

# Unipolar induction: a neglected topic in the teaching of electromagnetism

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**Abstract.** When a cylindrical magnet is rotated about its axis, electric fields develop on its surface which can be used to generate continuous currents. This effect is an example of electromagnetic induction, mechanical energy being converted into electrical energy, through the mediation of the magnetic field. It is suggested that the effect can be explained most simply in terms of the forces acting on conduction electrons inside the magnet, rather than in terms of flux linkage and the cutting of lines of force.

## 1. Introduction

In 1831, shortly after his discovery of electromagnetic induction, Michael Faraday carried out the experiment shown schematically in figure 1. A cylindrical steel magnet was hung vertically with its lower pole immersed in a bottle of mercury, and its upper pole connected to the circuit shown. When the magnet was made to rotate about its axis, a continuous current was observed in the galvanometer [1, p 204]. This should be compared with an earlier experiment performed by Ampère in 1821; if the galvanometer is replaced by a Voltaic pile so that a current is driven round the circuit, the magnet is found to rotate spontaneously about its own axis [1, p 168]. (The

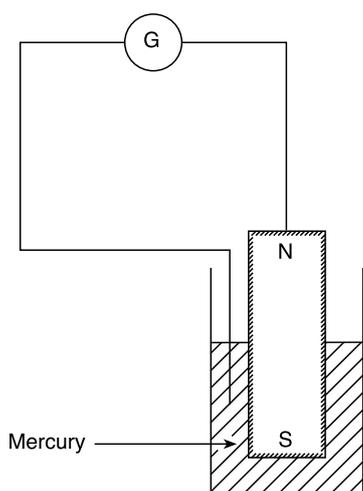


Figure 1. Diagram of Faraday's experiment of 1831.

close connection between these two experiments was not well understood at the time, because the principle of conservation and transformation of energy had not yet been established.)

Faraday's discovery sparked off a debate among physicists and electrical engineers which lasted the rest of the nineteenth century; this has been fully documented in a fine review by Miller [2]. In 1841 Weber christened the effect *unipolar induction*, because he believed that only one of the poles of the magnet was involved; he extended the term to include more general situations, in which a disc or hollow cylinder of copper rotated about the axis of the magnet. Attempts were made to explain unipolar induction in terms of circuit elements cutting through magnetic lines of force, but these explanations raised an immediate question. When a magnet rotates, do its lines of force rotate with it, and if so do they create an electromotive force as they pass through a stationary conductor?

Faraday himself believed that as the magnet rotates its lines of force remain stationary, and this brought him into conflict with Ampère's theory, in which magnetic properties arise from current loops within individual molecules in the body of the magnet; so that if lines of force exist they should be carried round with the magnet as it rotates. Physicists found themselves divided into roughly three camps: those who believed that the lines of force rotated, those who believed that they did not, and those who believed that lines of force were merely a representation of the field, so that the question had no physical meaning. It turned out to be very difficult to devise an experiment which could decide unambiguously between these hypotheses, and in the nineteenth century the theory of electromagnetism was not sufficiently established for it to be able to decide the matter on theoretical grounds alone. It was only in the twentieth century, with the general acceptance of Maxwell's equations, the electron theory of Lorentz and the principles of relativity, that a consensus on unipolar induction could emerge.

Although there was much confusion in the early years about the theory of unipolar induction, it was accepted that the effect actually exists, and there were some attempts to exploit it commercially. In 1912 the Westinghouse Corporation built a colossal unipolar generator, which delivered a direct current of 7700 A at a voltage of 264 V. However, such machines were never put into general production, because it was conceded that AC generators provided a better technology [2].

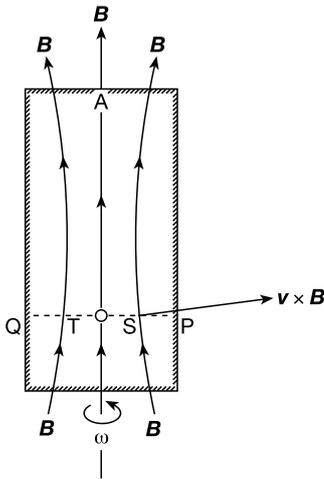
Students today still find unipolar induction a puzzling phenomenon, partly because the magnetic flux linking the circuit is constant in time. (I confess that when I first read an account of Faraday's experiment I was convinced that it was wrong, and I have found similar reactions among other teachers.) In the following sections I will describe the theory of unipolar induction in the simplest possible way, without invoking relativity, and then I will consider the impact of these ideas on the teaching of electromagnetism in general.

## 2. The origin of the electromotive force

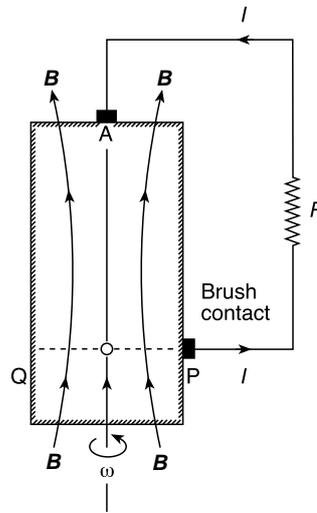
The fact that one can move or rotate a magnet does not always mean that one can move or rotate its magnetic field. The field is characterized by the value of the magnetic vector  $\mathbf{B}$  at every point in space; if all of these vectors are independent of time, the field is *constant*, regardless of whether the magnet producing the field is moving or not. In figure 1 the magnetic field is symmetrical about the cylindrical axis, and is unchanged when the magnet is set in rotation. Magnetic lines of force are a useful description of the magnetic field, but they are not real in themselves.

We shall stick to the laboratory frame of reference, in which the magnet is rotating and the external circuit is stationary. Within this frame the magnetic field is constant, and no electric fields can be *induced* in the electromagnetic sense.

Consider first the case of a cylindrical magnet which is rotating about its axis in complete electrical isolation (see figure 2). Let the radius of the magnet be  $a$ , and its angular velocity  $\omega$ . Conduction electrons within the magnet undergo collisions with the moving atoms, and this causes them to take up the rotational motion. At each point the electrons have a net drift



**Figure 2.** The case of a cylindrical magnet rotating about its axis in complete electrical isolation.



**Figure 3.** The rotating magnet of figure 2, but now connected to an external electrical circuit.

velocity  $v$ , given by

$$v = \omega \times r \quad (1)$$

where  $r$  is the position vector of the point in question, relative to an origin which is chosen to lie on the magnet axis.

The conduction electrons experience a force  $-ev \times B$  which is mainly directed *towards* the central axis; as a result a negative charge appears in the body of the magnet, and a positive charge on its curved outer surface. In equilibrium an electrostatic field is set up, such that the total Lorentz force on the conduction electrons is zero:

$$F = -eE - ev \times B = 0. \quad (2)$$

Therefore

$$E = -v \times B. \quad (3)$$

We now calculate the potential  $V$  corresponding to this electric field. Equation (3) shows that along the axis  $E$  is zero, and we can take  $V$  as zero at all points along this axis. Hence, at a general point P on the surface of the magnet

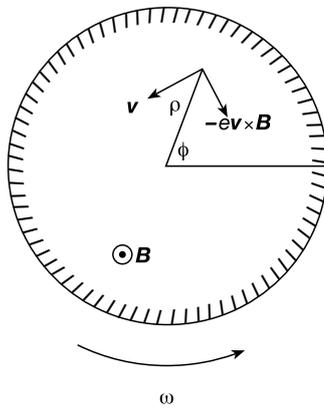
$$V_P = - \int_0^P E \cdot ds = + \int_0^P v \times B \cdot ds. \quad (4)$$

Expressing this integral in terms of cylindrical polar coordinates  $\{\rho, \phi, z\}$

$$\begin{aligned} V_P &= - \int_0^a \omega \rho B_z d\rho = \frac{\omega}{2\pi} \int_0^a B_z 2\pi \rho d\rho \\ &= \frac{\omega}{2\pi} \Phi_B \end{aligned} \quad (5)$$

where  $\Phi_B$  is the total magnetic flux passing through the horizontal circle QP.

This argument can be extended to any point inside the magnet; for example, at point S the potential  $V$  is given by equation (5), provided that  $\Phi_B$  is interpreted as the magnetic flux passing through the circle TS. It follows that the equipotential surfaces are obtained by rotating the magnetic lines of force about the axis of the cylinder, and one can also confirm that  $E$  is irrotational and is described completely by a scalar potential  $V$ .



**Figure 4.** Vector diagram in polar coordinates showing the velocity of and forces acting on the electrons in the rotating magnetic cylinder of figure 3 (see text for details).

Now consider the situation in figure 3, where brush contacts have been attached to points A and P, and they are connected by a stationary wire circuit of resistance  $R$ . There is a reduction in the voltage  $V_p$ , and inside the magnet the electric force becomes weaker than the magnetic force; hence electrons are drawn in towards the axis and a conventional current  $I$  circulates in the sense shown in figure 3.

In order to maintain this current indefinitely there has to be a continuous input of energy, and we now need to see where this energy is coming from. The lines of current flow inside the magnet are complicated, but fortunately we shall not have to calculate them in detail. In figure 2 the flow lines are circles, but in figure 3 they form inward spirals, and this introduces a new component into the magnetic force.

It is clear from figure 4 that if the drift velocity  $v$  has a component in the negative  $\rho$  direction, the magnetic force  $-ev \times B$  must have a component in the negative  $\phi$  direction, and this is directed *against* the angular velocity  $\omega$ . By means of their collisions the electrons exert a negative torque on the magnet, and to maintain its rotation a positive torque has to be supplied mechanically from outside. (We shall regard a torque as positive if it acts in the same sense as  $\omega$ .) Hence the mechanical work performed on the magnet is converted into electrical energy in the external circuit. Note that no energy is given to the electrons by the magnetic field itself, because the magnetic force  $-ev \times B$  is at right angles to the velocity  $v$ . The electrons acquire energy and net momentum from the collisions, and the role of the magnetic field is to *redirect* this momentum towards the axis, so that a current flows round the circuit.

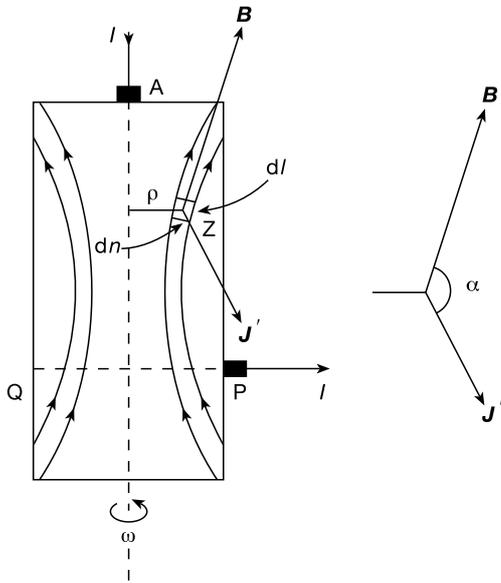
(Although we have concentrated on the unipolar generator, it is clear that the same system can act as a unipolar electric motor, as in Ampère's experiment [1, p 168]. If we replace the load resistor  $R$  in figure 3 by a power source providing a current  $I$  in the same direction, the magnet will rotate in an anticlockwise sense about the  $z$  axis, and will deliver a torque to an external mechanical system.)

### 3. The calculation of the electromotive force

We now return to the case of the unipolar generator. It is clear from figure 4 that equation (1) does not represent the drift velocity when current is being drawn from the system, and it does not even describe its tangential component  $v_\phi$ . We must be careful not to use equation (1) when the EMF is being calculated.

The EMF can be defined as the amount of mechanical energy converted into electrical energy per coulomb of charge circulated, and we begin by calculating the mechanical torque acting on the magnet. This can be done using the following argument due to Page and Adams [3]. The current density inside the magnet is given by

$$\mathbf{J} = -ne \mathbf{v} \quad (6)$$



**Figure 5.** Diagram showing the particular form of the volume element used in calculating the EMF for the unipolar generator (see text for details).

where  $n$  is the density of the conduction electrons.

The torque exerted by the electrons on the magnet is given by

$$\mathbf{G} = \iiint \mathbf{r} \times (\mathbf{J} \times \mathbf{B}) d\tau \quad (7)$$

integrated over the volume of the magnet.

In figure 5 we choose a particular form for the volume element  $d\tau$ . Take two neighbouring lines of force, and rotate them round the cylinder axis to form a curved sheath of magnetic flux. At the point  $Z$ ,  $d\tau$  is subtended between the inner and outer surfaces:

$$d\tau = \rho d\phi dl dn. \quad (8)$$

Let  $\mathbf{J}'$  be the projection of  $\mathbf{J}$  onto the  $\{\rho, z\}$  plane. ( $\mathbf{J}'$  is  $\mathbf{J}$  without its tangential component  $J_\phi$ .) The axial component of  $\mathbf{G}$  has the form

$$G_z = - \iiint \rho^2 B J' \sin \alpha d\phi dl dn \quad (9)$$

where  $\alpha$  is the angle between  $\mathbf{B}$  and  $\mathbf{J}'$ . (Note that  $J_\phi$ , the tangential component of  $\mathbf{J}$ , does not contribute to  $G_z$ : see figure 4.)

The flux enclosed in the walls of the sheath is given by

$$d\Phi_B = 2\pi\rho B dn \quad (10)$$

therefore

$$G_z = - \iiint \frac{\rho}{2\pi} J' \sin \alpha d\phi dl d\Phi_B. \quad (11)$$

The current enters at  $A$  and leaves at  $P$ , and it equals the total current across the walls of the sheath:

$$I = \iint \rho J' \sin \alpha d\phi dl \quad (12)$$

therefore

$$G_z = - \frac{I}{2\pi} \int d\Phi_B = - \frac{I}{2\pi} \Phi_B \quad (13)$$

where  $\Phi_B$  is the flux passing through the circle QP.

Let

$$\Gamma = \frac{I}{2\pi} \Phi_B. \quad (14)$$

Hence the conduction electrons exert a torque  $-\Gamma$  on the magnet, and it follows that the mechanical torque on the magnet is  $+\Gamma$ .

It is now easy to calculate the EMF  $\mathcal{E}$ . The mechanical power supplied is  $\Gamma\omega$  and this is equal to  $\mathcal{E}I$ :

$$\Gamma\omega = \frac{I\omega}{2\pi} \Phi_B = \mathcal{E}I \quad (15)$$

therefore

$$\mathcal{E} = \frac{\omega}{2\pi} \Phi_B. \quad (16)$$

(It is reassuring to see that  $\mathcal{E}$  equals the voltage between A and P when the system is on open circuit—see equation (5).)

The current density  $\mathbf{J}$  has a complicated structure inside the magnet, but it is important to realise that it is constant in time. No current filaments are cutting through stationary lines of force. Here we have a case of electromagnetic induction in which both the magnetic field  $\mathbf{B}$  and the current density  $\mathbf{J}$  are constant.

Before leaving the calculation of the EMF, we should compare the results we have obtained with the corresponding results for a relativistic theory. The basic assumptions of the relativistic treatment are of course different, but in the laboratory frame of reference the conclusions are very similar, at least to first order in  $a\omega/c$  where  $a$  is the radius of the magnet. Equation (3) is still valid for the electric field  $\mathbf{E}$  inside an isolated rotating magnet. However, we assumed in our discussion that this electric field is generated by a displacement of conduction electrons, and this is not entirely true in the relativistic theory;  $\mathbf{E}$  arises partly from the displacement of the conduction electrons, and partly from an electric polarization  $\mathbf{P}$  of the medium. This subject lies outside the scope of the present paper, and a good account of it is given by Rosser [4].

#### 4. The conservation of angular momentum

Figure 6 shows a conventional DC generator, which should be compared with the unipolar generator in figure 7. (Note that in the conventional generator the axis of rotation is at right angles to the magnetic field; in the unipolar generator it is parallel to the field.) In figure 6 there is a source of mechanical power or engine, which is connected by a shaft to the armature. Suppose that the engine exerts a torque  $+C$  on the armature, where  $C$  is a positive quantity. When the system is running at constant speed the magnet exerts a torque  $-C$  on the armature, and the armature reacts against the magnet and exerts a torque  $+C$  upon it. When the system is working backwards as an electric motor, it is the reaction of the armature against the magnet which provides the mechanical power.

In figure 7 the engine exerts a torque  $+\Gamma$  on the magnet, but there is no action and reaction between the magnet and armature, because they have become the same body. When the system is working backwards as an electric motor, one has the uncomfortable feeling that the magnet 'is pulling itself round by its own bootstraps' [5]†.

This paradox illustrates the point that one has to be extremely careful when one applies Newton's third law to electromagnetic systems ([7], [8, section 27-9]). The electromagnetic field itself possesses momentum, a concept which is more familiar to most of us in the case of photons than it is for static fields. The density of linear momentum is given by Poynting's vector, divided by  $c^2$ :

$$\mathbf{g} = \frac{1}{c^2} \mathbf{E} \times \mathbf{H}. \quad (17)$$

† There is a comment on this paper by Scanlon and Henriksen [6].

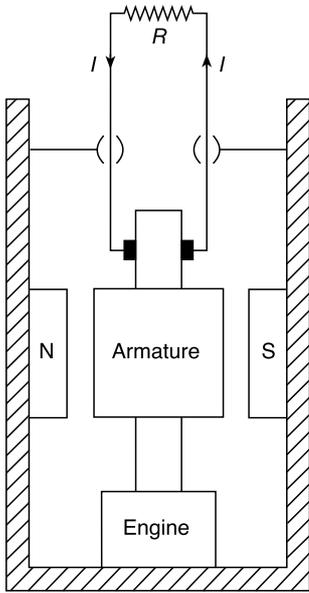


Figure 6. Conventional DC generator.

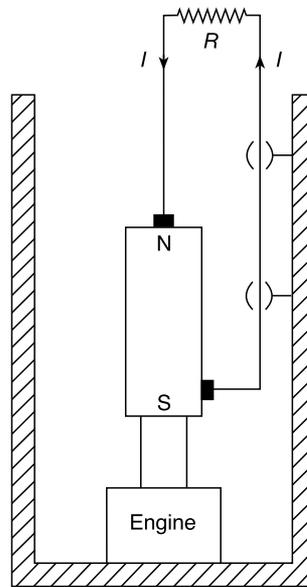


Figure 7. Unipolar generator.

Hence the angular momentum of the field is given by

$$L = \iiint r \times g \, d\tau \tag{18}$$

and this has to be added to the angular momentum of the mechanical parts of the system, to give the total angular momentum which is conserved. In all the cases we have considered both  $E$  and  $H$  are time-independent, and it follows that the electromagnetic and the mechanical angular momenta are both conserved individually. However, the existence of the field momentum implies that we cannot always divide the system into mechanical parts which are interacting with each other in action–reaction pairs. But if the complete system is mounted on a freestanding framework (shown shaded in the diagrams), then we can say that the total torque on this framework must be zero.

These considerations are important in the unipolar generator, because one can see from figure 2 that the field has an angular momentum directed along the axis of rotation, given by equation (18). In figure 7 the engine exerts a torque  $+\Gamma$  on the magnet, so that it exerts a torque  $-\Gamma$  on the framework. If the external circuit were to rotate with the magnet the EMF would be zero; for the generator to work the external circuit must be attached to the frame. It is shown in the appendix that the magnetic field exerts a torque on the external circuit, and the magnitude of this torque is  $+\Gamma$ . This is transmitted to the frame, and it cancels the torque on the frame exerted by the engine.

In the conventional generator the magnetic field is at right angles to the axis of rotation, and it seems that the field momentum can be neglected. In figure 6 the torque  $-C$  which the engine exerts on the frame is balanced by the torque  $+C$  which is transmitted through the magnet, so that the total torque on the frame is again zero. One might argue that the external circuit could experience a small torque in the fringing field of the magnet, and that in order to prevent the external circuit from rotating it has to be attached to the frame. However, this is not important, because any torque exerted by the external circuit on the frame would be cancelled by its torque on the magnet.

## 5. Discussion

Suppose we have a closed loop of thin wire which is moving or changing shape in a magnetic field which is constant in time. Standard arguments show that an EMF is induced in the wire, given by the equation

$$\mathcal{E} = \oint \mathbf{v} \times \mathbf{B} \cdot d\mathbf{s} = -\frac{d}{dt} \Phi_B \quad (19)$$

where  $d\mathbf{s}$  is a line element of the loop which is moving at velocity  $\mathbf{v}$ , and  $\Phi_B$  is the magnetic flux linking the loop.

Equation (19) is immensely useful, and it leads on naturally to the discussion of time-varying fields, Maxwell's equations and relativity. On the other hand, it can be misrepresented as the basic principle of electromagnetic induction, linking it inevitably with flux changes and the cutting of lines of force. The student can be forgiven for thinking:

'If the magnetic field  $\mathbf{B}$  is constant, and the current density  $\mathbf{J}$  is also constant, no EMF can be induced.'

This is an entirely false conclusion, as the counterexample of the unipolar generator shows. In the case of the moving wire, the current at any point is confined by the direction and velocity of the line element  $d\mathbf{s}$ . In the case of a continuous medium such as the magnet the current is not confined in this way, and there is no valid argument which takes us from the particular to the general case. It is not clear what meaning can be attached to equation (19) in the case of a unipolar generator which is delivering current to an external load.

A very clear but brief exposition of this argument is to be found in Feynman's *Lectures* [8, section 17-2]. He discusses a slightly different form of unipolar generator known as Faraday's disc, another of Faraday's discoveries in his great year of 1831 [1, p 196]. In this system the current density  $\mathbf{J}$  is constant in time, just as it is for the rotating magnet, and Feynman stresses that equation (19) is not a suitable starting point for the discussion. His conclusion is that when we are studying electromagnetic induction, the correct physics can always be obtained by considering Maxwell's equations and the Lorentz forces on the electrons.

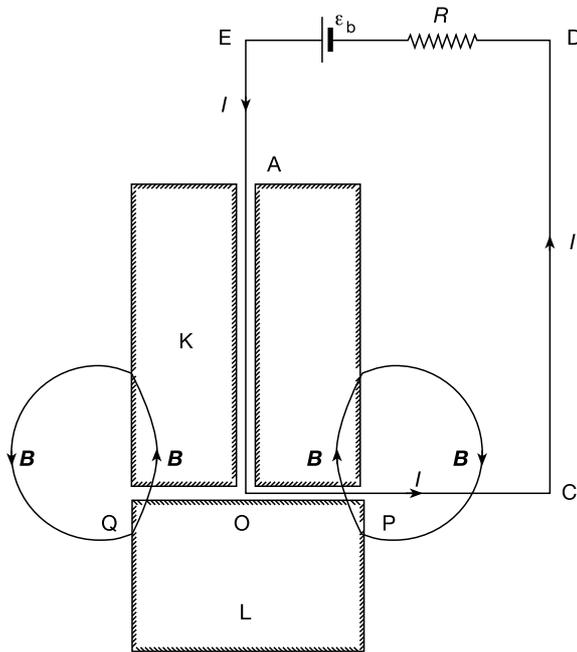
In these days of crowded syllabuses and examination deadlines, there are good arguments for omitting minor topics such as unipolar induction. However, I suggest that the material presented to students should be designed in such a way that the unipolar generator is not completely incomprehensible should they happen to come across it. It is better to speak of a magnetic field which is changing rather than moving or rotating, and lines of force should be presented as a good device for describing the field, but not as something real in themselves. Most of all, I would suggest that the electromotive force should be defined in terms of energy transfer, and not in terms of a line integral as in equation (19). The EMF of a chemical cell is the amount of chemical energy converted into electrical energy per coulomb circulated; the same definition works excellently for an electromagnetic generator, provided that one substitutes mechanical energy for chemical energy.

## Acknowledgment

I am grateful to Mr Stuart Leadstone for several lively discussions on this topic.

## Appendix. The torque acting on the external circuit

Suppose that the magnet in figure 2 has been cut into two cylindrical sections K and L, and that a narrow hole AO has been drilled down the axis of section K, as indicated in figure A1. A rigid insulated wire circuit OCDE is inserted as shown, and sections K and L are brought close together, so that  $B_z$ , the vertical component of the field in the gap, is equal to  $B_z$  inside



**Figure A1.** Diagram showing the set-up for calculating the torque acting on the external circuit (see text for details).

the magnet. A battery  $\mathcal{E}_b$  maintains a constant current  $I$  in the circuit. There is no need to rotate the magnet.

The field of the magnet exerts forces on the various parts of the circuit, and let the total axial component of the torque on the circuit be  $G_O$ . We can think of this torque as the sum of two terms; the torque  $G_{OP}$  acting on the section  $OP$ , and the torque  $G_{PE}$  acting on the rest of the circuit:

$$G_O = G_{OP} + G_{PE}. \quad (\text{A1})$$

Now make a virtual rotation  $\delta\phi$  of the circuit about the axis of the magnet. No EMF is induced because the flux linking the circuit is unchanged, and the current  $I$  remains constant. The work performed on the circuit is

$$\delta W = -G_O \delta\phi. \quad (\text{A2})$$

It is clear from the cylindrical symmetry that the total energy of the system is unchanged, and in fact no work is performed in the rotation:

$$\delta W = -G_O \delta\phi = 0$$

therefore

$$G_O = G_{OP} + G_{PE} = 0 \quad (\text{A3})$$

so that the total torque on the circuit is zero.  $G_{OP}$  is easily calculated:

$$\begin{aligned} G_{OP} &= - \int_0^a \rho B_z I d\rho = - \frac{I}{2\pi} \int_0^a 2\pi \rho B_z d\rho = - \frac{I}{2\pi} \Phi_B \\ &= -\Gamma \end{aligned} \quad (\text{A4})$$

where  $\Gamma$  is defined in equation (14).

Hence from equation (A3) we have

$$G_{PE} = +\Gamma. \quad (\text{A5})$$

This is a very general result, and it does not depend on the shape or dimensions of the external circuit. Note that the system in figure A1 could not operate as a unipolar electric motor. To get unipolar action we would have to make a mechanical break at point P, in such a way that the two parts of the circuit could move independently while maintaining electrical contact. For example, if a thin copper disc were placed in the gap between K and L, with brush contacts to the external circuit, it would rotate as a Faraday disc.

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