

Taking into account these definitions, Eqs. (5) and (6) can be written as follows:

$$E_0 \approx \tfrac{1}{2}ml_0^2(\dot{\theta}^2 + \omega_1^2\theta^2), \quad \text{for} \quad \theta \leqslant \alpha, \tag{8}$$

and

$$E_0 \approx \tfrac{1}{2}ml_0^2\left((1-y_0/l_0)^2\dot{\theta}^2 + \omega_2^2(1-y_0/l_0)^2\theta^2 + \omega_1^2\frac{z^2}{l_0y_0}\right), \quad \text{for} \quad \theta > \alpha. \tag{9}$$

If θ_{10} and θ_{20} are the absolute values of the maximum amplitudes on the left side ($\theta \leqslant \alpha$) and the right side ($\theta > \alpha$), respectively, the conservation of energy leads to the following equations of motion:

$$\dot{\theta}^2 + \omega_1^2(\theta^2 - \theta_{10}^2) = 0, \quad \text{for} \quad \theta \leqslant \alpha, \tag{10}$$

and

$$\dot{\theta}^2 + \omega_2^2(\theta^2 - \theta_{20}^2) = 0, \quad \text{for} \quad \theta > \alpha. \tag{11}$$

Note that $\alpha \leqslant \theta_{10} \leqslant \theta_{20}$. If this condition is not fulfilled, the interruption is not effective and the system becomes a simple pendulum. At the maximum amplitudes where $\dot{\theta} = 0$, combining Eqs. (7), (8), and (9) we obtain the following relation:

$$\theta_{20}^2 = \left(\frac{l_0}{l_0 - y_0}\right)\theta_{10}^2 - \frac{z^2}{y_0(l_0 - y_0)}. \tag{12}$$

The period of the interrupted pendulum can be calculated as follows:

$$T_{12}(\theta_{10}, \alpha) = 2\left[\int_{-\theta_{10}}^{\alpha}\frac{d\theta}{\dot{\theta}(\theta)} + \int_{\alpha}^{\theta_{20}}\frac{d\theta}{\dot{\theta}(\theta)}\right]. \tag{13}$$

Replacing Eqs. (10) and (11) in Eq. (13), we obtain

$$T_{12}(\theta_{10}, \alpha) = 2\left[\int_{-\theta_{10}}^{\alpha}\frac{d\theta}{\omega_1\sqrt{\theta_{10}^2 - \theta^2}} + \int_{\alpha}^{\theta_{20}}\frac{d\theta}{\omega_2\sqrt{\theta_{20}^2 - \theta^2}}\right]. \tag{14}$$

This expression can be analytically integrated and gives the following result:

$$T_{12}(\theta_{10}, \alpha) = \frac{2}{\omega_1}\left[\sin^{-1}\left(\frac{\alpha}{\theta_{10}}\right) + \frac{\pi}{2}\right] + \frac{2}{\omega_2}\left[-\sin^{-1}\left(\frac{\alpha}{\theta_{20}}\right) + \frac{\pi}{2}\right]. \tag{15}$$

For small angles, Eq. (12) can be approximated as follows:

$$\theta_{20} \approx \sqrt{\frac{l_0}{l_0 - y_0}}\,\theta_{10}\left[1 - \frac{1}{2}\frac{y_0}{l_0}\left(\frac{\alpha}{\theta_{10}}\right)^2\right], \tag{16}$$

and Eq. (15) reduces to

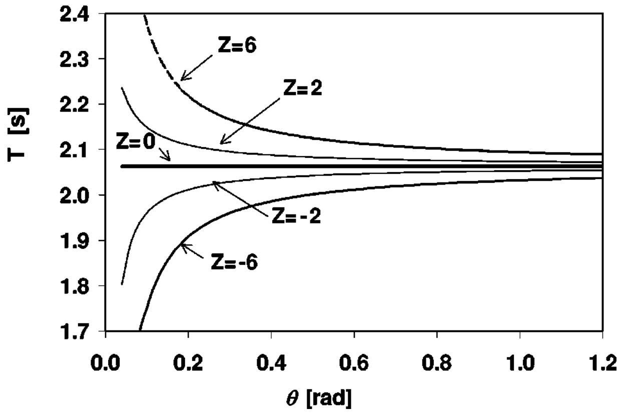

Fig. 2. The dependence of the period of an interrupted pendulum on the maximum amplitude, according to Eq. (18). The values of the parameters correspond to those used in our real experiment. The different lines correspond to predictions for different values of z. The values of z indicated in the figure are expressed in centimeters.

$$T_{12}(\theta_{10},\alpha)\approx\pi\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)+\frac{2\alpha}{\omega_1\theta_{10}}\left[1-\frac{\omega_1\theta_{10}}{\omega_2\theta_{20}}\right]. \tag{17}$$

Substituting Eqs. (1), (7), and (16) in Eq. (17), we obtain

$$T_{12}(\theta_0,z)\approx T_{12}^0\left\{1+\frac{2}{\pi}\left(\frac{z}{l_0\theta_{10}}\right)\left(\frac{\sqrt{l_0}}{\sqrt{l_0}+\sqrt{l_0-y_0}}\right)\times\left[1-\frac{1}{2}\frac{(l_0-y_0)}{l_0y_0^2}\left(\frac{z}{\theta_{10}}\right)^2\right]\right\}. \tag{18}$$

Inspecting Eq. (18) we observe that if $z\neq0$, $T_{12}(\theta_{10},z)$ diverges as θ_{10} decreases and, therefore, the interrupted pendulum is nonisochronous. On the other hand, if $z=0$, $T_{12}(\theta_{10},z)\rightarrow T_{12}^0$. This behavior is clearly illustrated in Fig. 2, where we plot $T_{12}(\theta_{10},z)$ as a function of θ_{10} for different values of z, according to Eq. (18). A more detailed discussion about the physics behind the dependence of the period of the interrupted pendulum on the coordinate z and a simple geometrical interpretation of the asymmetry of the period on this coordinate is presented in Appendix A. In the (usual) case when $\alpha/\theta_{20}\ll1$, Eq. (12) yields

$$\frac{\theta_{20}^2}{\theta_{10}^2}\approx1-\frac{y_0}{l_0}. \tag{19}$$

Combining Eqs. (7) and (19) with Eq. (17), we obtain

$$\Delta T(\theta_{10},\alpha)=T_{12}(\theta_{10},\alpha)-T_{12}^0\approx\frac{\alpha}{\theta_{10}}\frac{T_1}{\pi}\left(\frac{y_0}{l_0}\right). \tag{20}$$

This result indicates that the effect of a nonzero value of z, or equivalently a nonzero value of α, is to introduce a deviation or perturbation in the period of the interrupted pendulum with reference to the case $z=0$ (T_{12}^0) that is proportional to the ratio α/θ_{10}. Note that the perturbation is proportional to the parameters α or z, but is inversely proportional to the amplitude of the oscillation θ_{10}.

We shall see that the dependence of the period on amplitude when $z\neq0$ is different in nature to the conventional dependence of the period of the simple pendulum on amplitude for large angles, discussed in many introductory books.[3,9,10] This dependence of the period with amplitude is a consequence of the nonparabolic nature of the potential energy of the pendulum and is given by

$$T_1(\theta_{10})\approx T_1\left(1+\frac{\sin^2(\theta_{10}/2)}{4}+\cdots\right)\approx T_1\left(1+\frac{\theta_{10}^2}{16}+\cdots\right). \tag{21}$$

For moderately small angles, the quadratic term is sufficient. This correction is related to the fact that the potential energy in a pendulum is proportional to $(1-\cos\theta)$.[9,10] The parabolic potential energies assumed in Eqs. (5) and (6) are only first-order approximations. For the interrupted pendulum, this correction to first order becomes

$$T_{12}(\theta_{10})\approx\frac{T_1}{2}\left(1+\frac{\theta_{10}^2}{16}+\cdots\right)+\frac{T_2}{2}\left(1+\frac{\theta_{20}^2}{16}+\cdots\right). \tag{22}$$

Combining this expression with Eqs. (7) and (16), we obtain for $z=0$,

$$T_{12}(\theta_{10},z=0)\approx T_{12}^0\left(1+\frac{\theta_{10}^2}{16}\sqrt{\frac{l_0}{(l_0-y_0)}}\right). \tag{23}$$

The two corrections described by Eqs. (18) and (23) are independent and are due to different physical processes. The effect of the interruption with $z\neq0$ leads to the correction expressed by Eq. (18), and, because the actual potential of the pendulum is not strictly parabolic [see Eqs. (5) and (6)], we obtain the correction expressed by Eq. (23). These two effects can be combined to the lowest order in θ_{10} and z as follows:

$$T_{12}(\theta_{10},z)\approx T_{12}^0\cdot\left\{1+\frac{2}{\pi}\left(\frac{z}{l_0\theta_{10}}\right)\left(\frac{\sqrt{l_0}}{\sqrt{l_0}+\sqrt{l_0-y_0}}\right)\times\left[1-\frac{1}{2}\frac{(l_0-y_0)}{l_0y_0^2}\left(\frac{z}{\theta_{10}}\right)^2\right]\right\}\cdot\left(1+\frac{\theta_{10}^2}{16}\sqrt{\frac{l_0}{(l_0-y_0)}}\right), \tag{24}$$

where the first factor in parentheses on the right-hand side contains the dependence of the period with amplitude due to the misalignment of the interruption with the vertical axis of the pendulum, as described by Eq. (18), and the second factor on the right-hand side is the dependence of the period with amplitude due to the nonparabolic form of the potential. Figure 3 displays the dependence of the period with amplitude of the interrupted pendulum, as described by Eq. (24), for several values of z. This figure illustrates the way in which the period of the interrupted pendulum tends to diverge as θ_{10} decreases if $z\neq0$, as was shown in Fig. 2. The smooth positive slope of the different curves depicted in Fig. 3 for values of amplitude larger than 20° shows the effect of the non-strictly-parabolic shape of the real potential of the pendulum.

A better description of the behavior at lower angles can be obtained using the exact expression for $z\neq0$, given by Eq. (15):

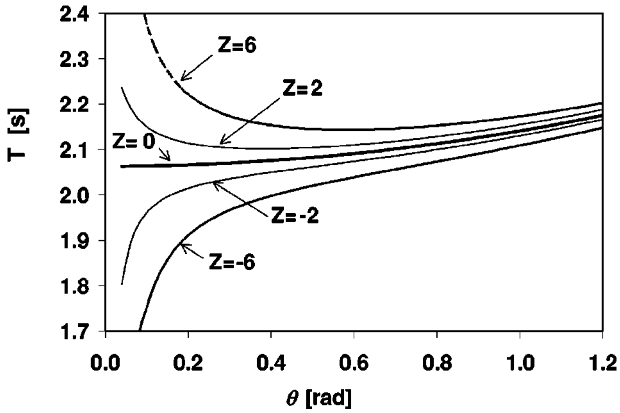

Fig. 3. The dependence of the period of an interrupted pendulum on the maximum amplitude, according to Eq. (24). The values of the parameters correspond to those used in our real experiment. The different lines correspond to predictions for different values of z. The values of z indicated in the figure are expressed in centimeters.

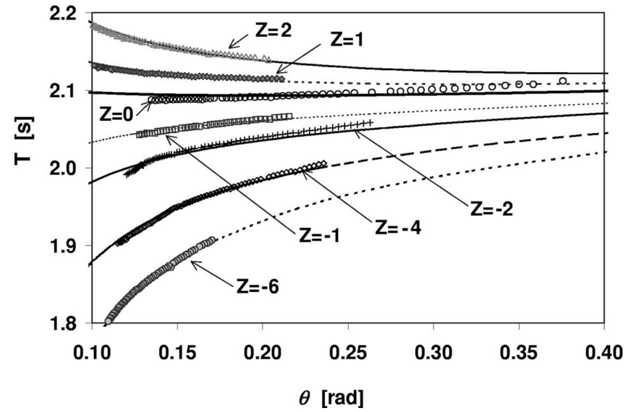

Fig. 4. Experimental results of the period of the interrupted pendulum as a function of the amplitude, for different values of the offset parameter z. The values of z indicated in the figure are expressed in cm. The continuous curves are the corresponding predictions of our model calculated using Eq. (25). The angular range in this figure spans 23°.

$$T_{12}(\theta_{10},\alpha)\approx T_{12}^{0}\left\{1+\frac{1}{\pi T_{12}^{0}}\left[T_1\sin^{-1}\left(\frac{\alpha}{\theta_{10}}\right)-T_2\sin^{-1}\left(\frac{\alpha}{\theta_{20}}\right)\right]\right\}\left(1+\frac{\theta_{10}^{2}}{16}\sqrt{\frac{l_0}{(l_0-y_0)}}\right). \tag{25}$$

The physical meaning of the two correction terms in this equation are the same as those discussed in Eq. (24).

III. EXPERIMENTAL RESULTS AND DISCUSSION

The interrupted pendulum that we built was essentially the one that is illustrated in Fig. 1, with a photogate connected to a PC and placed at the lowest position of the bob. The photogate was set up to measure the complete period of the oscillation. A horizontal meter stick was placed just below the bob, and by visual inspection we were able to read the maximum amplitude, θ_{10}. The total length of our pendulum was $l_0=(152.0\pm 0.2)$ cm and the interruption was located at $y_0=(86.9\pm 0.1)$ cm. The distance z was varied from 6 to -2 cm. For each position z, we measured the period T_{12} and maximum amplitude θ_{10}, as the system slowly damped out.

In Fig. 4 we present the results of measurements for different values of z and the theoretical expectations calculated by using Eq. (25). There is a very good agreement between our model and the experimental results. The agreement for $z=0$ is the least satisfactory, most likely because for this position the relative error in z is the largest.

IV. CONCLUSIONS

We have found that the interrupted pendulum does not converge to a definite period as the maximum amplitude of the oscillation approaches zero, if the interruption is not placed along the vertical axis that contains the point of suspension. We have developed a simple model to account for this dependence of the period with amplitude that agrees very well with the experimental results. This property of the interrupted pendulum is a general property of the parabolic potential, consisting of two quadratic forms with different curvatures that joint at a point different from the apex ($z\neq 0$). If the transition between the two parabolas of different curvature occurs at the apex or vertex of the potential, the period is indeed isochronous, otherwise the period does not converge to a definite value as the amplitude approaches zero.

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APPENDIX: PHYSICAL AND GEOMETRICAL INTERPRETATION OF THE PHENOMENA

In this appendix we explore the physical and geometrical origin of the dependence of the period of the interrupted pendulum on the maximum amplitude θ_{10} and on the horizontal offset in the position of the interruption z or equivalently on $\alpha=z/y_0$.

According to Eqs. (8) and (9) the potential energy $V(\theta)$ for our system can be written as

$$V(\theta)=\begin{cases}\frac{1}{2}\mathfrak{T}\omega_1^2\theta^2, & \text{if } \theta\leqslant\alpha,\\ \frac{1}{2}\mathfrak{T}\omega_1^2\left(\frac{l_0-y_0}{l_0}\right)\left(\theta^2+\left(\frac{y_0}{l_0-y_0}\right)\alpha^2\right), & \text{if } \theta>\alpha,\end{cases} \tag{A1}$$

where we have introduced the parameter $\mathfrak{T}\equiv ml_0^2$. The kinetic energy $T(\theta)$ and $V(\theta)$ as a function of θ are displayed in the upper panel of Fig. 5. Taking the variable θ as a generalized coordinate, according to Eqs. (8) and (9), the associated momentum[9] p_θ is

$$p_\theta=\frac{\partial T}{\partial\dot\theta}=\begin{cases}\mathfrak{T}\dot\theta, & \text{if } \theta\leqslant\alpha,\\ \mathfrak{T}\left(\frac{y_0}{l_0-y_0}\right)\dot\theta, & \text{if } \theta>\alpha.\end{cases} \tag{A2}$$

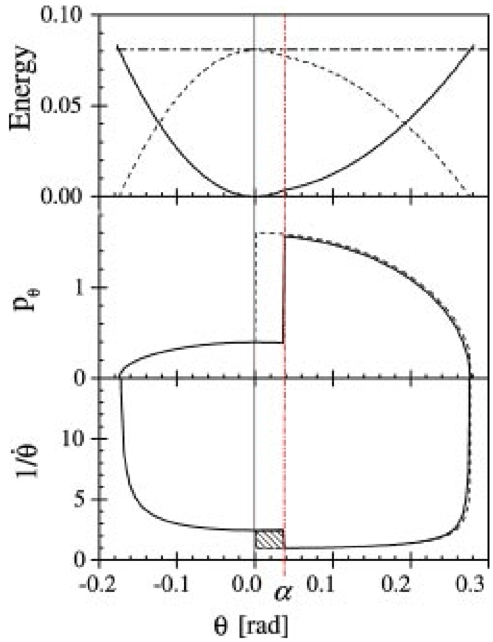

Fig. 5. In the upper panel, the solid line represents the potential energy $V(\theta)$ as a function of the generalized coordinate θ. The dashed line represents the kinetic energy $T(\theta)$. The horizontal dot–dashed line indicates the total energy of the system. The vertical dot–dashed line indicates the angular position of the interruption at $\theta=\alpha$. In the middle panel, we present the phase space diagram (p_θ vs θ) of the motion of the interrupted pendulum, both for $z=0$ (or $\alpha=0$) in dashed lines, and for $z>0$ (or $\alpha>0$) in solid lines. In the lower panel, we plot $1/\dot\theta$ vs θ. The dashed lines correspond to the case of $z=0$ and the solid lines (superimposed on the dashed lines) represent the corresponding case for $z>0$. The area of the hatched rectangle is the geometrical representation of the perturbation on the period $\Delta T(\theta_{10},\alpha)$, in first order given by Eq. (20). The units used for the vertical scales are in MKS for each variable for the system under study.

Note that p_θ is related to the *angular momentum* of the system, but is not the *conventional* angular momentum, since in our case the axis of rotation undergoes a sudden change when $\theta=\alpha$.

Also, from Eqs. (10) and (11) we can write a generalized velocity $\dot\theta(\theta)$ as

$$\dot\theta(\theta)=\begin{cases}\omega_1\sqrt{\theta_{10}^2-\theta^2}, & \text{if } \theta\leqslant\alpha,\\ \omega_1\sqrt{\dfrac{l_0}{l_0-y_0}}\sqrt{\theta_{20}^2-\theta^2}, & \text{if } \theta>\alpha.\end{cases}\tag{A3}$$

It is interesting to make a phase diagram[9] of the motion of the system, i.e., a plot of p_θ vs θ. In the middle panel of Fig. 5, we show the phase diagram for $z>0$ (or $\alpha>0$) and $z=0$ (or $\alpha=0$). Because the phase diagram for our system is symmetric with respect to the horizontal axis θ, only the upper half of this diagram has been plotted. Also, in the lower panel of the figure, we display $1/\dot\theta$ vs θ. Note that while $V(\theta)$ and $T(\theta)$ are continuous in the neighborhood of $\theta=\alpha$, the variables p_θ, $\dot\theta$ and $1/\dot\theta$, are discontinuous at $\theta=\alpha$. At first glance, a discontinuity in a velocity ($\dot\theta$) may appear physically paradoxical. To gain a deeper insight into the problem, let us consider the case for $z=0$ in more detail. From the conservation of energy we can write

$$mgl\,\frac{\theta_{10}^2}{2}=\frac{1}{2}ml^2(\dot\theta(0^-))^2,\quad \text{for } \theta\leqslant 0;\tag{A4}$$

therefore

$$\dot\theta(\theta=0^-)\equiv\dot\theta(0^-)=\omega_1\theta_{10},\quad\text{for } \theta\leqslant 0.\tag{A5}$$

Similarly, we have

$$\dot\theta(\theta=0^+)\equiv\dot\theta(0^+)=\sqrt{\frac{l_0}{l_0-y_0}}\,\omega_1\theta_{20},\quad\text{for } \theta>0.\tag{A6}$$

Equation (12) also leads in this case ($z=0$) to

$$\theta_{20}=\theta_{10}\sqrt{\frac{l_0}{l_0-y_0}};\tag{A7}$$

consequently,

$$\dot\theta(0^-)=\left(\frac{l_0-y_0}{l_0}\right)\dot\theta(0^+),\tag{A8}$$

and the generalized velocity $\dot\theta$ is discontinuous at $\theta=\alpha=0$. Nonetheless, the tangential or linear velocity ν in the neighborhood of this point ($\theta=0$) is

$$\begin{aligned}\nu(\theta=0^-)&=l_0\dot\theta(0^-)=l_0\omega_1\theta_{10},\quad\text{if } \theta\leqslant 0,\\ \nu(\theta=0^+)&=(l_0-y_0)\dot\theta(0^+)=l_0\omega_1\theta_{10},\quad\text{if } \theta>0.\end{aligned}\tag{A9}$$

Therefore, the linear velocity and the kinetic energy are continuous at $\theta=0$, as expected from a physical point of view. A discontinuity in energy would imply infinite power. This discussion makes the discontinuities observed in the middle and lower panels of Fig. 5 at $\theta=\alpha$, physically consistent. Moreover, we can understand the discontinuities of p_θ, and $1/\dot\theta$ as a consequence of the change in the axis of rotation of the system at $\theta=\alpha$. All the above observations hold true in general, even if $\alpha\neq 0$.

Coming back to the question of the variation of the period of the interrupted pendulum $T_{12}(\theta_0,\alpha)$; according to Eq. (13), it is associated with twice the area of the plot of $1/\dot\theta$ vs θ shown in the lower panel of Fig. 5.

Therefore, the variation of the period of the interrupted pendulum for the case of $z\neq 0$ (or $\alpha\neq 0$) with reference to the case of $z=0$ (T_{12}^0) is given in first approximation by the area of the hatched region depicted in the lower panel of Fig. 5. Of course, this will be the case if the contribution to the integrals indicated in Eq. (13) at $\theta\approx\theta_{10}$ and $\theta\approx\theta_{20}$ are independent of α. To test this ansatz, we estimated $\Delta T(\theta_{10},\alpha)$ given by Eq. (20) for the case where the condition $\alpha/\theta_{20}\ll 1$ is satisfied, using the value of the area of the hatched region in Fig. 5, i.e.,

$$\begin{aligned}\Delta T(\theta_{10},\alpha)&\approx 2\alpha\cdot\left(\frac{1}{\dot\theta(\alpha^+)}-\frac{1}{\dot\theta(\alpha^-)}\right)\\ &\approx 2\alpha\left(\frac{1}{\dot\theta(0^+)}-\frac{1}{\dot\theta(0^-)}\right),\end{aligned}\tag{A10}$$

the value of the last parenthesis (the height of the hatched rectangle) can be estimated using Eqs. (A5) and (A8) as

$$\left(\frac{1}{\dot{\theta}(0^+)}-\frac{1}{\dot{\theta}(0^-)}\right)=\frac{1}{\theta_{10}\omega_1}\left(\frac{y_0}{l_0}\right); \tag{A11}$$

consequently,

$$\Delta T(\theta_{10},\alpha)\approx\frac{\alpha}{\theta_{10}}\frac{T_1}{\pi}\left(\frac{y_0}{l_0}\right). \tag{A12}$$

This expression is identical to Eq. (20) and, therefore, it indicates that our ansatz of associating $\Delta T(\theta_{10},\alpha)$ with the hatched area in the lower panel of Fig. 5 is correct. Furthermore, this analysis allows us to obtain a physical interpretation of the value of perturbation of the period of the interrupted pendulum $\Delta T(\theta_{10},\alpha)$. It is originated in the sudden jump of the generalized (angular) velocity at the position of the interruption ($\theta=\alpha$). If $\alpha>0$ (or $z>0$) the period of interrupted pendulum is larger compared to the case of $\alpha=0$, due to the fact that the jump to a higher velocity occurs later in the motion than for $\alpha=0$. In other words, the interrupted pendulum spends a larger fraction of its motion as a longer pendulum, which has a larger period. For a similar reason, the period for $\alpha<0$ is smaller than for $\alpha=0$. Furthermore, the origin of the linear dependence of $\Delta T(\theta_{10},\alpha)$ on α is readily connected to the base of the rectangular hatched region in Fig. 5 (lower panel). On the other hand, the dependence of $\Delta T(\theta_{10},\alpha)$ on $1/\theta_{10}$ is related to the jump in the angular velocity $\dot{\theta}(\theta)$ in the neighborhood of the discontinuity at $\theta=\alpha$, as Eq. (A11) indicates. This last result is related to the fact that the maximum velocity of the pendulum $\dot{\theta}(0^-)$ is proportional to the amplitude θ_{10}, as Eq. (A5) indicates. In more simple terms, for a given value of $\alpha>0$, the fraction of time that the interrupted pendulum spends as a longer pendulum will be larger as θ_{10} decreases and consequently its total period will increase.

[a]Electronic mail: sgil@df.uba.ar

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CONCISE WRITING

Vigorous writing is concise. A sentence should contain no unnecessary words, a paragraph no unnecessary sentences, for the same reason that a drawing should have no unnecessary lines and a machine no unnecessary parts. This requires not that the writer make all sentences short, or that he avoid all detail and treat his subjects only in outline, but that every word tell.

William Strunk, Jr., *The Elements of Style, 3rd Edition* (MacMillan Publishing Company, Inc., New York, 1979), p. 23.