The construction of quantum mechanics from electromagnetism. Part II: the Hydrogen atom.

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Abstract

In Part I we constructed the Quantum Mechanics of a charged unitary entity and prescribed the form in which such a particle interacts with other charged particles and matter in general. In this second part we extend the description to the hydrogen atom testing the correctness and accuracy of the general description. The relation between electron and proton in the atom is described systematically in a construction that is free from analogies or ad-hoc derivations and it supersedes conventional Quantum Mechanics (whose equations linked to measurements can be recovered). We briefly discuss why the concept of isolation built in Schrödinger's time evolution is not acceptable and how it immediately results in the well known measurement paradoxes of quantum mechanics. We also discuss the epistemic grounds of the development as well as those of conventional Quantum Mechanics.

Keywords: Atomic energy levels; consilience; unity of thought; epistemic praxis; instrumentalism;

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1 Introduction

In the First Part of "The Construction of Quantum Mechanics (QM) from Electromagnetism (EM)" (Solari & Natiello, 2024) we have presented a version of Hamilton's principle for a charged quantum particle that encodes (a) Newton's first and second law of motion under the EM force described by Lorentz; (b) Maxwell's equations that propagate the EM action from the locus of its maximum intensity (matter) to any other material element in relational space (Lorenz-Lorentz perspective (Solari & Natiello, 2022b)) and (c) the relation between frequencies of stationary states of the quantum object and the energy (kinetic plus EM) of the object.

The conservation of energy of a quantum particle is described in the form in which Faraday conceived matter as the inferred part of the matter-action pair and it leads directly to the transition rule: $\mathcal{E}_I^f + \mathcal{E}_{EM}^f = \mathcal{E}_I^i + \mathcal{E}_{EM}^i$ with i, f meaning initial and final states while I, EM mean internal and EM energies. In the early times of QM, Bohr showed that the Balmer and Rydberg lines of the Hydrogen atom corresponded to this rule. This is to say that the observed values summarised in Rydberg's formula agree in value with the equation, yet the equation was not part of the theory because the "emission" of light was conceived as the result of external influences since the isolated atom could not decay and in such way it would not collapse with the electron falling onto the proton.

The standard QM then has to find a meaning for its formulae and such task has been called "interpretation of QM". All the interpretations that we know about, rest on the idea of point particles. An idea that allows for the immediate creation of classical images that suggest new formulae such as those entering in the atomic spectra of the Hydrogen atom. Since the accelerated electron (and proton) cannot radiate in QM by ukase[1] the radiation rule has to be set as a separated rule. The atom, according to the standard theory, can then be in an "mixed energy" state until it is measured (meaning to put it in contact with the Laboratory) which makes the state represented by the wave-function to collapse (in the simplest versions). While there are more sophist-

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[1] ukase: edict of the tsar

icated[2]interpretations, none of them incorporates the measurement process to QM. It remains outside QM.

The idea of isolating an atom deserves examination. One can regard as possible to set up a region of space in which EM influences from outside the region can be compensated to produce an effective zero influence. In the simplest form we think of a Faraday cage and perhaps similar cages shielding magnetic fields. In the limit, idealised following Galileo, we have an atom isolated from external influences. But this procedure is only one half of what is needed. We need the atom to be unable to influence the laboratory, including the Faraday cage. And we cannot do anything onto the atom because so doing would imply it is not isolated. This is, we must rest upon the voluntary cooperation of the atom to have it isolated. The notion of an isolated atom is thus shown to be a fantasy. If the atom, as the EM system that it is, is in the condition of producing radiation, it will produce it whether the Faraday cage is in place or not. The condition imposed by the ukase is metaphysical (i.e., not physical, nor the limit of physical situations). Actually, if such condition is imposed to Hamilton's principle of Part I, the Schrödinger's time-evolution equation is recovered to the price that the consilience with Electromagnetism and Classical Mechanics is lost.

It was Schrödinger who dug more deeply into the epistemological problems of QM. His cat, now routinely killed in every QM course/book was a form of ridiculing the, socially accepted, statistical interpretation of QM:

> One can even set up quite ridiculous cases. A cat is penned up in a steel chamber, along with the following device (which must be secured against direct interference by the cat): in a Geiger counter there is a tiny bit of radioactive substance, so small, that perhaps in the course of the hour one of the atoms decays, but also, with equal probability, perhaps none; if it happens, the counter tube discharges and through a relay releases a hammer which shatters a small flask of hydrocyanic acid. If one has left this entire system to itself for

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[2] We imply the double meaning as elaborated and false. The first as current use and the second implied in the etymological root "sophism" in the ancient Greek: σοφιστικός (sophistikós), latin: sofisticus.

> an hour, one would say that the cat still lives if meanwhile no atom has decayed. The psi-function of the entire system would express this by having in it the living and dead cat (pardon the expression) mixed or smeared out in equal parts.(Schrödinger, 1980)

Schrödinger understood as well that the need for a statistical interpretation was rooted in the assumption of point particles (Schrödinger, 1995).

The alternative QM set forth in Part I goes along Schrödinger thoughts and begins by indicating the way in which an EM quantum object acts upon other EM objects and it is acted on as well. This bold step puts immediately the problem of measurement within the theory and protects the life of Schröndiger's cat. We will show in this Part II how it can be used to produce the Hamiltonian of the Hydrogen atom and its stationary states without the need to resort to analogies. We will show how the construction is performed and how it results in formulae that are in agreement with standard QM (after the latter has been patched with "relativistic corrections" as it is usually done). The construction allows to understand better the so-called Spin-orbit interaction, which in standard QM couples the spin and the "orbit" of the electron, being then a "self-interaction" (an oxymoron) of the electron. We will show that when properly and systematically considered it represents a coupling between the proton and the electron as all other couplings considered.

2 Constructive ideas for Quantum Mechanics from Electromagnetism

We borrow some basic ingredients from Maxwell's electrodynamics and from traditional Quantum Mechanics. These elements will be combined in dynamical equations in the next Section.

1. The atom is the microscopic entity susceptible of measurements. Proton and electron within the atom do not have a sharp separable identity, they are inferences (or "shadows"). The identity of the atom consists of charge, current, magnetisation(s) and masses.

2. The experiments of Uhlenbeck & Goudsmidt and Zeeman as well as that of Stern and Gerlach suggest the existence of intrinsic magnetisations. We introduce it following the standard form relying in Pauli's seminal work (Pauli Jr, 1927).
3. The interactions within the atom and of the atom with detectors are electromagnetic.
4. Elementary entities such as the proton and the electron have no self-interactions. There is no evidence regarding the existence of self-energies (constitutive energies) at this level of description, correspondingly, they will not be included.
5. The experimental detection of properties belonging to the shadows within the atom depends on the actual possibility of coupling them with measurement devices (usually based upon electromagnetic properties). We will stay on the safe side and will not assume as measurable things like the distance from proton to electron or the probability distribution of the relative position between proton and electron. Such "measurements" remain in the domain of fantasy unless the actual procedure is offered for examination. In this respect we severely depart from textbook expositions of Quantum Mechanics.
6. The atom as an EM-entity is described with wave-functions. Any quantity associated to an atom, such as its EM fields, are represented by operators on the wave-function's space.

2.1 Expression of the electromagnetic potentials as integrated values on operators

2.1.1 Wave-functions

Two bodies that are spatially separated are idealised as represented by a *wave-function* which is the product of independent wave-functions on different coordinates. For the case of a proton and an electron we write $\Psi(x_e, x_p) = \psi_e(x_e)\psi_p(x_p)$ and call it the *electromagnetic limit*. In contrast, a single body with an internal structure is represented by the product of a wave-function associated to the body and another one representing the internal degrees of freedom. For the hydrogen atom $\Phi(x_r, x_{cm}) = \phi_r(x_r)\phi_{cm}(x_{cm})$ represents the *atomic limit*.

The election of the centre of mass, cm, to represent the position of the body will be shown later to be consistent with the form of the relational kinetic energy introduced. The variable x_r (r for *relative* or *relational*) stands for $x_e - x_p$. Our constructive postulate is that, when proton and electron are conceived as (spatially) separated entities, the laws of motion correspond to electromagnetism and when conceived as a hydrogen atom, the usual QM is recovered as a limit case[3]. In fact, the present approach is broader than the usual QM in at least two aspects: (a) The decomposition of the wave-function as a product is not imposed, but rather regarded as a limit and (b) the proton is regarded as something more than just the (point-charge) source of electrostatic potential. Hence, starting from the laws of EM, the structure of a general quantum theory for $|\Psi\rangle$ can be postulated as surpassing EM. Specialising this theory to the atomic case, the QM of the hydrogen atom is obtained.

2.1.2 Integrals on operators

We will call the value of the integral

$$\int d^3x_a\, d^3x_b\, \Big(\Psi^*(x_a, x_b)(\hat{O}\Psi(x_a, x_b))\Big) \equiv \Big\langle \Psi^*(\hat{O}\Psi)\Big\rangle \equiv \Big\langle \Psi^*|\hat{O}|\Psi)\Big\rangle \qquad (1)$$

the *integrated value* of the operator $\hat{O}$, where $\hat{O}$ is an operator that maps the wave-function Ψ as $\hat{O}\Psi(x_a, x_b) = \Psi'(x_a, x_b)$[4].

2.1.3 Physical Background

In this section our goal is to develop an electromagnetic theory of microscopic systems encompassing both QM and Maxwell's electrodynamics, starting with the principles stated in the previous Section

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[3] Conventional QM also assumes whenever necessary that $\psi_{CM}(x_{CM})$ corresponds to a point-like particle at rest.

[4] We will drop the time-argument of the wave-function whenever possible to lighten the notation.

and in Part I of this work (Solari & Natiello, 2024). Hence, we will compute Maxwell's EM interaction energy, either between the atom and external EM-fields or between different fields arising within the atom. We display first some classical results that are relevant for the coming computations.

Maxwell's electromagnetic *interaction energy* between entities 1 and 2 reads:

$$\mathcal{E} = \int d^3x \left[\epsilon_0 E_1 \cdot E_2 + \frac{1}{\mu_0} B_1 \cdot B_2\right]. \tag{2}$$

As such, this result is independent of the choice of gauge for the electromagnetic potentials A, V. In the sequel, we will adopt the Coulomb gauge whenever required.

The vector and scalar potentials associated to a charge-current density $(Cq\rho, j)$ in a (electro- and magneto-) static situation are

$$\left(\frac{V}{C}, A\right)(x,t) \;=\; \frac{\mu_0}{4\pi}\int d^3y\, \frac{(Cq\rho, j)(y,t)}{|x-y|}.$$

The vector potential associated to a magnetisation M at the coordinate y reads:

$$A_M(x,t) \;=\; \frac{\mu_0}{4\pi}\nabla_x \times \int d^3y\, \frac{M(y,t)}{|x-y|}.$$

For potentials as defined in the Lorenz gauge there is a gauge transformation that brings them to Coulomb form i.e., ρ electrostatic and $\nabla \cdot A = 0$. Finally, electric and magnetic fields are obtained through Maxwell's equations $E = -\nabla V - \dfrac{\partial A}{\partial t}$ and $B = \nabla \times A$.

2.1.4 Electromagnetic static potentials

We define the density within the atom as $\rho(x_a, x_b) = |\Psi(x_a, x_b)|^2$. When appropriate, protonic and electronic densities may be defined as $\int d^3x_b \, |\Psi(x_a, x_b)|^2 \equiv \rho_a(x_a)$ for each component, $a = e, p$. In the electromagnetic limit of full separation between electron and proton ($|\Psi\rangle = |\psi_a\rangle|\psi_b\rangle$), with a wave-function for particle b normalised to unity, we recover $\rho_a(x_a) = |\psi_a(x_a)|^2$. Protonic and electronic current-densities within the atom are defined similarly for $a = e, p$ by

$$j_a(x_p, x_e) = -\frac{i\hbar q_a}{2m_a}\left(\Psi^*(x_p, x_e)\left(\nabla_a \Psi(x_p, x_e)\right) - \Psi(x_p, x_e)\left(\nabla_a \Psi^*(x_p, x_e)\right)\right)$$

with $q_e = -q_p$.

The electrostatic potential reads

$$V_a(x, t) \;=\; \frac{q_a}{4\pi\epsilon_0}\int d^3x_a\, d^3x_b\, \frac{1}{|x - x_a|}\,|\Psi(x_a, x_b)|^2 = \frac{q_a}{4\pi\epsilon_0}\langle\Psi|\frac{1}{|x - x_a|}|\Psi\rangle.$$

Moreover,

$$V_a(x, t) = \frac{q_a}{4\pi\epsilon_0}\int d^3x_a\, \frac{1}{|x - x_a|}\left(\int d^3x_b\, |\Psi(x_a, x_b)|^2\right) = \frac{q_a}{4\pi\epsilon_0}\int d^3x_a\, \frac{\rho_a(x_a)}{|x - x_a|}$$

showing that the electromagnetic structure is preserved in general, not just in the limit of full separation.

Similarly, the vector potential (Coulomb gauge) associated to an intrinsic microscopic magnetisation $M_a(x_a, x_b) = M_a(x_a)\rho(x_a, x_b)$ reads:

$$\begin{aligned} A_{M_a}(x, t) &= \frac{\mu_0}{4\pi}\nabla_x \times \int d^3x_b\, d^3x_a\, \frac{M_a(x_a, x_b)}{|x - x_a|} \\ &= \frac{\mu_0}{4\pi}\nabla_x \times \int d^3x_a\, \frac{M_a(x_a)}{|x - x_a|}\rho(x_a) \\ &= \frac{\mu_0}{4\pi}\nabla_x \times \langle\Psi|\frac{M_a}{|x - x_a|}|\Psi\rangle \end{aligned}$$

(the index in ∇ indicates the derivation variable) where Ψ is the wavefunction. In the electromagnetic limit, the vector potential reads:

$$\begin{aligned} A_{M_a}(x, t) &= \frac{\mu_0}{4\pi}\nabla_x \times \int d^3x_a\, \frac{M_a}{|x - x_a|}\,|\psi_a(x_a)|^2 \\ &= \frac{\mu_0}{4\pi}\nabla_x \times \langle\psi_a|\frac{M_a}{|x - x_a|}|\psi_a\rangle, \end{aligned}$$

which is the classical EM potential for a density $|\psi_a(x_a)|^2$.

For a static current we have

$$A_a(x, t) \;=\; \frac{1}{2}\frac{\mu_0 q_a}{4\pi m_a}\int d^3x_a\, d^3x_b\, \left(\Psi^*(x_a, x_b)\frac{1}{|x - x_a|}(-i\hbar\nabla_a)\Psi(x_a, x_b) + c.c.\right).$$

Here we may also define

$$\widetilde{j}_a(x_a) = \frac{q_a}{2m_a}\int d^3x_b\ \left(\Psi^*(x_a,x_b)(-i\hbar\nabla_a)\Psi(x_a,x_b)\right) + c.c.$$

and set

$$A_a(x,t) \ = \ \frac{\mu_0}{4\pi}\int d^3x_a\ \left(\frac{\widetilde{j}_a(x_a)}{|x-x_a|}\right).$$

2.1.5 On atomic magnetic moments

We associate an intrinsic magnetisation to proton and electron, following Pauli's original work (Pauli Jr, 1927). A general form for the intrinsic magnetisation for a microscopic constituent a reads

$$M_a = \frac{q_a}{m_a}S_a$$

where $S_a = \frac{\hbar}{2}\sigma^a$ and σ^a are the Pauli matrices for constituent a.

2.2 Variational formulation

Both Newtonian mechanics (Arnold, 1989) and Maxwell's electrodynamics (Solari & Natiello, 2022b) can be reformulated as variational theories, satisfying the *least action principle.* The action (or action integral) is the integral of the Lagrangian over time. From Part I,

$$\mathcal{L} = \int d^3y\ \left[\frac{1}{2}\left(\Psi^*\left(i\hbar\dot{\Psi}\right) + (i\hbar\dot{\Psi})^*\Psi\right) + \frac{\hbar^2}{2m}\Psi^*T\Psi + \frac{1}{2}\sum_{i\neq j}\left(\frac{1}{\mu_0}B_i\cdot B_j - \epsilon_0 E_i\cdot E_j\right)\right] \qquad (3)$$

with $i,j \in \{e,p,L\}$ which stand for electron, proton and Laboratory. The symbol T stands for kinetic energy and its form will be presented later. Actually "d^3y" is a symbol indicating that the integral is performed over all non-temporal arguments of the wave-function, which in the case of the hydrogen atom will be at least six.

In this Part of the work we are interested in the internal energies associated to stationary states of the microscopic entity (the Hydrogen atom). In Part I we assumed that the microscopic entity did not have an internal structure and therefore there was no “self-energy”. In the present situation the atomic energy will have an internal part and an interaction part, as implied by eq.(3).

We propose therefore that the stationary states of the hydrogen atom have an associated wave-function $\Psi_\omega(x_e, x_p)$, where the indices e and p correspond to the inferred electron and proton, whose unifying entity is the atom. x represents appropriate arguments of the wave-function, satisfying

$$(\hbar\omega\Psi_\omega) - \frac{\hbar^2}{2m}T\Psi_\omega - \hat{H}\Psi_\omega = 0 \tag{4}$$

(we drop the index ω in the sequel) where

$$\left\langle \delta\Psi \hat{H}_{EM}\Psi \right\rangle = \delta_{\Psi^*}\left\langle \frac{1}{\mu_0}\left(B_L \cdot (B_e + B_p) + B_e \cdot B_p\right) + \epsilon_0\left(E_L \cdot (E_e + E_p) + E_e \cdot E_p\right)\right\rangle$$

the variation of the wave-function is $\delta\Psi$ and $\langle \cdots \rangle$ indicates integration over all coordinates as in eq.(1).

3 Contributions to the integrated value of H

3.1 Relational kinetic energy

In terms of classical mechanics, the kinetic energy associated to the hydrogen atom can be organised as follows. To fix ideas let index b correspond to the proton and a to the electron. The quantum-mechanical wave-function is $\Psi(x_a, x_b)$. The center-of-mass coordinate (relative to some external reference such as “the laboratory”) and the relative coordinate (which is an invariant of the description, independent of an external reference) can be defined as usual, satisfying

$$\begin{aligned}
x_{cm} &= \frac{m_a x_a + m_b x_b}{m_a + m_b} \\
x_r &= x_a - x_b \\
x_a &= x_{cm} + x_r \frac{m_b}{m_a + m_b} \\
x_b &= x_{cm} - x_r \frac{m_a}{m_a + m_b}.
\end{aligned}$$

The wave-function and the gradient (we also make use of $p_x = -i\hbar\nabla_x$) read:

$$\begin{aligned}
\Psi(x_a, x_b) &= \Psi(x_{cm} + x_r \frac{m_b}{m_a + m_b}, x_{cm} - x_r \frac{m_a}{m_a + m_b}) = \Phi(x_{cm}, x_r) \\
\nabla_{cm} &= \nabla_a + \nabla_b \\
\nabla_r &= \frac{m_b}{m_a + m_b}\nabla_a - \frac{m_a}{m_a + m_b}\nabla_b \\
\Phi(x_{cm}, x_r) &= \phi(\frac{m_a x_a + m_b x_b}{m_a + m_b}, x_a - x_b) = \Psi(x_a, x_b) \\
\nabla_a &= \nabla_{cm}\frac{m_a}{m_a + m_b} + \nabla_r \\
\nabla_b &= \nabla_{cm}\frac{m_b}{m_a + m_b} - \nabla_r
\end{aligned}$$

Hence,

$$\begin{aligned}
T_a + T_b = \frac{p_a^2}{2m_a} + \frac{p_b^2}{2m_b} &= -\hbar^2 \left(\nabla_{cm}^2 \frac{1}{2(m_a + m_b)} + \nabla_r^2 \frac{(m_a + m_b)}{2m_a m_b} \right) \\
&= \frac{-\hbar^2}{2(m_a + m_b)}\nabla_{cm}^2 + \frac{-\hbar^2}{2m_r}\nabla_r^2 = T_{cm} + T_r .
\end{aligned}$$

The associated integrated values read:

$$\mathcal{E}_k = \langle\Psi|T_a|\Psi\rangle + \langle\Psi|T_b|\Psi\rangle = \langle\Phi|T_{cm}|\Phi\rangle + \langle\Phi|T_r|\Phi\rangle$$

3.2 Relative electrostatic potential energy

The Coulomb electrostatic interaction energy for an atomic density $|\Psi(x_a, x_b)|^2$ reads:

$$\begin{aligned}
\mathcal{E}_C^{x_a,x_b} &= \epsilon_0 \int d^3x\, E_1 \cdot E_2 \, |\Psi(x_a, x_b)|^2 = -\epsilon_0 \int d^3x\, E_1 \cdot \nabla V_2 \, |\Psi(x_a, x_b)|^2 \\
&= \int d^3x\, \epsilon_0 \left[-\nabla \cdot (E_1 V_2) + (\nabla \cdot E_1)\, V_2 \right] |\Psi(x_a, x_b)|^2 .
\end{aligned}$$

The full energy is given by averaging over the wave function Ψ. The first contribution vanishes when integrated over all space by Gauss theorem, and the second term can be recast as $-\epsilon_0(\nabla \cdot \nabla V_1)V_2$, namely

$$\begin{aligned}\mathcal{E}_C &= \int d^3x_a\, d^3x_b\; |\Psi(x_a,x_b)|^2 \int d^3x\, \epsilon_0 \left(\frac{q_a q_b}{(4\pi\epsilon_0)^2}\right)\left(-\frac{1}{|x-x_b|}\Delta\frac{1}{|x-x_a|}\right)\\ &= -\frac{e^2}{4\pi\epsilon_0}\int d^3x_e\, d^3x_p\; |\Psi(x_e,x_p)|^2\, \frac{1}{|x_e-x_p|} = -\langle\Psi|\frac{e^2}{4\pi\epsilon_0}\frac{1}{|x_e-x_p|}|\Psi\rangle\end{aligned}$$

where $q_a = e = -q_b$ and e is the electron charge (negative), i.e., a is the electron, b the proton. In the electromagnetic limit it corresponds to the potential energy between two separate charge densities $\rho_{k=}|\psi_k(x_k)|^2$ of opposite sign, corresponding to Maxwell's equation $q_k\rho_k = \epsilon_0\nabla \cdot E_k$.

3.3 Interaction of relative current with external magnetic field

For this interaction *relative current* means the currents (j_a, j_b) of both electron and proton regarding their motion relative to the (source of) external field. Recall that we associate $j = qv = \dfrac{q}{m}p = \dfrac{q}{m}(-i\hbar\nabla)$, i.e., current as a property of charge in (perceived) motion. The electromagnetic energy in the external field $B(x)$ is

$$\begin{aligned}\mathcal{E}_{jB} &= \frac{1}{\mu_0}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^* \int d^3x\, B(x)\cdot\nabla\times(A_a+A_b)\,\Psi(x_a,x_b)\\ &= \frac{1}{2}\frac{1}{\mu_0}\frac{\mu_0}{4\pi}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^* \int d^3x\, B(x)\cdot\nabla\times\left(\frac{1}{|x-x_a|}\frac{q_a}{m_a}p_a\right)\Psi(x_a,x_b) + c.c.\\ &+ \frac{1}{2}\frac{1}{\mu_0}\frac{\mu_0}{4\pi}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^* \int d^3x\, B(x)\cdot\nabla\times\left(\frac{1}{|x-x_b|}\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b) + c.c.\end{aligned}$$

Having in mind the (normal) Zeeman effect, we perform an explicit calculation using that the vector potential for the approximately constant external field B reads $B \equiv \nabla\times A_L = \nabla\times\big(\chi(x)\frac{1}{2}x\times B\big)$, with χ a smoothed version of the characteristic function for the experiment's region. We place the origin of coordinates inside the apparatus and assume the characteristic function to be 1 inside a macroscopically large ball K around the origin, so that

$$\int_{K\times K} dx_a\, dx_b\, |\Psi(x_a,x_b)|^2 \simeq 1$$

(i.e., the atom is inside the measurement device). In such a case,

$$\begin{aligned}
\mathcal{E}_{jB} &\sim \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^* \left(\frac{1}{2}x_a \times B \cdot \frac{q_a}{m_a}p_a + \frac{1}{2}x_b \times B \cdot \frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b) + c.c. \\
&= -\frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^* \frac{1}{2}B \cdot \left(x_a \times \frac{q_a}{m_a}p_a + x_b \times \frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b) + c.c. \\
&= -\frac{e}{2m_r}\langle B \cdot \left(\frac{m_r}{m_e+m_p}x_r \times p_{CM} + x_{CM} \times p_r + \frac{m_p - m_e}{m_p + m_e}L_r\right)\rangle \\
&\sim -\left(\frac{e}{2m_e} + O(\frac{m_e}{m_p})\right)\langle\Psi|B \cdot L_e|\Psi\rangle
\end{aligned}$$

where in the last line we have switched to CM, r coordinates and used that $\langle p_r\rangle = \langle p_{CM}\rangle = 0$ and $[x_r, \nabla_{CM}] = [x_{CM}, \nabla_r] = 0$.

Further, the argument of the above integral reads,

$$\nabla \times A_L \cdot \nabla \times A_a = \nabla \cdot (A_L \times (\nabla \times A_a)) + A_L \cdot \nabla \times (\nabla \times A_a)\,.$$

The first term vanishes by Gauss theorem. The experimental conditions assume that the atom lies well inside the region of magnetic field and that the border effects $\nabla\chi(x - x_{CM}) \times \left(\frac{1}{2}x \times B\right)$ can be disregarded. Moreover, $\nabla \times (\nabla \times A_a) = \nabla(\nabla \cdot A_a) - \Delta A_a$. For the first term, we integrate over x_a, x_b, using that $\nabla_x \frac{1}{|x - x_a|} = -\nabla_{x_a}\frac{1}{|x - x_a|}$, arriving to the integral of $\frac{1}{|x - x_a|}\nabla_{x_a} \cdot \left(\Psi(x_a,x_b)^* \frac{q_a}{m_a}p_a\Psi(x_a,x_b)\right) + c.c.$, which is zero for an atom in a stationary state, since by the continuity equation it corresponds to the time-derivative of the atomic charge density. For the other term, recall that $-\Delta\frac{1}{|x - x_a|} = 4\pi\delta(x - x_a)$. Hence,

$$\begin{aligned}
\mathcal{E}_{jB} &= \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^* \int d^3x\, A_L \cdot \left(\delta(x - x_a)\frac{q_a}{m_a}p_a + \delta(x - x_b)\frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b) + c.c. \\
&= \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^* \left(A_L(x_a) \cdot \frac{q_a}{m_a}p_a + A_L(x_b) \cdot \frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b) + c.c. \\
&\sim \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^* \left(\frac{1}{2}x_a \times B \cdot \frac{q_a}{m_a}p_a + \frac{1}{2}x_b \times B \cdot \frac{q_b}{m_b}p_b\right)\Psi(x_a,x_b) + c.c.
\end{aligned}$$

assuming that the atomic wave-function is negligibly small outside the border of the experiment's region. We now move to center-of-mass and relative coordinates, assuming further that the atom as a whole is at rest (at the centre of mass location) during the experiment (i.e., $\langle\Psi|p_{CM}|\Psi\rangle = 0$, $p_r \sim p_e$, $x_r \sim x_e$ and $\frac{m_p-m_e}{m_p+m_e} \sim 1$). Hence,

$$\begin{aligned}
\mathcal{E}_{jB} &\sim \frac{1}{2}\int d^3x_a\, d^3x_b\, \Psi(x_a,x_b)^* \left(\frac{1}{2}(x_a - x_{CM}) \times B \cdot \frac{q_a}{m_a}p_a + \frac{1}{2}(x_b - x_{CM}) \times B \cdot \frac{q_b}{m_b}p_b\right) \Psi(x_a,x_b) \\
&\quad +c.c. \\
&= \frac{1}{2}\left(-\frac{e}{2(m_p+m_e)}\right)\int d^3x_{CM}\, d^3x_r\, \Psi(x_{CM},x_r)^*\, B \cdot \left(\frac{m_p}{m_e} - \frac{m_e}{m_p}\right) L_r \Psi(x_{CM},x_r) + c.c. \\
&= -\frac{e}{2m_r}\frac{m_p - m_e}{m_p + m_e}\langle\Psi|B \cdot L_r|\Psi\rangle \sim -\left(\frac{e}{2m_e} + O(\frac{m_e}{m_p})\right)\langle\Psi|B \cdot L_e|\Psi\rangle
\end{aligned}$$

3.4 Interaction of spin(s) with external magnetic field

We regard spin as an intrinsic magnetisation $\dfrac{q_a}{m_a}S_a$, leading to a vector potential

$$A_s(x,t) \;=\; \frac{\mu_0}{4\pi}\nabla \times \langle\Psi|\left(\frac{q_a}{m_a}\frac{S_a}{|x - x_a|} - \frac{q_a}{m_b}\frac{S_b}{|x - x_b|}\right)|\Psi\rangle \qquad (5)$$

whose interaction energy with an external magnetic field $B(x,t)$ reads

$$\begin{aligned}
\mathcal{E}_{SB} &= \frac{1}{\mu_0}\int d^3x\; B(x,t) \cdot \nabla \times A_s(x,t) \\
&= \frac{e}{4\pi}\langle\Psi|\int d^3x\; B(x,t) \cdot \left(\nabla \times \nabla \times \left(\frac{1}{m_e}\frac{S_e}{|x - x_e|} - \frac{1}{m_p}\frac{S_p}{|x - x_p|}\right)\right)|\Psi\rangle \\
&= \frac{e}{4\pi}\langle\Psi|\int d^3x\; B(x,t) \cdot \left[\nabla\left(\nabla \cdot \left(\frac{1}{m_e}\frac{S_e}{|x - x_e|} - \frac{1}{m_p}\frac{S_p}{|x - x_p|}\right)\right)\right]|\Psi\rangle \\
&- \frac{e}{4\pi}\langle\Psi|\int d^3x\; B(x,t) \cdot \left(\Delta\left(\frac{1}{m_e}\frac{S_e}{|x - x_e|} - \frac{1}{m_p}\frac{S_p}{|x - x_p|}\right)\right)|\Psi\rangle
\end{aligned}$$

Since $\nabla \cdot B = 0$, the first contribution integrates to zero by Gauss theorem since $B \cdot \nabla\phi = \nabla \cdot (B\phi) - \phi\nabla \cdot B$. Hence,

$$\mathcal{E}_{SB} \;=\; e\langle\Psi|\left(\frac{1}{m_e}B(x_e,t) \cdot S_e - \frac{1}{m_p}B(x_p,t) \cdot S_p\right)|\Psi\rangle$$

thus completing the description of Zeeman effect. In the atomic limit the usual expression is recovered.

Stern-Gerlach effect. All forces, including the Lorentz force (Solari & Natiello, 2022b), are obtained from actions as the response of the action integral to a variation of the relative position of the interacting bodies. Displacing by δx the position of the field B one gets

$$\delta\mathcal{E}_{SB} \;=\; e\langle\Psi|\left(\frac{1}{m_e}B(x_e+\delta x,t)\cdot S_e - \frac{1}{m_p}B(x_p+\delta x,t)\cdot S_p\right)|\Psi\rangle = \delta x\cdot F$$

and then $F \sim \langle\Psi|\nabla\left(\frac{e}{m_e}B(x,t)\cdot S\right)|\Psi\rangle$. The force is nonzero only for spatially varying magnetic fields as observed in the experiment Bauer (2023).

3.5 Spin-orbit interaction

If we follow the standard discourse of Quantum physics, spin-orbit represents mainly the interaction between the electron's orbit and spin. Such heuristic approach would break one of our fundamental propositions: all the energies in the hydrogen atom are interaction energies between proton and electron, no self energy is involved. A relational view is actually forced to recognise that the orbit of the electron is a motion relative to the proton. Hence, only their relative velocity can matter. Such observation does not solve our problem, ... but let us follow its lead.

The coupling of spin and relative current has two parts, namely the interaction of proton spin $\dfrac{-e}{m_p}S_p$ with the relative current $\dfrac{ep_r}{m_r}$ as perceived by the proton and the corresponding interaction of electron spin $\dfrac{e}{m_e}S_e$ with relative current $\dfrac{-e(-p_r)}{m_r}$ as perceived by the electron. $m_r = \dfrac{m_e m_p}{m_e+m_p}$ stands for the reduced mass and $e\left(\dfrac{p_e}{m_e} - \dfrac{p_p}{m_p}\right) = e\dfrac{p_r}{m_r}$. The magnetic field operator associated to the relative current as seen by the proton is hence the curl of the vector potential operator associated to that current (recall that the energy contribution is the integrated value of the operators over the wave-function), and correspondingly for the current perceived by the electron:

$$\begin{aligned}
\hat{B}_{jp}(x)|\Psi\rangle &= \frac{\mu_0 e}{4\pi}\nabla_x \times \left(\frac{1}{|x-x_p|}\left(\frac{p_e}{m_e}-\frac{p_p}{m_p}\right)\right)|\Psi\rangle \\
&= \frac{\mu_0 e}{4\pi}\nabla_x \frac{1}{|x-x_p|} \times \left(\frac{p_e}{m_e}-\frac{p_p}{m_p}\right)|\Psi\rangle \qquad (6)\\
\hat{B}_{je}(x)|\Psi\rangle &= \frac{\mu_0 e}{4\pi}\nabla_x \frac{1}{|x-x_e|} \times \left(\frac{p_e}{m_e}-\frac{p_p}{m_p}\right)|\Psi\rangle.
\end{aligned}$$

The magnetic field operators associated to the spin are:

$$\begin{aligned}
\hat{B}_{se}(x) &= \frac{\mu_0 e}{4\pi}\nabla_x \times \left(\nabla_x \times \left(\frac{1}{|x-x_e|}\frac{S_e}{m_e}\right)\right) \\
\hat{B}_{sp}(x) &= -\frac{\mu_0 e}{4\pi}\nabla_x \times \left(\nabla_x \times \left(\frac{1}{|x-x_p|}\frac{S_p}{m_p}\right)\right)
\end{aligned}$$

and the energy contribution is

$$\mathcal{E}_{SO} = \kappa \frac{1}{\mu_0}\langle\Psi| \int d^3x \ \hat{B}_{se}(x)\cdot \hat{B}_{jp}(x) + \hat{B}_{sp}(x)\cdot\hat{B}_{je}(x)|\Psi\rangle$$

where κ is a numerical constant that needs to be determined. We transform the spin field operators as

$$\begin{aligned}
\hat{B}_{se}(x)\cdot\hat{B}_{jp}(x) &= \frac{\mu_0 e}{4\pi m_e}\hat{B}_{jp}(x)\cdot\left(\nabla\left(\nabla\cdot\frac{S_e}{|x-x_e|}\right)-\Delta\left(\frac{S_e}{|x-x_e|}\right)\right) \\
&= \frac{\mu_0 e}{4\pi m_e}\left(\nabla\cdot\left(\hat{B}_{jp}(x)\left(\nabla\cdot\frac{S_e}{|x-x_e|}\right)\right)-\hat{B}_{jp}(x)\cdot\Delta\left(\frac{S_e}{|x-x_e|}\right)\right) \\
&= \frac{\mu_0 e}{4\pi m_e}\hat{B}_{jp}(x)\cdot(4\pi S_e \delta(x-x_e)) \\
\hat{B}_{sp}(x)\cdot\hat{B}_{je}(x) &= -\frac{\mu_0 e}{4\pi m_p}\hat{B}_{je}(x)\cdot(4\pi S_p\delta(x-x_p)) \qquad (7)
\end{aligned}$$

using Gauss theorem and that $\nabla\cdot B = 0$. Performing the $x-$integral first, we obtain

$$\begin{aligned}
\mathcal{E}_{SO} &= -\kappa\frac{\mu_0 e^2}{4\pi}\langle\Psi|\left(\frac{S_e}{m_e}+\frac{S_p}{m_p}\right)\cdot\left(\frac{-(x_p-x_e)}{|x_e-x_p|^3}\times\frac{p_r}{m_r}\right)|\Psi\rangle \\
&= -\kappa\frac{\mu_0 e^2}{4\pi}\langle\Psi|\left(\frac{S_e}{m_e}+\frac{S_p}{m_p}\right)\cdot\left(\frac{L_r}{m_r|x_e-x_p|^3}\right)|\Psi\rangle, \qquad (8)
\end{aligned}$$

where $L_r = x_r \times p_r$ is the relative angular momentum operator.

Let us complete the expression determining the value of κ before we turn back to the question that opens this subsection. The expression for the electromagnetic energy (Equation 2) was introduced by Maxwell considering the kind of interactions known at his time. It is symmetric in the indexes of the two interacting systems, hence, if $\mathcal{P}(12)$ is a permutation of indexes, the energy can be written as:

$$\mathcal{E}_{SO} = \frac{1}{2}\left\{\int d^3x \left[\epsilon_0 E_1 \cdot E_2 + \frac{1}{\mu_0} B_1 \cdot B_2\right] + \mathcal{P}(12) \int d^3x \left[\epsilon_0 E_1 \cdot E_2 + \frac{1}{\mu_0} B_1 \cdot B_2\right]\right\}$$

We may call the first integral the way in which system one acts upon system two and the second integral is the reciprocal action. The energy can then be obtained by first establishing one action and next “symmetrising”. The action of $\mathcal{P}(12)$ is merely changing the point of view in a somewhat arbitrary form, while the imposition of symmetry removes the arbitrariness since the acting group is a group of arbitrariness (see (Solari & Natiello, 2018)). The operation of taking two different view points corresponds to operating with $\mathcal{P}(ep)$ as it can be easily verified in all the previous expressions. Hence, unless $\kappa = \frac{1}{2}$ we would not be counting the interaction properly.

It remains to show that we are dealing with an interaction between two different entities and not an internal interaction. Consider our final expression, Eq. (8) and write it as:

$$\begin{aligned}
\mathcal{E}_{SO} &= -\frac{1}{2}\frac{\mu_0 e^2}{4\pi}\langle\Psi| \left(\frac{S_e}{m_e} + \frac{S_p}{m_p}\right) \cdot \left(\frac{(x_e - x_p) \times (v_e - v_p)}{|x_e - x_p|^3}\right) |\Psi\rangle \\
&= \frac{1}{2}\int d^3x_e d^3x_p \left\{\Psi^\dagger \left(\frac{(x_e - x_p)}{|x_e - x_p|^3}\right) \cdot \frac{e^2}{4\pi\epsilon_0 C^2} \left(\frac{S_e \times v_e}{m_e} - \frac{S_p \times v_p}{m_p}\right) \Psi\right\} \\
&\quad -\frac{1}{2}\int d^3x_e d^3x_p \left\{\Psi^\dagger \left(\frac{(x_e - x_p)}{|x_e - x_p|^3}\right) \cdot \frac{\mu_0 e^2}{4\pi} \left(\frac{S_e \times v_p}{m_e} - \frac{S_p \times v_e}{m_p}\right) \Psi\right\}
\end{aligned}$$

($\Psi^\dagger$ stands for the row matrix that is the transpose and complex conjugate of Ψ). Consider further the case $\Psi = \psi_e\psi_p$, i.e., the case when $\Psi = \psi_e\psi_p$ and in addition the distance $|x_e - x_p|$ is macroscopic. In such a case we can consider that the variation of the relative position with respect to $\langle x_e - x_p\rangle$ is negligible in front of the macroscopic distance. The main contribution of the first term reads

$$\sim \frac{1}{2}\frac{1}{4\pi\epsilon_0 C^2}\frac{(x_p - x_e)}{|x_e - x_p|^3} \cdot \int d^3x_e \left(e\psi_e^\dagger \frac{S_e \times v_e}{m_e}\psi_e\right) \int d^3x_p \left(-e|\psi_p|^2\right),$$

which represents the interaction between an electric dipole $\int d^3x_e \left(e\psi_e^\dagger \frac{S_e \times v_e}{m_e}\psi_e\right)$, located in the electron (now approximated as a point) and the proton (approximated as a point in front of the macroscopic distance). This view is compatible with the idea that magnetic dipoles in motion produce electric dipole fields. Up to a certain point, $\left(e\psi_e^\dagger \frac{S_e \times v_e}{m_e}\psi_e\right)$ represents a density of electric dipoles and $-e|\psi_p|^2$ a density of charge. The difference with such densities is that it is not possible to limit the interaction to "part of the electron" or "part of a proton", hence a integration over full space is always mandatory., thus the view of $\left(e\psi_e^\dagger \frac{S_e \times v_e}{m_e}\psi_e\right)$ as density of classical dipoles is only an analogy, it leaves aside the unity of the electron.

Actually, the dominant term in Spin-orbit interactions corresponds to the electric dipole of the electron acting upon the proton. Thus, what was described as an interaction between the electron and its own orbit is now identified as the action of the electric dipole associated with the moving electron with the proton charge. An equivalent contribution appears with the action of $\mathcal{P}(ep)$. The last two terms are magnetic field interactions between proton and electron.

The expression 8 corresponds well with the final expression in textbook derivations. However, we have not resorted to analogy, nor have we patched this or any other expression with gyromagnetic numbers and have not further patched the expression with "relativistic corrections" (such as Thomas' correction) in the need to agree with experiments. The derivation of the spin-orbit contribution highlights the differences between the utilitarian/instrumentalist and the old style approach.

Related experiments One of the best known results of the atomic limit was the prediction of the hydrogen spectroscopic lines with Schrödinger's wave equation, providing a full theoretical expression of the Rydberg constant. The transition $n = 2$ to $n = 1$ (Kramida *et al.*, 2023) is reported as the spectroscopic line at $\lambda = 1215.6699\mathring{A}$. The linewidth allows for a calculated resolution into $\lambda_1 = 1215.668237310\mathring{A}$ and $\lambda_2 = 1215.673644608\mathring{A}$ The state $n = 2$ is an octuplet, partially resolved by the Spin-orbit Hamiltonian of Section 3.5 into a quadruplet and a doublet for $l = 1$ and two unresolved states with $l = 0$. The difference between the two energy levels is calculated as

$$\begin{aligned}\Delta E_{SO} &= \frac{3}{2}\hbar^2\frac{1}{24a_0^3}\left(\frac{1}{2}\frac{\mu_0 e^2}{4\pi m_e^2}\right)\frac{m_e}{m_r}\\ &= \frac{3}{96}\frac{m_e}{m_r}\alpha^4 m_e C^2\\ &= 7.259023470408092\cdot 10^{-24}J\end{aligned}$$

while the calculated experimental energy difference amounts to $\Delta E_{exp} = hC(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}) = 7.26816814178113\cdot 10^{-24}J$. The contribution consists of: $\frac{3}{2}\hbar^2$ coming from the level difference in $\langle L\cdot S\rangle$ between $j = 3/2$ and $j = 1/2$ (for $l = 1$ and $s = 1/2$), $\frac{1}{24a_0^3} = \langle R_{21}(r)|\frac{1}{r^3}|R_{21}(r)\rangle$ for the associated Hydrogen (n, l)-radial wave-function, $a_0 = \frac{\hbar}{\alpha m_e C}$ is the Bohr radius, m_e is the electron mass and $\alpha = \frac{\mu_0 e^2 C}{4\pi\hbar}$ is the fine-structure constant. We have disregarded the nuclear spin contribution.

3.6 Spin-spin interaction (part of hyperfine structure)

The interaction between spins within the atom uses an expression for the vector potential corresponding to that in Section 3.4 and reads

$$\begin{aligned}\mathcal{E}_{SS} &= \frac{1}{\mu_0}\langle\Psi|\int d^3x\; B_e(x,t)\cdot B_p(x,t)|\Psi\rangle\\ B_e &= \nabla\times\left(\nabla\times\left(\frac{e}{m_e}\frac{\mu_0}{4\pi}\frac{S_e(x_e)}{|x-x_e|}\right)\right)\\ B_p &= \nabla\times\left(\nabla\times\left(\frac{-e}{m_p}\frac{\mu_0}{4\pi}\frac{S_p(x_p)}{|x-x_p|}\right)\right)\end{aligned}$$

To proceed with the calculation we need to break the $e-p$ symmetry, namely $\mathcal{E}_{SS} = \frac{1}{2}\,(\mathcal{E}_{SS}^{e\to p} + \mathcal{E}_{SS}^{p\to e})$. We pick the electron to proceed with the calculation, applying Gauss theorem again as in Section 3.4:

$$\begin{aligned}
\mathcal{E}_{SS}^{e\to p} &= \frac{1}{\mu_0}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\int d^3x\ B_p(x)\cdot\left(-\Delta\frac{S_e(x_e)}{|x-x_e|}\right)|\Psi\rangle \\
&= -\frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\int d^3x\ \left(\nabla_x\times\left(\nabla_x\times\frac{S_p}{|x-x_p|}\right)\right)\cdot S_e\delta(x-x_e)|\Psi\rangle \\
&= -\frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\left[\left(\nabla_x\times\left(\nabla_x\times\frac{S_p}{|x-x_p|}\right)\right)\cdot S_e\right]_{x=x_e}|\Psi\rangle
\end{aligned}$$

Through the vector calculus identities

$$\begin{aligned}
-\left(\nabla_x\times\left(\nabla_x\times\varphi(x)S_p\right)\right)\cdot S_e &= \left(S_e\times\nabla_x\right)\cdot\left(S_p\times\nabla_x\right)\varphi(x) \\
\left(S_e\times\nabla_x\right)\cdot\left(S_p\times\nabla_x\right)\frac{1}{|x-x_p|} &= \left[\left(S_e\cdot S_p\right)\Delta_x-\left(S_e\cdot\nabla_x\right)\left(S_p\cdot\nabla_x\right)\right]\frac{1}{|x-x_p|}
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathcal{E}_{SS}^{e\to p} &= \frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\left[\left[\left(S_e\cdot S_p\right)\Delta_x-\left(S_e\cdot\nabla_x\right)\left(S_p\cdot\nabla_x\right)\right]\frac{1}{|x-x_p|}\right]_{x=x_e}|\Psi\rangle \\
\mathcal{E}_{SS} &= \frac{e}{m_p}\frac{e}{m_e}\frac{\mu_0}{4\pi}\langle\Psi|\left[\left[\left(S_e\cdot S_p\right)\Delta_x-\left(S_e\cdot\nabla_x\right)\left(S_p\cdot\nabla_x\right)\right]\frac{1}{|x-x_e|}\right]_{x=x_p}|\Psi\rangle
\end{aligned}$$

since both contributions are identical. The spin-spin interaction is symmetric in front of the interchange of electron and proton and then carries a factor one in front.

3.7 Interaction with an external electric field

Suppose the atom is in interaction with an external electric field, E_Z. The electromagnetic interaction is then:

$$\mathcal{E}_{AZ}=\epsilon_0\int dx\ \left(\left(E_A+E_A^{am}\right)\cdot E_Z\right)$$

where $E_A=-\nabla V$ and $E_A^{am}=-\frac{\partial A}{\partial t}$ being (V,A) the electromagnetic potentials associated to the atom:

$$\begin{aligned}
A &= \frac{\mu_0}{4\pi}\nabla\times\langle\Psi|\left(\frac{e}{m_e}\frac{M_e}{|x-x_e|}-\frac{e}{m_p}\frac{M_p}{|x-x_p|}\right)|\Psi\rangle \\
V &= \frac{e}{4\pi\epsilon_0}\langle\Psi|\left(\frac{1}{|x-x_e|}-\frac{1}{|x-x_p|}\right)|\Psi\rangle
\end{aligned}$$

4 Concluding remarks

The quantum mechanics that emerges from electromagnetism has been tested in the construction of the internal energies for the Hydrogen atom producing outstanding results, superior to those of standard quantum mechanics. Equal in precision but definitely superior in consistence and consilience. Another insight gained is that the moving spin reveals its role as the carrier of *electric* polarisation in the spin-orbit interaction.

The path followed in this construction is blocked for the standard presentation of QM as extension from mechanics by the early assumption of point-particles. The usual form used to incorporate the Lorentz force in the case of point particles was the constructive basis for several authors such as Pauli (Pauli Jr, 1927) and Dirac (Dirac, 1928). This led to the identification of the momentum p with the operator $-i\hbar\nabla$ (in all situations), a compensating error sustained in Bohr's principle of correspondence. In turn, this chain of decisions made impossible to associate the operator with the current and prevented the production of EM interactions in QM except by resource to analogy.

In contrast, in the present work the unity of concepts is sought in the fundamentals. The Lorentz force was integrated to Maxwell equations in (Solari & Natiello, 2022b) (see Part I as well), without restrictions to point-particles, using Lorentz' Lagrangian formulation. The integration with mechanics was then performed by incorporating the kinetic part of the action integral in the form suggested by the QM of the isolated particle, thus reuniting the *material* and *action* sides of the entity. After this synthetic effort was completed, Hamilton's principle showed the extraordinary power of consilience which was its initial justification (Hamilton, 1834). It is worth mentioning that so far there has been no need to request the quantification of the EM field. All results are consistent with Planck's view (Brush, 2002) who recognised that the exchange of energy in absorption and emission processes was quantified yet, this fact does not imply that the field itself needs to be quantified a priori, being the latter an idea usually attributed to Einstein (Einstein, 1905).

The view of inferred matter as having identity by itself with forces that abut around was the standard view used in the construction of physics in the late XIX century ([529] Maxwell, 1873) and propagates there on, without being given reconsideration. For Einstein Einstein (1936) the concept of material point is fundamental for Newton's mechanics (not just an idealisation). In his work Einstein criticises Lorentz' proposal of electrons extending in the space (with soft borders in (Lorentz, 1892, §33) and spherical in (Lorentz, 1904, §8)) in the form:

> The weakness of this theory lies in the fact that it tried to determine the phenomena by a combination of partial differential equations (Maxwell's field equations for empty space) and total differential equations (equations of motion of points), which procedure was obviously unnatural. The unsatisfactory part of the theory showed up externally by the necessity of assuming finite dimensions for the particles in order to prevent the electromagnetic field existing at their surfaces from becoming infinitely great. The theory failed moreover to give any explanation concerning the tremendous forces which hold the electric charges on the individual particles. (Einstein, 1936)

The criticism is faulty in several accounts. As we have shown in Part I, a single mathematical entity, Hamilton's principle, is able to produce the equations of Electromagnetism and mechanics. Einstein gives no argument to justify the character of "unnatural" for Lorentz' procedure, actually, "unnatural" appears to stand for "alien to my scientific habit". In contrast, mathematical statements require proofs as we offer in Part I. The argument concerning a need for a force to held together the electron is equally faulty, it assumes that there are parts that need to be held together, that the quantum entity is a composite in which one part can act on another part. Actually, acquaintance with Maxwell's arguments, make us to realise that the electrostatic energy computed in electromagnetism requires the possibility of dividing the charge in infinitesimal amounts. Something that can be considered a reasonable limit in macroscopic physics (Maxwell's situation) but is ridiculous when it comes to the electron.

Maxwell's concept of science and our own concept differ radically from Einstein's. For Maxwell and the present work, the construction is a relevant part of the theory, for Einstein:

> There is no inductive method which could lead to the fundamental concepts of physics. Failure to understand this fact constituted the basic philosophical error of so many investigators of the nineteenth century. It was probably the reason why the molecular theory, and Maxwell's theory were able to establish themselves only at a relatively late date. Logical thinking is necessarily deductive; it is based upon hypothetical concepts and axioms. How can we hope to choose the latter in such a manner as to justify us in expecting success as a consequence? (Einstein, 1936).

Einstein's utterly instrumentalist statements contrast with the legend of his early reading of Kant's "Critic of pure reason" which elaborates in the opposite direction through the concept of *synthetic judgement*. The final question posed was answered by Peirce's abduction (retroduction)(Peirce, 1994) and the whole issue was considered in Whewell's philosophy of science(Whewell, 2016), also here in contrasting terms.

In contrast, Einstein accepted without criticism that "Everywhere (including the interior of ponderable bodies) the seat of the field is the empty space. (Einstein, 1936)" (referring to the EM fields) despite being conscious that the idea perpetuated the essence of the ether (Einstein, 1924).

The reason we emphasise Einstein's positions is that he has been socially instituted as the "most perfect scientist", something "known" even by the illiterate. Actually, acceptance of his statement requires the previous acceptance of the utilitarian view of science.

We believe we have shown by example that when we adopt a different *epistemic praxis* such as we have done in this work and in our reconstruction of EM (Solari & Natiello, 2022b), we reach a different theoretical understanding, actually a deeper understanding when the epistemic praxis is more demanding, as it is the relation between our dualist phenomenology (Solari & Natiello, 2022a) and instrumentalism. Thus, physics depends not only on the observable natural phenomena but it depends as well on our philosophical disposition. Yet, if the concept of theory is weakened so as to be reduced to the equations (cf. Hertz(p.21 Hertz, 1893)), the distinctions between theories mostly fade out, the phenomenological link disappears and interpretation crops up to guide the use of the formulae after sacrificing the unity of thought, critical and phenomenological actions.

In summary, there is no compelling reason to believe scientific results that have been achieved by weakening the old scientific demand for Truth into the current demand of usefulness. Newtonian physics is not incompatible with EM, as it is usually preached; and both of them can live in harmony with quantum mechanics. There is no reason for abandoning (relational) Cartesian geometry or to consider time as an "odd" spatial coordinate. There is no license to use formulae outside their range of validity as it is too often done.

Too often in the course of this investigation we have encountered misrepresentations of the thoughts and writings of the natural philosophers that developed physics until the middle of the XIX century, when the *scientist* was born. Abridged representations of their ideas and writings creep in as soon as the creators fade out. Absolute space is attributed to Newton's mechanics and "true motion" disappears (Solari & Natiello, 2021), Faraday becomes a supporter of the ether rather than the careful philosopher he was, absolutely inclined to entertain doubts as much as possible avoiding to precipitate into simplifying inventions. Faraday writes in a letter to R Taylor:

> But it is always safe and philosophic to distinguish, as much as is in our power, fact from theory; the experience of past ages is sufficient to show us the wisdom of such a course; and considering the constant tendency of the mind to rest on an assumption, and, when it answers every present purpose, to forget that it is an assumption, we ought to remember that it, in such cases, becomes a prejudice, and inevitably interferes, more or less, with a clear-sighted judgment [...]

Maxwell's plead to consider the hypothesis of the ether as worth of research (p. 493 Maxwell, 1873) was transformed as well into a belief when the story was told. Yielding to the needs of the new social position of science (and its emerging epistemic praxis), Maxwell's theory was deprived of its mathematical construction lowering the acceptance standard from correct into plausible.

In our own experience, we have spent more time undoing promissory hunches than constructing correct reasoning. It is worth noticing that such hunches trigger exploratory actions and as such are important. It is believing, instead of doubting them, what makes them prejudice. What appears to us as correct is made of the debris of our errors.

The great minds that constructed Special Relativity and Quantum Theory were passengers of their epoch, a time when “conquering nature” (the old imperialist dream of Francis Bacon) had finally taken prevalence over the “understanding nature,” sought by his contemporary Galileo Galilei. Our own work cannot escape the rule, even though we are not “great minds”. We live in a time where Nature reminds us that we must understand ki [5] and, consequently, love ki. Science must then rescue the teaching of the old masters, the natural philosophers, and their practice of critical thinking.

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Declaration of Interest

The authors declare that there exists no actual or potential conflict of interest including any financial, personal or other relationships with other people or organizations within three years of beginning the submitted work that could inappropriately influence, or be perceived to influence, their work.

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[5] The word ki has been proposed as a new and respectful pronoun for Nature and all what is part of Nature. The word comes from Anishinaabe’s language but relates as well to the Japanese and Chinese Ki and to the old English kin (like in kinship).

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