

Making Atwood's Machine "Work"

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Atwood's machine is an experiment often used to determine the acceleration due to gravity. A convenient pulley for this experiment is the PASCO Smart Pulley™, which consists of a light, plastic pulley with low-friction bearings and a photogate through which the spokes of the pulley pass. Timings can easily be converted into distances, speeds, and accelerations. Because of the pulley's low rotational inertia and the quality of the bearings, it seems to be a prime candidate for use in an Atwood's machine. When the inertia and friction of the pulley and the mass of the string are ignored, the acceleration of gravity g can be calculated from the well-known equation¹

$$g = a(m_2 + m_1)/(m_2 - m_1); \text{ (assuming } m_2 > m_1\text{),} \quad (1)$$

where m_2 and m_1 are the two masses hanging on the ends of the string passing over the pulley and a is the magnitude of the acceleration of both masses. Typical values of g achieved with

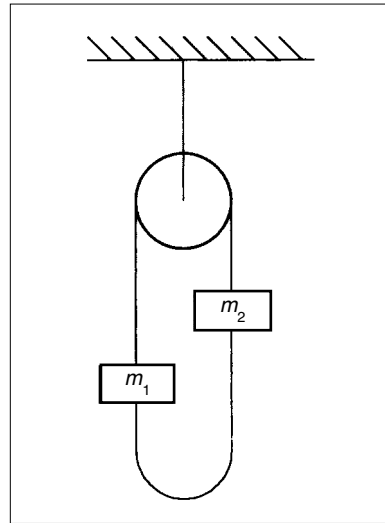


Fig. 1. Simplified diagram of apparatus showing the additional string connecting the bottoms of two hanging masses.

this apparatus using Eq. (1) are shown in Table I.

The results are disappointing. We observe two trends. The first is that the values of g obtained are closer to the accepted value of 9.81 m/s^2 when the difference between the two hanging masses increases. Secondly, the values of g are "better" when the sum of the two hanging masses is greater. These trends suggest that friction and/or rotational inertia play a significant role in spite of the low-friction bearings and the small mass of the pulley. The first trend suggests that friction is important since any inertia effect behaves like an addition to the total mass involved and thus is independent of the mass difference. The second trend shows that the total mass does affect the outcome. Unfortunately, since the friction may also change as the loading on the bearing changes, the second observed trend does not distinguish between the two mechanisms. Both mechanisms will be included in the more detailed analysis to follow.

Another potential problem is the mass of the string itself. As the string moves over the pulley,

Table I. Values of g [from Eq. (1)] for two values of total mass with varying differences in mass for each case. Bearing friction, rotational inertia, and string mass are assumed negligible.

$m_2 + m_1$ (kg)	$m_2 - m_1$ (kg)	g (m/s^2)
0.37626	0.00197	8.68
0.37626	0.00408	9.08
0.37626	0.00995	9.51
0.37626	0.02003	9.61
0.02601	0.00199	8.21
0.02601	0.00402	8.45
0.02601	0.00602	8.57

the mass of the string moves from one side to the other, in effect causing a varying mass difference between the two sides. Using very light string or thread can reduce the effect but never remove it. To eliminate this problem, an additional length of string is connected from the bottom of one hanging mass to the bottom of the other one (see Fig. 1). With this configuration the mass of the string on each side of the pulley remains the same as the masses move. Due to the string's mass, there is a small increase in mass of the system that we will include in the effective mass of the pulley. An equivalent alternative scheme suggested by another author² is to attach a length of string to the bottom of each hanging mass, making sure that the lower ends of the strings touch the ground at all times during the motion.

The Modified Approach

The effects of the bearing friction and the combined pulley and string inertias can be included as effective masses. The rationale for this comes from analyzing the sums of the forces on the masses and the sum of the torques on the pulley. Assuming that $m_2 > m_1$, the pertinent equations are:

$$\text{(for } m_2) \quad m_2 g - T_2 = m_2 a, \quad (2)$$

where T_2 is the tension in the string and a is the acceleration.

$$\text{(for } m_1) \quad T_1 - m_1 g = m_1 a, \quad (3)$$

where T_1 is the tension in the string on the other side of the pulley.

$$\text{(for pulley)} \quad T_2 R - T_1 R - \tau_f = I_p \alpha, \quad (4)$$

where R is the radius of the pulley, τ_f is the frictional torque of the pulley bearing, I_p is the pulley moment of inertia, and α is the angular acceleration of the pulley. Note that $\alpha R = a$.

The frictional torque is assumed to be a constant for a given total mass of the system. Experimentally this assumption can be checked. If you simply spin a Smart Pulley™ and record the timing data as it slows down, you will find that

the speed-versus-time graph is essentially linear, indicating a constant-deceleration. When the two masses are present, the speed-versus-time data is also linear. A nonconstant torque will not result in constant acceleration. It is also easy to show that the friction depends on the loading of the pulley. For instance, masses of 0.0202 kg and 0.0200 kg hung on the pulley will accelerate while masses of 0.3002 kg and 0.3000 kg will remain stationary. The force difference is the same in both cases, but friction is overcome only for the case of the lighter masses. As expected, the frictional force is greater when the bearings carry a heavier load.

Since the frictional torque seems to be constant, let us write it in the form $\tau_f = m_f g R$, where m_f can be thought of as a small equivalent mass that has been added to the lighter mass, m_1 , to account for the small frictional torque of the pulley. Let us also write the moment of inertia $I_p = m_p R^2$ where m_p is an effective mass of the pulley.³ It is not equal to the actual mass of the pulley because the mass of the real pulley is not all distributed on the outer rim at radius R .

The mass of the string is included in this quantity because the string also accelerates and is at a radial distance R . With these definitions Eq. (4) can be rewritten as

$$\text{(for pulley)} \quad T_2 R - T_1 R - m_f g R = m_p R^2 \alpha. \quad (5)$$

If we divide Eq. (5) by R and write α in terms of a , then

$$\text{(for pulley)} \quad T_2 - T_1 - m_f g = m_p a. \quad (6)$$

Now Eqs. (2), (3), and (6) can be solved simultaneously for the acceleration a , resulting in the following equation for the acceleration of gravity:

$$g = a(m_2 + m_1 + m_p)/(m_2 - m_1 - m_f). \quad (7)$$

An equivalent form of this equation is found in the literature.²

Experimental Design and Data Analysis

To obtain values of g we need to determine all the quantities in Eq. (7). This is straightforward except for m_p and m_f . We must remember that m_p is constant independent of the hanging masses used while m_f varies. However, m_f re-

mains nearly constant if the sum of the hanging masses stays constant. When the hanging masses are imbalanced and accelerating, the tension in the string is less than when the masses are balanced and stationary. Resorting to the simple model that ignores friction and pulley inertia for guidance here, it is easily shown that the tension goes to zero as the acceleration of the masses approaches g . For an acceleration of $0.3g$ (the maximum observed in the data to follow) the tension has only decreased by 9% so the load on the pulley bearings stays relatively constant for accelerations $\leq 0.3g$. This basic conclusion is unchanged by small values of m_p and m_f in Eq. (7).

Solving Eq. (7) for $m_2 - m_1$ gives

$$m_2 - m_1 = a(m_2 + m_1 + m_p)/g + m_f, \quad (8)$$

which implies a linear relationship between $m_2 - m_1$ and a , with an intercept of m_f if $m_2 + m_1$ is kept constant. Seven data sets were taken for various total values of hanging mass ($m_2 + m_1$) ranging from 0.02601 kg to 0.37626 kg. Values of acceleration are obtained from the pulley timings using a least-squares fit to find the slope of the speed-versus-time graph. To obtain more reliable acceleration values, each experimental run is repeated three or four times. In the sub-

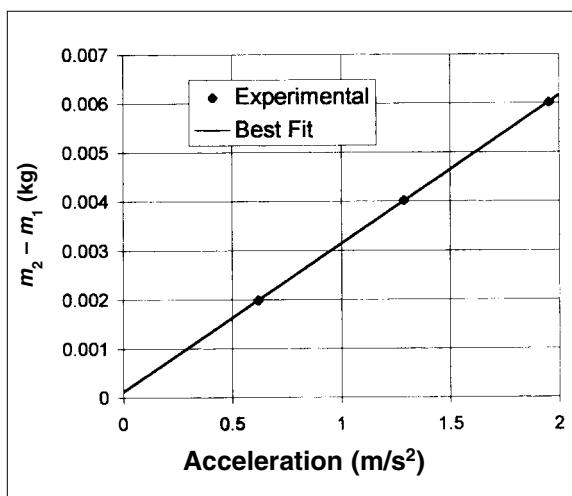


Fig. 2. Relationship between mass difference and acceleration for the case of $m_1 + m_2 = 0.02601$ kg. For the best fit line, the slope (S) = $3.019 \pm 0.012 \times 10^{-3}$ kg-s²/m and the intercept (m_f) = $1.26 \pm 0.16 \times 10^{-4}$ kg.

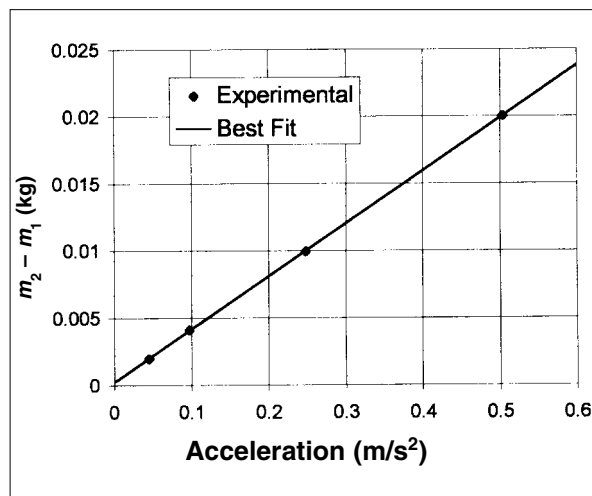


Fig. 3. Relationship between mass difference and acceleration for the case of $m_1 + m_2 = 0.37626$ kg. For the best fit line, the slope (S) = $3.928 \pm 0.009 \times 10^{-2}$ kg-s²/m and the intercept (m_f) = $2.34 \pm 0.26 \times 10^{-4}$ kg.

sequent analysis only the mean values of the accelerations are used. Representative graphs of Eq. (8) are shown in Figs. 2 and 3 for the minimum and maximum values of the total hanging mass. Because all of the masses were measured to the nearest 0.00001 kg, no vertical error bars are shown. Typical standard deviations of the acceleration values plotted on the horizontal axes are less than 1% of the mean values. The largest is 3%, which would be approximately the size of the plot symbol on the graph. Thus, no horizontal error bars are displayed.

For convenience we define the slopes of the graphs from Eq. (8) as

$$S = (m_2 + m_1 + m_p)/g. \quad (9)$$

For each value of $m_2 + m_1$, the slope S is determined by a least-squares fitting routine. Solving Eq. (9) for $m_2 + m_1$, we can write

$$m_2 + m_1 = gS - m_p. \quad (10)$$

Since we have obtained seven pairs of $m_2 + m_1$ and S values (see for example Figs. 2 and 3), the relationship in Eq. (10) can be graphed to determine g . The slope of the graph in Fig. 4 gives the value of $g = 9.663 \pm 0.006 \text{ m/s}^2$. This value has a high precision ($\pm 0.06\%$) but differs from the accepted value of $g = 9.81 \text{ m/s}^2$ by 1.5%.

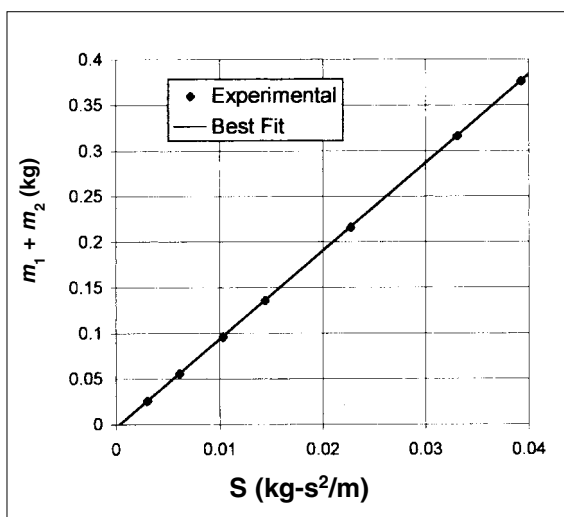


Fig. 4. Relationship between $m_1 + m_2$ and S [see Eq. (10)]. Best fit line has a slope (g) = $9.663 \pm 0.006 \text{ m/s}^2$ and an intercept ($-m_p$) = $-3.36 \pm 0.14 \times 10^{-3} \text{ kg}$.

Again, error bars are too small to be seen on the figure.

Resolution and Conclusion

The software used for obtaining all of the

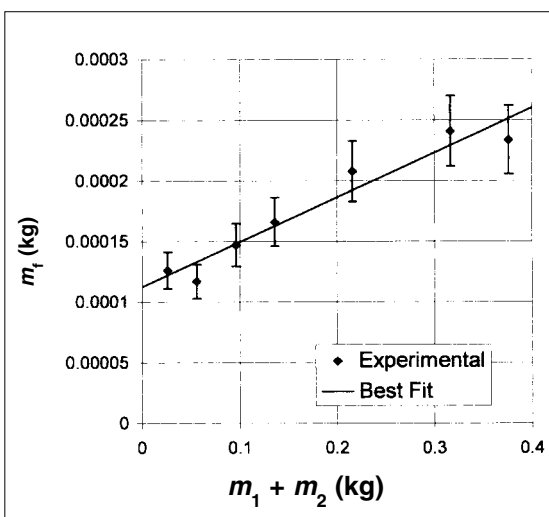


Fig. 5. Relationship between effective frictional mass m_f and $m_1 + m_2$. Best fit line has a slope = $3.7 \pm 0.4 \times 10^{-4}$ and an intercept = $1.12 \pm 0.09 \times 10^{-4} \text{ kg}$.

Table II. Values of g [from Eq. (10)] before and after applying the correction factor for the circumference of the Smart Pulley™. The effects of pulley friction and inertia and string mass are included in the analysis.

Quantity of Interest	Uncorrected Value	Corrected Value
Pulley circumference (m)	0.15000	0.15225 ± 0.00008
Acceleration of gravity (m/s^2)	9.663 ± 0.006	9.808 ± 0.011

acceleration data assumes that the circumference of the pulley is exactly 0.15 m when the string runs in the groove. As a double check on this value, we passed the string over the pulley and measured the length for six pulley revolutions to be 0.9135 ± 0.0005 m. From this measurement the circumference is 0.15225 ± 0.00008 m. The ratio of the measured circumference to the assumed value is 1.0150 ± 0.0005 and provides a scale factor that applies to all quantities determined with the pulley that have length dimensions. Table II summarizes the final results for the acceleration of gravity.

This experiment shows that a Smart Pulley™ can be used to measure the acceleration of gravity very accurately but only if bearing friction and pulley inertia are included. The concepts involved are relatively straightforward and can be appreciated by undergraduate students. It is appropriate for a project lab emphasizing experimental technique and computer-data handling. C. Wang² also proposed a method for including the friction and inertia of the pulley, but it required an independent measurement of the inertia using a physical pendulum and a determination of the “frictional mass” by searching for the zero acceleration condition. As outlined in the next section, the technique described in this paper not only determines g but also m_p and m_f without any additional measurements.

An Added Bonus

Although we did not set out to determine the moment of inertia of the pulley or the value of the friction, these were byproducts of the methodology. The effective mass of the pulley,

m_p , is determined from the intercept of the graph in Fig. 4 to be 0.00336 ± 0.00014 kg. The mass of the string, 0.00013 ± 0.00001 kg, must be subtracted from this, leaving a value of 0.00323 ± 0.00015 kg. Using the value of R determined from the measured circumference (see Table II), $I_p = m_p R^2 = 1.90 \pm 0.09 \times 10^{-6}$ kg-m² that is in good agreement with the value of 1.8×10^{-6} kg-m² given on the PASCO website (www.pasco.com). The effective “frictional mass” values obtained from the intercepts of graphs like those shown in Figs. 2 and 3 do indeed vary. Figure 5 shows a graph of m_f as a function of $m_2 + m_1$. The friction appears to have an inherent value with no external load on the bearings and then to increase almost linearly as the loading is increased by the added mass. Not all pulleys are identical. If this experiment is performed, be sure to use the same pulley for the duration of the data-taking phase.

Acknowledgment

I give special thanks to Beckie Fulton and Jim Edwards, two of my physics students who demonstrated as part of a laboratory project the basic feasibility of the method presented here.

References

1. This equation is found in the mechanics sections of many introductory physics textbooks. For instance, see E. Hecht, *Physics: Algebra/Trig*, 2nd ed. (Brooks/Cole, Pacific Grove, CA, 1998), p. 133.
2. C. Wang, “The improved determination of acceleration in Atwood’s machine,” *Am. J. Phys.* **41**, 917-919 (1973).
3. Thomas B. Greenslade, “Atwood’s machine,” *Phys. Teach.* **23**, 24-28 (Jan. 1985).